A note on $L^1$-bounded martingales. II

By Toshitada SHINTANI *
(Received November 25, 1992)

Abstract.
Let $f$ be an $X$-valued martingale when a Banach space $X$ has the Radon-Nikodým Property. If $f$ is $L^1_X$-bounded then $f$ is of bounded variation. Using this the martingale transform $g$ of $f$ converges a. e. in $X$.

1. Notations.
Let $(\Omega, \sigma, P)$ be a probability space and $a_1, a_2, \ldots$ a nondecreasing sequences of sub-$\sigma$-fields of $a$. Let $X$ be a Banach space with norm $|\cdot|$ and the Radon-Nikodým property. Let $f = (f_1, f_2, \ldots)$ be an $X$-valued martingale with norm $\|f\|_1 = \sup \mathbb{E} |f_n| < \infty$. Let $\nu = (\nu_1, \nu_2, \ldots)$ be a real-valued predictable sequence, that is, $\nu_k: \Omega \to \mathbb{R}$ is $\mathcal{A}_k$ measurable, $k \geq 1$. Then $g = (g_1, g_2, \ldots)$, defined by $g_n = \sum_{k=1}^n \nu_k (f_{k+1} - f_k)$ with $|\nu_k| \leq 1$ in absolute value, is the transform of the martingale $f$ by $\nu$. Write $\|f\|_p = \sup |g_n(\omega)|$ and define the maximal function $g^*$ of $g$ by $g^*(\omega) = \sup |g_n(\omega)|$.

2. Results and the proofs.

Theorem 1. If $\|f\|_1 < \infty$ then $\sum_{n=1}^\infty |f_{n+1} - f_n| < \infty$ a. e., that is, $f$ is of bounded variation.

Proof. By Chatterji’s result, $f_n \to f_\infty$ a. e. in $L^1_X$ as $n \to \infty$.

Now, for almost all $\omega \in \Omega$
$$
\sum_{n=1}^\infty |f_{n+1}(\omega) - f_n(\omega)| \leq 2 \cdot \sum_{n=1}^\infty |f_n(\omega) - f_\infty(\omega)|.
$$
Let $a_0(\omega) = |f_0(\omega) - f_\infty(\omega)|$ then $\lim_{n \to \infty} a_0(\omega) = 0$.

Thus, for $0 < \varepsilon < 1$, there exists a number $N = N(\varepsilon, \omega) > 0$ such that $0 \leq a_0(\omega) < \varepsilon < 1$ (\forall n \geq N).

Let $x_0(\omega) = \sqrt[n]{a_0(\omega)}$ and $S = \{x_1(\omega), x_2(\omega), \ldots\}$.

Since $S$ is a bounded infinite set of real numbers in general, by the Bolzano-Weierstrass’ theorem, $S$ has the accumulation point $\lambda = \lambda(\omega) = 0$ and $\lambda = \lim_{n \to \infty} x_n(\omega) < 1$.

In fact, if $\lim_{n \to \infty} x_n(\omega) = \lambda = 1$ then, for $0 \leq \varepsilon < 1$, there exists a number $N = N(\varepsilon, \omega) > 0$ such that $|\sqrt[n]{a_0(\omega)} - 1| \leq \varepsilon$ for all $n \geq N$. So $0 \leq (1 - \varepsilon)^n \leq a_0(\omega) \leq (1 + \varepsilon)^n$ (\forall n \geq N).

If $\varepsilon$ does not near to 0 then $\lim_{n \to \infty} x_n(\omega) = 1$ does not hold so let $\varepsilon \to 0$ then $N$ is nondecreasing and $N = N(0, \omega)$ is finite or infinite.

Since for $\varepsilon = 0$ the above inequality holds for all $n \geq N$ when $N = N(0, \omega) \leq \infty$, so

* 数学，一般教科助教授
Thus it is not that \( a_n(\omega) \to 0 \) as \( n \to \infty \). This contradicts to the fact that \( a_n(\omega) \to 0 \) as \( n \to \infty \). Thus \( \lambda \neq 1 \).

So, by Cauchy’s result, \( \sum_{n=1}^{\infty} |a_n(\omega)| \) converges for almost all \( \omega \). Therefore \( \sum_{n=1}^{\infty} |f_n + 1 - f_n| < \infty \) a.e.

**Theorem 2.** If \( \|f\|_1 < \infty \) then the martingale transform \( g \) converges a.e. in \( X \).

In fact,
\[
|g(\omega)| \leq \sum_{n=1}^{\infty} |v_n(\omega)| \cdot |f_n + 1(\omega) - f_n(\omega)| \leq \sum_{n=1}^{\infty} |f_n + 1(\omega) - f_n(\omega)| < \infty
\]

for almost all \( \omega \).

**Corollary.** Let \( v = (v_1, v_2, \ldots) \) be a sequence of any random variables with \( |v| \leq 1 \).

Then \( h_n = \sum_{k=1}^{n} v_k (f_k + 1 - f_k) \) converges a.e.

**Theorem 3.** Let \( 1 < p < \infty \). For a Banach space \( X \) with the Radon-Nikodým property,
\[
\lambda \cdot P(g^* > \lambda) \leq c \cdot \|f\|_1, \quad \lambda > 0 \text{ and } \|g\|_p \leq c_p \cdot \|f\|_p \text{ hold.}
\]

Proof. For any Banach space \( X \), by a result of Burkholder (Theorem 1.1 of [2]), the following statements, each to hold for all such \( f \) and \( g \), are equivalent:
\[
\|f\|_1 < \infty \Rightarrow g \text{ converges a.e.,}
\]
\[
\lambda \cdot P(g^* > \lambda) \leq c \cdot \|f\|_1, \quad \lambda > 0,
\]
\[
\|g\|_p \leq c_p \cdot \|f\|_p.
\]

Combine this result with Theorem 2.

**Acknowledgement.** The author is very grateful to professor D. L. Burkholder for his kindly discussions in details.

**References**


