\textbf{L}^p\text{-convergence of an extended stochastic integral}

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(Received, 15 November, 1994)

\textbf{Abstract} Let $1 < p < \infty$. Let $f = \{ f(t), 0 \leq t \leq 1 \}$ be an L\textsuperscript{p}-integrable martingale and $v = \{ v(t), 0 \leq t \leq 1 \}$ a family of random variables with a continuous parameter $t$. Suppose $|v| \leq 1$ in absolute value and that $v(t)$ is continuous. Put

$$\theta_m = \sum_{k=0}^{s_m-1} v(\xi_{m,k}) [f(t_{m,k+1}) - f(t_{m,k})].$$

Here, $\forall \xi_{m,k} \in [t_{m,k}, t_{m,k+1}]$, $k \geq 0$, and $\text{Max}_k (t_{m,k+1} - t_{m,k}) \to 0$ as $m \to \infty$.

Then $\theta_m$ converges in $L^p$ and $\theta_\infty$ defines a new stochastic integral $\int_0^1 v(t) \, df(t)$.

Let $(\Omega, a, P)$ be a probability space and $\{a_t\}_{t \geq 0}$ a nonincreasing family of sub-$\sigma$-fields of $a$. Let $f = \{ f(t), 0 \leq t \leq 1 \}$ be an L\textsuperscript{p}-integrable martingale where $1 < p < \infty$ on a probability space $(\Omega, a, \{a_t\}, P)$ and $v = \{ v(t), 0 \leq t \leq 1 \}$ a family of random variables with a continuous parameter $t$. Suppose that $|v| \leq 1$ in absolute value, $v(t)$ is continuous and $v(t)$ is $a_t$-adapated.

Let $\Delta = \{ \Delta_m : 0 = t_{m,0} < t_{m,1} < \cdots < t_{m,s_m} = 1 \}$, be a sequence of subdivisions of $[0, 1]$ with $|\Delta_m| = \text{Max}_k (t_{m,k+1} - t_{m,k}) \to 0$ as $m \to \infty$. Here notice that if $m \uparrow \infty$ then $s_m \uparrow \infty$.

Put $\bar{\theta}_m = \sum_{k=0}^{s_m-1} v(\xi_{m,k}) [f(t_{m,k+1}) - f(t_{m,k})] (\forall \xi_{m,k} \in [t_{m,k}, t_{m,k+1}], k \geq 0)$

and $\bar{\theta}_m = \sum_{k=0}^{s_m-1} v(t_{m,k}) [f(t_{m,k+1}) - f(t_{m,k})]$.

By the results of R. C. James [4] and G. Pisier [8],

\textbf{Theorem.} (G. Pisier [8, Theorem 1.3, (iv)])

Let $X$ be a Banach space and $f = (f_n)_{n \geq 0}$ an arbitrary $X$-valued martingale.

Then

\begin{itemize}
  \item[(\ast)] $X$ is super-reflexive (= super-Radon-Nikodým)
  \item[(\ast\ast)] $\sum_{n \geq 0} \|f_n + 1 - f_n\|_p \leq C \cdot \sup_n \|f_n\|_p$ $(1 < p < \infty)$.
\end{itemize}

(Here, $C$ is a constant which does not depend on $f$.)

Since $X = R$ is super-reflexive, (\ast\ast) holds.

(\ast\ast) will be called by the name of Pisier's inequality.

In this paper, it is proved that the following theorem holds:

\textbf{Theorem.} $\theta_m$ converges in $L^p$ and $\theta_\infty = \bar{\theta}_\infty = \int_0^1 v(t) \, df(t)$.

$\theta_\infty$ defines a new stochastic integral.

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Proof. Let $1 < p < \infty$.

\[ \| \theta_m \|_p = E^{1/p} \left[ \left( \sum_{k \geq m} |v(m, k) | f(t_{m, k+1}) - f(t_{m, k}) \right)^p \right] \leq E^{1/p} \left[ \left( \sum_{k \geq m} |v(m, k) | f(t_{m, k+1}) - f(t_{m, k}) \right)^p \right] \text{ (Here, } |v| \leq 1) \]

\[ \leq E^{1/p} \left[ \left( \sum_{k \geq m} |f(t_{m, k+1}) - f(t_{m, k}) \right)^p \right] \]

(Since $L^p$ is a Banach lattice. See [9].)

\[ \leq \sum_{k \geq m} \| f(t_{m, k+1}) - f(t_{m, k}) \|_p \]

\[ \leq C \sup_{p \in [0, 1]} \| f(t_{m, k}) \|_p \text{ (m = 0, 1, 2, \ldots \ldots) (By Pisier's inequality)} \]

(Since $C$ does not depend on $f$, $C$ does not depend on $m$.)

\[ \leq C \cdot \sup_{t \in [0, 1]} \| f(t) \|_p \text{ (Since } t_{m, k} \in [0, 1] \text{ .)} \]

\[ \leq C \cdot \| f(1) \|_p \text{ (Since } |f(t)|^p \text{ is a submartingale, } E \ |f(t)|^p \leq E \ |f(1)|^p \text{ so } E^{1/p} \left[ \left. |f(t)|^p \right\} \leq E^{1/p} \left[ \left. |f(1)|^p \right\} < \infty \right). \]

Thus, $E \left[ |\theta_m|^p \right] \leq C^p \cdot \| f(1) \|_p^p$.

If $\theta_\infty$ exists then

\[ \| \theta_\infty \|_p = \lim_{m \to \infty} \| \theta_m \|_p \]

\[ = \lim_{m \to \infty} \| \theta_m \|_p \leq C \cdot \| f(1) \|_p < \infty. \]

Therefore $\theta_\infty \in L^p$.

Next, it is proved that the existence of $\theta_\infty$.

By a result of P. W. Millar [6], $\theta_m$ converges in $L^p$, that is, $\lim_{m \to \infty} \| \theta_m - \theta_n \|_p = 0$.

Take arbitrary $\epsilon > 0$ and fix this.

Since $v(t)$ is uniformly continuous on $[0, 1]$, for sufficiently large $m_0 = m_0(\epsilon, \omega) = m_0(\omega) \gg m$

\[ \| v(\xi_{m, k}) (\omega) - v(t_{m, k}) (\omega) \| \leq \epsilon \quad (\forall k \gg 0, \forall m' \gg m_0) \]

(Because, since $v(t)$ is uniformly continuous, for $\epsilon > 0$ there is a $\delta > 0$.

Since $|\Delta_{m}| \to 0$ as $m \to \infty$, for sufficiently large $m_0(\omega)$,

\[ \delta > |\Delta_{m}| = \max_k (t_{m, k+1} - t_{m, k}) \quad (\forall m' \gg m_0(\omega)) \]

\[ \gg \gg \xi_{m, k} - t_{m, k} \gg 0 \quad (\forall k \gg 0) \]

So $\epsilon \gg |v(\xi_{m, k}) (\omega) - v(t_{m, k}) (\omega) | \quad (\forall k \gg 0)$.

Therefore,

\[ \| \theta_m - \theta_n \|_p \leq \sum_{k \geq n} \| |v(m, k) - v(t_{m, k}) [f(t_{m, k+1}) - f(t_{m, k})]| \|_p \]

\[ \leq \sum_{k \geq n} E^{1/p} \left[ \left( |v(\xi_{m, k}) - v(t_{m, k}) | [f(t_{m, k+1}) - f(t_{m, k})] \right)^p \right] \]

\[ \leq \sum_{k \geq n} E^{1/p} \left[ \left( \epsilon \cdot |f(t_{m, k+1}) - f(t_{m, k})| \right)^p \right] \]

\[ \leq \epsilon \cdot \sup_{m \geq n} \sum_{k \geq n} \| f(t_{m, k+1}) - f(t_{m, k}) \|_p \]

\[ \leq \epsilon \cdot \sup_{m \geq n} \| f(t_{m, n}) \|_p \}

\[ \leq \epsilon \cdot \sup_{m \geq n} \| f(t_{m, n}) \|_p \}

\[ \leq \epsilon \cdot \| f(1) \|_p \}

\[ \leq \epsilon \cdot C \cdot \| f(1) \|_p \}.

(Here, \( \{ f(t_m^+; k) \} \) \( k > 0 \) is a martingale.

In fact, take an any \( \omega \in \Omega \) and fix \( m'(\omega) \).

In general, since \( f = \{ f(t) \} \) is a martingale, for any \( m \geq 0 \)
\( \{ f(t_m; k) \} k > 0 \) is a martingale. So for almost all \( \omega' \in \Omega \)
\[
E \left( f(t_m^+(\omega), k(\omega)) \right) = \frac{\alpha_{m'}(\omega', k(\omega))}{\alpha_m(\omega')} \left( \omega' \right)
\]
Here, let \( \omega' = \omega \) then \( \{ f(t_m^+(\omega), k(\omega)) \} \) \( k > 0 \) is a martingale.)

Thus,
\[
\lim_{m \to \infty} \| \theta_m - \theta_m' \|_p = \lim_{m \to \infty} \| \theta_m' - \theta_m'' \|_p \quad (m \geq \text{Max } m' (\geq m'))
\]
\[
\leq \epsilon \cdot C \cdot \| f(1) \|_p \quad \text{for all } \epsilon > 0.
\]

Therefore, from \( \| \theta_m - \theta_n \|_p \leq \| \theta_m - \theta_m'' \|_p + \| \theta_m'' - \theta_n \|_p + \| \theta_m - \theta_m'' \|_p \)
it follows that
\[
\lim_{m \to \infty} \| \theta_m - \theta_n \|_p
\]
\[
\leq 2 \cdot \lim_{m \to \infty} \| \theta_m - \theta_m'' \|_p + \lim_{m \to \infty} \| \theta_m'' - \theta_n \|_p \quad (m, n \geq \text{Max } m'(\omega'))
\]
\[
< 2 \epsilon \cdot C \cdot \| f(1) \|_p + 0 \quad \text{for all } \epsilon > 0.
\]

So, by the completeness of \( L^p \), \( \theta_m \) converges in \( L^p \) so that \( \theta_{\infty} \) exists.

From this proof \( \theta_{\infty} = \bar{\theta}_m = \int_0^1 v(t) \, df(t) \) follows.

**Remark.** When \( p > 1 \) the Pisci{e}r's inequality implies the Burkholder's \( L^p \)-inequality in \( [1] \) so that the Millar's results \( [6] \) hold without that \( v(t) \) is \( \Lambda \)-adapted. Therefore, it may be that \( v \) is any uniformly bounded and continuous random variable.

**Corollary.** \( \int_0^1 v(t) \, dB(t) \) converges in \( L^2 \).

(The convergence of this integral cannot be proved by the method of R. L. Stratonovich. See \([2]\) and \([13]\).)

**References**


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