Martingale Transforms in a Banach Space

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Abstract. If \(f=(f_1,f_2,\cdots)\) is a real \(L^1\)-bounded martingale then \(\sum_{n=1}^{\infty} |f_{n+1}-f_n| < \infty \) a.e. The same result holds for \(X\)-valued martingales, where \(X\) is a Banach space, provided \(X\) has the Radon-Nikodým property. Using this the martingale transform \(g\) of \(f\) by \(v\) converges almost everywhere without assuming that \(v\) is predictable.

1. Notations. Let \((\Omega,\mathcal{A},P)\) be a probability space and \(\alpha_1, \alpha_2, \cdots\) a nondecreasing sequence of sub-\(\sigma\)-fields of \(\alpha\). Let \(X\) be a Banach space with norm \(\|\cdot\|\) and the Radon-Nikodým property. Let \(f=(f_1,f_2,\cdots)\) be an \(X\)-valued martingale with norm \(\|f\|_1=\text{sup} E|f_t|<\infty\). Let \(v=(v_1,v_2,\cdots)\) be a real-valued predictable sequence, that is, \(v_k:\Omega\to\mathbb{R}\) is \(\alpha_k\)-measurable, \(k\geq 1\). Then \(g=(g_1,g_2,\cdots)\), defined by \(g_n=\sum_{k=1}^{n} v_k (f_{k+1}-f_k)\) with \(|v|<1\) in absolute value, is the transform of the martingale \(f\) by \(v\). Write \(\|f\|_p=\text{sup} \|f_n\|_p\) and define the maximal function \(g^*\) of \(g\) by \(g^*(\omega)=\text{sup} |g_n(\omega)|\).

2. Real-valued case. Let \(\beta\) be a sub-\(\sigma\)-field of \(\alpha\). If \(Z\) is a random variable with finite mean, by the Radon-Nikodým theorem, for \(Z\) there is a \(\beta\)-measurable function \(\varphi\) which is satisfying

\[
\int Z(\omega)\,dP = \int \varphi(\omega)\,dP \quad \text{for every } A \in \beta
\]

and which decides the correspondence \(Z\to \varphi(\text{i.e.}, Z(\omega)\to \varphi(\omega))\).

This function \(\varphi\) is unique up to a set of \(P\)-measure zero, and any such function, denoted by \(E(Z/\beta)\), is called the conditional expectation of \(Z\) relative to \(\beta\). Therefore, the above correspondence is written by

\[
E(Z/\beta)(\omega)=E(Z(\omega)/\beta)=\varphi(\omega) \quad \text{for almost all } \omega \in \Omega.
\]

If \(f=(f_1,f_2,\cdots)\) is a martingale then, for almost all \(\omega\),

\[
E(f_{n+1}(\omega)/\alpha_n)=f_n(\omega) \quad (n=1,2,\cdots).
\]

Let \(X=\mathbb{R}\), that is, let \(f=(f_1,f_2,\cdots)\) be an \(L^1\)-bounded and real-valued martingale. Then \(|\cdot|\) denotes the absolute value.

Theorem 1. If \(\|f\|_1<\infty\) then \(\sum_{n=1}^{\infty} |f_{n+1}-f_n| < \infty \) a.e., that is, \(f\) is of bounded variation.

Proof. Suppose that there exists a subset \(M\) of \(\Omega\) such that \(P(M)\neq 0\) and

\[
\sum_{n=1}^{\infty} |f_{n+1}(\omega)-f_n(\omega)| = \infty \quad \text{for all } \omega \in M.
\]

Then, for any \(G=G(\omega)>0\) there is a number \(N=N(G,\omega)>0\) such that

\[
\sum_{k=1}^{N} |f_{k+1}(\omega)-f_k(\omega)| > G \text{ on } M \quad (\forall n>N).
\]

So there are a number \(k=k(\omega)<n\) and a positive real number \(G'=G'(\omega)\) such that

\[
|f_{k+1}(\omega)-f_k(\omega)| = G' > 0 \quad \text{for each } \omega \in M.
\]

Here, set

\[
G'=G'(\omega') = |f_{k(\omega)+1}(\omega')-f_{k(\omega)}(\omega')| \quad \text{for each } \omega \in M \quad (\omega' \in \Omega, M\subset \Omega).
\]

\(G'\) is well-defined on \(\Omega\) and \(G'>0\) when \(\omega'\omega\), i.e., \(G'>0\) on \(M\).

Now, when \(\omega'=\omega\), \(|f_{k+1}(\omega)-f_k(\omega)|\) is defined on \(M\).
By the definition of the absolute value

\[ |f_{k+1}(\omega) - f_k(\omega)| = \{-f_{k+1}(\omega) - f_k(\omega) \} \text{ on } A \overset{\text{def}}{=} \{ \omega : f_{k+1}(\omega) \geq f_k(\omega) \} \subset M \backslash A. \]

Since \( k(\omega) = k < \infty \), \( \{k(\omega) : \omega \in M\} \subset \{1, 2, \ldots, n, \ldots\} \).

Thus,

\[ E \left| f_{k(\omega)}(\omega') \right| \leq \sup_{\lambda \in \{k(\omega) : \omega \in M\}} E \left| f_{\lambda} \right| \leq \sup_{\lambda \in \{1, 2, \ldots, n, \ldots\}} E \left| f_{\lambda} \right| = \sup E \left| f_n \right| < \infty. \]

So \( |f_{k+1} - f_k| \in L'. \)

For almost all \( \omega \in A \)

\[ E(\left| f_{k+1} - f_k \right| / \alpha_k)(\omega) = E(\left| f_{k+1} - f_k \right| / \alpha_k) \]

\[ = E((f_{k+1} - f_k)^* + (f_{k+1} - f_k)^-)(\omega) / \alpha_k) \]

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\[ = E(\left| f_{k+1} - f_k \right| / \alpha_k) \]

\[ = E(\left| f_{k+1} - f_k \right| / \alpha_k) \]

\[ = E((f_{k+1} - f_k)(\omega) / \alpha_k) \]

\[ = E(\left| f_{k+1} - f_k \right| / \alpha_k)(\omega) \]

In general, since \( f \) is a martingale \( E(f_{k+1} / \alpha_k) = f_k \) a.e. for any \( k \). Take any \( \omega \in \Omega \) and fix this. Let \( k = k(\omega) \).

Then \( E(f_{k(\omega)} / \alpha_{k(\omega)})(\omega') = f_{k(\omega)}(\omega') \) for almost all \( \omega' \in \Omega \).

Here take \( \omega' = \omega \) then \( E(f_{k(\omega)} / \alpha_{k(\omega)})(\omega) = f_{k(\omega)}(\omega) \)

for almost all \( \omega \). Thus, \( E(f_{k+1} / \alpha_k) = f_k \) a.e. for almost all \( \omega \in A \).

So for almost all \( \omega \in A \)

\[ E(\left| f_{k+1} - f_k \right| / \alpha_k)(\omega) = E(\left| f_{k+1} - f_k \right| / \alpha_k) \]

\[ = E(f_{k+1} - f_k)(\omega) / \alpha_k \]

\[ = E(f_{k+1} / \alpha_k)(\omega) \]

\[ = f_{k+1} / \alpha_k \]

\[ = f_k \]

\[ = 0. \]

Therefore \( E(\left| f_{k+1} - f_k \right| / \alpha_k)(\omega) = 0 \) for almost all \( \omega \in M \).

On the other hand, for almost all \( \omega' \in \Omega \)

\[ E(G'(\omega') / \{ \phi, \Omega \}) = E(G' / \{ \phi, \Omega \})(\omega') \]

\[ = E(E(G' / \alpha_{k(\omega)})(\omega') / \{ \phi, \Omega \}) \]

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\[ = E(E(G' / \alpha_{k(\omega)})(\omega') / \{ \phi, \Omega \}) \]

If \( E(G'(\omega') / \alpha_{k(\omega)} = 0 \) \( (k=k(\omega)) \) for almost all \( \omega' \in \Omega \)

then

\[ E(G'(\omega')) = E(G'(\omega') / \{ \phi, \Omega \}) \]

\[ = E(E(G'(\omega') / \alpha_{k(\omega)})(\omega') / \{ \phi, \Omega \}) \]

\[ = E(0 / \{ \phi, \Omega \}) \]

\[ = E(0) \]

\[ = 0. \]

Thus, \( G' = 0 \) a.e. This contradicts to \( G' > 0 \) on \( M \).
So \( E(G'(\omega')/\alpha_{\omega}) \neq 0 \) when \( \omega' = \omega \) on \( M \).

Then \( 0 = E( |f_{n+1}(\omega) - f_n(\omega)|/\alpha_\omega) \)
\[ = E(G'(\omega)/\alpha_\omega) \]
\[ \neq 0 \ \text{for some} \ \omega \in M. \]

This is a contradiction on \( M \). Thus there is not such \( M \).

Therefore \( \sum_{\omega \in \Omega} |f_{n+1}(\omega) - f_n(\omega)| < \infty \) for almost all \( \omega \in \Omega \).

**Corollary 1.** If \( f = (f_n)_{n \geq 1} \) is an \( L^1 \)-bounded martingale then
\( E( |f_{n+1} - f_n|/\alpha_n) = 0 \) a.e. and \( \|f_{n+1} - f_n\|_1 = E |f_{n+1} - f_n| = 0 \) for \( n < \infty \).

In fact, let \( M = \Omega \) in above proof.

**Corollary 2.** Under the above condition \( \sum_{n=1}^\infty \|f_{n+1} - f_n\|_1 < \infty \).

In fact, \( \lim_{n \to \infty} \|f_{n+1} - f_n\|_1 = 0 \).

### 3. Vector-valued case.

Let \( Z(\omega) \) be a Bochner-integrable function on a probability space \((\Omega, \alpha, P)\) taking values in \( X \).

Let \( \beta \) be a sub-\( \sigma \)-field contained in \( \alpha \). Then the conditional expectation \( E(Z/\beta) \) of \( z(\omega) \) relative to \( \beta \) is defined as a Bochner-integrable function on \((\Omega, \alpha, P)\) such that \( E(Z/\beta) \) is \( \beta \)-measurable and that

\[
\int A \cdot E(Z/\beta) \, dP = \int A \cdot E(Z) \, dP, \quad \forall A \in \beta, \text{ where the integrals are Bochner-integrals.}
\]

Therefore, by above correspondence \( Z(\omega) \to E(Z/\beta)(\omega) \), similarly in the real-valued case

\( E(Z/\beta)(\omega) \) is written by \( E(Z(\omega)/\beta) \)

for almost all \( \omega \in \Omega \).

(See [4], p.395 and p.396, Theorem 1. And also see [5], p.22.)

Let \( f \) be an \( X \)-valued and \( L^1 \)-bounded martingale.

Then \( E(f_{n+1}(\omega)/\alpha_n) = f_0(\omega) \) (\( n=1,2,\cdots \)).

**Theorem 2.** If \( \|f\|_1 < \infty \) then \( \sum_{n=1}^\infty |f_{n+1} - f_n| < \infty \) a.e.

**Proof.** Suppose that there exists a subset \( M \) of \( \Omega \) such that \( P(M) \neq 0 \) and \( \sum_{n=1}^\infty |f_{n+1}(\omega) - f_n(\omega)| = \infty \) for all \( \omega \in M \).

Then, for any \( G = G(\omega) > 0 \) there is a number \( N = N(G, \omega) > 0 \) such that \( \sum_{n=1}^N |f_{n+1}(\omega) - f_n(\omega)| > G \) on \( M \) (\( \forall n \geq N \)).

So there are a number \( k = k(\omega) \leq n \) and a positive real number \( G' = G'(\omega) \) such that \( |f_{k+1}(\omega) - f_k(\omega)| = G' > 0 \) for each \( \omega \in M \).

Then, \( \tilde{g}(\omega') = \frac{f_{k+1}(\omega') - f_k(\omega')}{\alpha_k(\omega')} \) for each \( \omega' \in M \) (\( \omega' \in \Omega, M \subseteq \Omega \)) such that \( |\tilde{g}(\omega)| = G'(\omega) > 0 \) when \( \omega' = \omega \), i.e., \( \tilde{g} = \tilde{g}(\omega) \neq \tilde{0} \) on \( M \).

Since \( f \) is a martingale, for almost all \( \omega' \in \Omega \)
\[
E(f_{k+1}(\omega') - f_k(\omega'))/\alpha_k(\omega') = E(f_{k+1}(\omega') - f_k(\omega'))/\alpha_k(\omega') = \tilde{0}.
\]

So \( \int_M E(\tilde{g}(\omega')/\alpha_k(\omega')) \, dP(\omega') = \tilde{0} \) and \( \int_{\delta_M} E(\tilde{g}(\omega')/\alpha_k(\omega')) \, dP(\omega') = \tilde{0} \).

Thus, \( E(\tilde{g}) = E(\tilde{g}(\omega')/\alpha_k(\omega')) \, dP(\omega') \)
\[ = \int_M E(\tilde{g}(\omega')/\alpha_k(\omega')) \, dP(\omega') + \int_{\delta_M} E(\tilde{g}(\omega')/\alpha_k(\omega')) \, dP(\omega') \]
\[ = \tilde{0} \] (Here \( E \) denotes the Bochner integral. See [5].)

\[ \iff E \cdot |\tilde{g}| = 0 \] (The \( E \) is the Lebesgue integral)
\[ \iff |\tilde{g}| = 0 \ a.e. \]
\[ \iff \tilde{g}(\omega') = \tilde{0} \] for almost all \( \omega' \in \Omega \) and for each \( \omega \in M \).

So \( \tilde{g}(\omega) = \tilde{0} \) on \( M \) (\( \subseteq \Omega \)).
This is a contradiction on M. Thus, there is not such M.
Therefore \[ \sum_{n=1}^{\infty} |f_{n+1}(\omega) - f_n(\omega)| < \infty \] for almost all \( \omega \in \Omega \).


**Theorem 3.** If \( \|f\|_1 < \infty \) then the martingale transform \( g \) converges a. e. in \( X \) without the assumption that \( v \) is predictable.

In fact,
\[
|g_n(\omega)| \leq \sum_{n=1}^{\infty} |v_n(\omega)| \cdot |f_{n+1}(\omega) - f_n(\omega)| \leq \sum_{n=1}^{\infty} |f_{n+1}(\omega) - f_n(\omega)| < \infty
\]
for almost all \( \omega \).

**Theorem 4.** Let \( 1 < p < \infty \) and \( \|f\|_1 < \infty \). For a Banach space \( X \) with the Radon-Nikodým property, \( \lambda \cdot P(g^* > \lambda) \leq c \cdot \|f\|_1, \lambda > 0 \), and \( \|g\|_p \leq c_p \cdot \|f\|_p \)
hold under the assumption that \( v \) is predictable.

**Proof.** For any Banach space \( X \), by a result of Burkholder (Theorem 1.1 of [2]), the following statements, each to hold for all such \( f \) and \( g \) are equivalent:
\[
\|f\|_1 < \infty \iff \text{g converges a. e.},
\]
\[
\lambda \cdot P(g^* > \lambda) \leq c \cdot \|f\|_1, \lambda > 0,
\]
\[
\|g\|_p \leq c_p \cdot \|f\|_p.
\]
Combine this result with Theorem 3.

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**References**


[2] ____________ : A geometric characterization of Banach spaces in which martingale difference sequences are unconditional.


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