A Note on Conditional Expectations

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Abstract. If $f$ is any real-valued function in $L^1(\Omega, a, P)$ and $B$ is any sub-$\sigma$-field of $a$ then $E(f/B) = f$ a.e. Here, in general, the exceptional set $\notin B$. By using this it is shown that the paths of Brownian Motion are a.e. differentiable.

Let $(\Omega, a, P)$ be a probability space and $E(f/B)$ the conditional expectation of $f$ with respect to the sub-$\sigma$-field $B$ of $a$. Let $B \not\equiv (\phi, \Omega)$.

Theorem 1. Let $A \subseteq a$ and let $B$ be any sub-$\sigma$-field of $a$ then $E(\chi_A/B) = \chi_A$ a.e.
Here the exceptional set $e$, in general, $e \subseteq B$.

Proof. Let $A \subseteq a$, $\forall B \subseteq a$ and $A \not\subseteq B$. (If $A \subseteq B$ then the Theorem is well-known.)

\[
\int_\omega E(\chi_A/B)(\omega) \, dP = \int_\omega \chi_A(\omega) \, dP
\]
\[
= \int_\omega \chi_A(\omega) \, dP + \int_{A^c} \chi_A(\omega) \, dP
\]
\[
= 1 \cdot P(A) + 0 \cdot P(A^c)
\]
\[
= P(A) \quad (\text{It may be supposed that } 0 < P(A) < 1.)
\]

On the other hand, since $A \subseteq a$ and $B \subseteq a$,

\[
\int_\omega E(\chi_A/B)(\omega) \, dP = \int_\omega E(\chi_A/B)(\omega) \, dP + \int_{A^c} E(\chi_A/B)(\omega) \, dP
\]
Let $Z(\omega)$ be any $a$-measurable random variable. Then, by the mapping when $B$ is given

\[
F : Z(\omega) \mapsto E(Z/B)(\omega) \quad (\forall \omega \in \Omega),
\]

$E(\chi_A/B)(\omega) = a.$ e. constant $a$ on $A$
and $E(\chi_A/B)(\omega) = a.$ e. constant $b$ on $A^c$.

Notice that the exceptional sets are $a$-measurable so that these union $\notin B$ since $A \not\subseteq B$, in general.

Then

\[ P(A) = \int_\omega E(\chi_A/B)(\omega) \, dP = a \cdot P(A) + b \cdot P(A^c) \]
\[ = a \cdot P(A) + b \cdot (1 - P(A)) \cdot (1 - P(A) \neq 0). \]

By (*) \quad $b = 0 \implies a = 1$ and \quad $a = 1 \implies b = 0$

So \quad $a \neq 1 \iff b \neq 0$.

By contraposition of the above statement,

$\iff a \neq 1 \iff b \neq 0$.

We shall show that if we suppose that $a \neq 1$ and $b \neq 0$ then we have a contradiction. For instance, suppose $a = 1/k$ and $b = 1/k$ ($k > 1$) (so suppose $a = b$). Then, by (*),

\[ 1 > P(A) = 1/k \cdot P(A) + 1/k \cdot (1 - P(A)) = 1/k. \]

So take $k$ such that $k > \frac{1}{P(A)}$ then $P(A) < P(A)$.

This is a contradiction. So it is not $a \neq 1$ thus it is not $b \neq 0$. That is, $a = 1$ and $b = 0$.

Therefore $E(\chi_A/B)(\omega) = \chi_A(\omega)$ a.e. and the exceptional set $e$, in general, $e \subseteq B$. (q.e.d.)

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Remark. If \( a = b \) then \( E(\omega, B)(\omega) = a \) a.e. constant on \( \Omega \).
As \( f \in L^1(\Omega, a, P) \) is a limit of sequence of simple functions, \( E(f/B)(\omega) = a \) a.e. constant on \( \Omega \). This can not happen since \( E(f/B)(\omega) \) is the function of \( \omega \).
Therefore \( E(\omega, B) = a \) a.e. constant on \( \Omega \).

Theorem 2. Let \( f \in L^1(\Omega, a, P) \) and \( \forall B \subseteq a \).
Then \( E(f/B) = f \) a.e. (The exceptional set \( e \), in general, \( e \subseteq B \)).

Proof. Since \( f \in L^1 \iff |f| = f^+ + f^- \in L^1 \), so that \( f^+, f^- \in L^1 \), \( f^+ \) and \( f^- \) are \( a \)-measurable.
For \( f^+ \) and \( f^- \) there are sequences of \( a \)-measurable simple functions \( \{g_n\}_{n \geq 1} \) and \( \{h_n\}_{n \geq 1} \) such that \( g_n(\omega) \geq 0, g_n(\omega) \uparrow f^+(\omega) \); \( h_n(\omega) \geq 0, h_n(\omega) \downarrow f^-(\omega) \) (\( \forall \omega \in \Omega \)).
Set \( f_n = g_n - h_n \) (\( n = 1, 2, \ldots \)).
Then, for \( a \)-measurable function \( f \), \( \{f_n\}_{n \geq 1} \) is the sequence of \( a \)-measurable simple functions such that
\[
|f_n(\omega)| \leq |f(\omega)| \in L^1 \quad \text{and} \quad \lim_{n \to \infty} f_n(\omega) = f(\omega) \quad (\forall \omega \in \Omega).
\]
Here, for \( f^+ = g_n \) is defined as follows:

for \( i = 0, 1, \ldots, n2^n - 1 \) and \( n = 1, 2, \ldots \),

set \( A_n = \left\{ \frac{i}{2^n} \leq f < \frac{i + 1}{2^n} \right\} \), \( A_n \in \mathcal{A} \in \mathcal{F}_n \)

and define \( g_n \) by
\[
g_n = \sum_{i=1}^{2^n} \frac{i}{2^n} \cdot \chi_{A_n}
\]
Then \( A_n \subseteq a \) and \( \{g_n\}_{n \geq 1} \) satisfies above property.
Define similarly \( h_n \) for \( f^- \). Next, by \( E(\omega, B) = \chi_B \) a.e.,
\[
E(g_n/B) = g_n \quad \text{a.e. and} \quad E(h_n/B) = h_n \quad \text{a.e.}
\]
so that \( E(f_n/B) = f_n \) on \( \Omega \setminus\{\varepsilon_n\} \quad \text{and so} \quad \Omega \setminus\{\varepsilon_n\} \quad \text{on} \quad \Omega \setminus\{\varepsilon_n\}.
\]
By the Lebesgue's convergence theorem
\[
\lim_{n \to \infty} E(f_n/B) = E(\lim_{n \to \infty} f_n/B) = E(f/B)
\]
and the most left side \( \lim_{n \to \infty} f_n = f \).
Then \( E(f/B) = f \) a.e. for \( \forall B \subseteq a \). (q.e.d.)

Theorem 3. Let \( f = (f_t)_{t \geq 0} \) be any real-valued martingale on the probability space \( (\Omega, a, (a_t), P) \).
Then \( P\left( \lim_{s \to \infty} \frac{f_t(\omega) - f_s(\omega)}{t - s} = 0 \right) = 1. \)

Proof. As \( f \) is a martingale it may be supposed that the paths are continuous.
Since \( \lim_{t \to s, a} E(f_t/a_s) = f_s, f_t(\omega) = f_s(\omega) \) a.e.
So take any \( s \geq 0 \) and fix this and let
\[
f_t(\omega) = f_s(\omega) \quad \text{on} \quad \Omega \setminus\{\varepsilon_t\} \quad P(\varepsilon_t) = 0,
\]
and
\[
f_{t+s}(\omega) = f_s(\omega) \quad \text{on} \quad \Omega \setminus\{\varepsilon_t\} \quad P(\varepsilon_t) = 0,
\]
So \( \frac{f_t(\omega) - f_s(\omega)}{t - s} = 0 \) on \( \Omega \setminus\{\varepsilon_t\} \quad (t \neq s) \quad (i., \quad \forall \omega \in \Omega \setminus\{\varepsilon_t\}) \)
and this holds also for any \( t' \) instead of \( t \).
Take any \( \omega \in \Omega \setminus\{\varepsilon_t \cup \varepsilon_{t'}\} \). Then
\[
\lim_{t \to s} \frac{f_t(\omega) - f_s(\omega)}{t - s} = 0.
\]
In fact, when \( t \to t' \), \( f_t(\omega) \) becomes \( f_{t'}(\omega) \) since paths are continuous.
\[
\lim_{t \to t'} \frac{f_t(\omega) - f_{t'-s}(\omega)}{t' - s} = \frac{f_{t'}(\omega) - f_s(\omega)}{t' - s} = 0
\]

on $\Omega \setminus e_r$ (thus, $= 0$ on $\Omega \setminus e_i \cup e_r$).

So, for $\forall \ t' \geq 0$, on $\Omega \setminus e_i \cup e_r$

\[
\left( * \right) \quad \lim_{t \to s} \frac{f_t(\omega) - f_s(\omega)}{t - s} = 0 = \lim_{t \to t'} \frac{f_t(\omega) - f_s(\omega)}{t - s}.
\]

(Notice that $t \to t'$ is, in general, $t \neq t'$ and $t \to t'$.)

Now, $f_s(\omega) = f_t(\omega)$ on $\Omega$ so that $e_s = e_t$.

Let $t' = s$ in $(*)$ then $e_r = e_i = e_s = \emptyset$ and

\[
0 = f_s(\omega) - f_s(\omega) = \lim_{t \to s} \frac{f_t(\omega) - f_s(\omega)}{t - s} \quad \text{on} \quad \Omega \setminus e_i, \quad P(e_i) = 0.
\]

Therefore, $\lim_{t \to s} \frac{f_t(\omega) - f_s(\omega)}{t - s} = 0$ a.e.

that is,

\[
P\left( \lim_{t \to s} \frac{f_t(\omega) - f_s(\omega)}{t - s} = 0 \right) = 1. \quad (q. \ e. \ d.)
\]

**Corollary.** Let $B = (B_t)_{t \geq 0}$ be any real-valued Brownian motion then

\[
P\left( \lim_{t \to s} \frac{B_t(\omega) - B_s(\omega)}{t - s} = 0 \right) = 1.
\]

**Remark.** Since $B_t(\omega) = B_s(\omega)$ a.e., the proofs of Paley-Wiener-Zygmund theorem fail.

**References.**


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