On the Semiranked Group (I)

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Synopsis

In this paper we will give a definition of the \( SR \)-group (namely, Semiranked Group) that is a new notion, and will attempt its general theory.

Introduction:

An abstract space with a mathematical structure\(^5\) \( S \) is called \( S \)-space. If so, \textit{What is the method of \( R \)-spaces?}\(^5\) It is to replace the structure \( T \) in the \( T \)-space (i.e. Topological space) with the structure \( R \).

In this paper we will define a new notion, \( SR \)-group, by the same method as is taken in the definition of the semitopological group.\(^6\) We shall use the same terminology that is introduced in [1] and [2]. And throughout this paper, we shall treat only \( R \)-spaces with indicator \( \omega_0 \). We shall denote the point of an \( R \)-space by \( x, y, z, \ldots \), the family of neighborhoods of \( x \) with rank \( n \) by \( \mathcal{B}_n(x) \), and fundamental sequences of neighborhoods with respect to \( x \)\(^4\) by \( \{u_n(x)\}, \{v_n(x)\}, \ldots \).

\( \S \) 1. Continuous, Homeomorphism.

In this section we will define two new notions, \( r \)-continuous, and \( r \)-homeomorphism.

Definition 1. \( r \)-continuous.

Let \( G \) and \( H \) be two \( R \)-spaces. A mapping \( f \) of \( G \) into \( H \) is said to be \( r \)-continuous if it satisfy next condition:

\[ (\ast \ast) \text{ for each } x \in G \text{ and any } \{u_n(x)\}, \text{ there exists a } \{v_n(f(x))\} \text{ such that } f(u_n(x)) \subseteq v_n(f(x)). \]

Remark 1. \( (\ast \ast) \) implies if \( x \in \lim x_n \) then \( f(x) \in \lim f(x_n) \).

Definition 2. \( r \)-homeomorphism, \( r \)-homeomorphic.

Let \( G \) and \( H \) be two \( R \)-spaces with same indicator \( \omega_0 \). A mapping \( f \) of \( G \) onto \( H \) is said to be \( r \)-homeomorphism if it satisfies next conditions:

1) \( f \) is a bijection.\(^5\)
2) \( f \) is (bi)-continuous.
3) For any \( \{u_n(x)\}, \{v_n(f(x))\} \) (such that \( v_n(f(x)) = f(u_n(x)) \)) is a fundamental sequence of neighborhoods with respect to \( f(x) \in H \).

If there is a homeomorphism between two \( R \)-spaces, then they are called homeomorphic with each other.

\( \S \) 2. The definition of \( SR \)-group and \( R \)-group.

Definition 3. (i) An \( R \)-space \( G \) that is also a group is called a \( SR \)-group (i.e. Semiranked group) if the operation \( (x, y) \rightarrow xy \) is continuous as follows:

(a) Let \( x, y \) be \( \forall x, y \in G \). Then for any \( \{u_n(x)\}, \{v_n(y)\} \), there exists a \( \{w_n(xy)\} \) such that \( u_n(x)v_n(y) \subseteq w_n(xy) \).

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1) [12].
2) [1], [2].
3) [8].
4) [2], II, p. 551.
(ii) An $R$-space $G$ that is also a group is called an $R$-group (i.e. Ranked group) if the mapping $(x,y)\mapsto xy^{-1}$ is continuous as follows:

(b) Let $x, y$ be $\forall x, y \in G$. Then for any $\{u_n(x), \{v_n(y)\}$, there exists a $\{w_n(xy^{-1})\}$ such that $u_n(x)v_n(y)^{-1} \subseteq w_n(xy^{-1})$.

Remark 2. (a) implies that, if $x \in \{\lim n x_n\}$ and $y \in \{\lim n y_n\}$, then $xy \in \{\lim n x_ny_n\}$. (b) implies that, if $x \in \{\lim n x_n\}$, $y \in \{\lim n y_n\}$, then $xy^{-1} \in \{\lim n x_ny_n^{-1}\}$.


Evidently, we get following proposition:

Proposition 1. Every $R$-group is a $SR$-group. But the converse is not true.

Theorem 1. Let $a$ be a fixed element of a $SR$-group $G$. Then the mappings

\[ r_a : x \mapsto xa, \quad l_a : x \mapsto ax \]

of $G$ onto $G$ are homeomorphisms of $G$.

Proof. It is clear that $r_a$ is a one-to-one and onto mapping. Since $G$ is a $SR$-group, for any $\{u_n(x), \{v_n(a)\}$ there exists a $\{w_n(xa)\}$ such that $u_n(x)v_n(a) \subseteq w_n(xa)$. Moreover $r_a(u_n(x)a) = u_n(x)a \subseteq u_n(x)v_n(a) \subseteq w_n(xa)$. Hence, $r_a$ is continuous. By the same argument, $r_a^{-1} : x \mapsto xa^{-1}$ is continuous.

Furthermore, $\{r_a(u_n(x))\}$ is a fundamental sequence of neighborhoods with respect to $xa \in G$. Therefore, $r_a$ is a homeomorphism. The fact that $l_a$ is a homeomorphism follows similarly. (Q.E.D.)

Definition 4. Translation. $r_a$ and $l_a$ are, respectively, called the right and left translation of $G$.

Corollary 1. Let $O$ be an $r$-open, $F$ an $r$-closed, and $A$ any subset of a $SR$-group $G$ and let $a \in G$.

Then:

(i) $Oa$, $aO$, $AO$ and $OA$ are $r$-open.

(ii) $Fa$, $aF$ are $r$-closed.

Proof. Since the mappings in Theorem 1 are homeomorphisms, (i) is obvious. By the same argument, $Fa$ and $aF$ are $r$-closed in (ii).

Since $AO = \bigcup_{a \in A} aO$, $OA = \bigcup_{a \in A} Oa$, and the union of $r$-open sets is $r$-open. (Q.E.D.)

Therefore,

Remark 4. $r_a$ and $l_a$ can be considered $r$-open and $r$-closed mappings.

Corollary 2. Let $G$ be a $SR$-group. For $\forall x_1, x_2 \in G$, there exists a homeomorphism of $G$ such that $f(x_1) = x_2$.

Namely, $G$ is homogeneous.$^{30}$

Proof. Let $x_1^{-1}x_2 = a \in G$, and consider the mapping $f : x \mapsto xa$. (Q.E.D.)

Theorem 2. If $SR$-group $G$ satisfying F. Hausdorff's axiom $(O)^{30}$ is complete,$^{11}$ then $G$ is of the second Category.

§ 3. The neighborhoods of identity of a $SR$-group.

Let $G$ be a $SR$-group, and $e$ be its identity. $e_n$ will denote the family of neighborhoods of $e$ with rank $n$, and $\{U_n\}$, $\{V_n\}$, ... fundamental sequences of neighborhoods with respect to $e$.

The system $\{e_n\}$ possesses the following properties:

(A) for every $V$ in $e$, $e \in V$ (where $e = \bigcup_{n=0}^{\infty} e_n$)

(B) for any $U$, $V$ in $e$, there is a $W$ in $e$ such that $W \subseteq U \cap V$.

(a) for any $V$ in $e$ and for any integer $n$, there is a $m, m \geq n$, and a $U$ in $e_m$ such that $U \subseteq V$.

6) [5].

7) [7], II, p. 788.

9) [14], p. 28.

10) F. Hausdorff: Grundzüge der Mengenlehre, 1914, p. 213.

11) [11], I, pp. 554-555.
(β) \( G \in \varepsilon_e \).

These are obvious as the properties of neighborhoods in an \( R \)-space. This \( \{ \varepsilon_e \} \) has been introduced in [5]. We shall call this system \( \{ \varepsilon_e \} \) a fundamental system of neighborhoods of \( e \).

Furthermore, from (α), we get following properties:

\((SR_1)\) For any \( \{ U_n \} \), \( \{ V_n \} \), there exists a \( \{ W_n \} \) such that \( U_n V_n \subseteq W_n \).

\((SR_2)\) For any \( \{ U_n \} \) and for any \( x \in G \), there exists a \( \{ V_n \} \) such that \( x U_n x^{-1} \subseteq V_n \).

\((SR_3) (resp. (SR_4))\) Let \( x \) be any point of \( G \). For any \( \{ U_n \} \) there exists a \( \{ v_n(x) \} \) such that \( x U_n \subseteq v_n(x) \) (resp. \( U_n x \subseteq v_n(x) \)), and, conversely, for any \( \{ u_n(x) \} \), there exists a \( \{ V_n \} \) such that \( u_n(x) \subseteq x V_n \) (resp. \( u_n(x) \subseteq V_n x \)).

Proof. \((SR_1)\) is immediate consequences of (α), putting \( x = y = e \). We shall prove \((SR_3)\). Let \( \{ u_n(x) \} \) be some fundamental sequence of neighborhoods with respect to \( x \). Because of (α), there is a \( \{ v_n(x) \} \) such that \( u_n(x) U_n \subseteq v_n(x) \).

Since \( x \in u_n(x) \), \( x U_n \subseteq v_n(x) \). Conversely, taking some fundamental sequence of neighborhoods with respect to \( x^{-1} \), say \( \{ v_n(x^{-1}) \} \), and applying (α), there exists a \( \{ V_n \} \) such that \( v_n(x^{-1}) U_n(x) \subseteq V_n \). Since \( x^{-1} \in v_n(x^{-1}) \), \( x^{-1} U_n(x) \subseteq V_n \), i.e., \( u_n(x) \subseteq x V_n \). Similarly we can prove \((SR_3)\).

Next, we shall prove \((SR_2)\). For any \( \{ U_n \} \) and for any \( x \in G \), because of \((SR_3)\), we get a \( \{ v_n(x) \} \) such that \( x U_n \subseteq v_n(x) \).

Then, from \((SR_4)\), there exists a \( \{ V_n \} \) such that \( v_n(x) \subseteq V_n x \). Hence, \( x U_n x^{-1} \subseteq V_n \).

Remark 5. \((α)\) follows from \((SR_1)\), \((SR_2)\), \((SR_3)\), (or \((SR_4)\)). Therefore the three conditions above are not only necessary, but sufficient for a group \( G \) which is also an \( R \)-space to be a \( SR \)-group.

Proof. The proof is similar in [5]:

Take any \( \{ u_n(x) \} \), \( \{ v_n(y) \} \). From \((SR_3)\) and \((SR_4)\), there are \( \{ U_n \} \), \( \{ V_n \} \) such that \( u_n(x) \subseteq x U_n \), \( v_n(y) \subseteq V_n y \). Applying \((SR_1)\), we get \( \{ W_n \} \) such that \( U_n V_n \subseteq W_n \) and moreover, by \((SR_2)\), a \( \{ W_n \} \) such that \( x W_n x^{-1} \subseteq W_n \). From \((SR_3)\) again, there is a \( \{ w_n(x) y \} \) such that \( W_n x y w_n(x) y \subseteq w_n(x) y \). Then, \( u_n(x) v_n(y) \subseteq x U_n V_n y \subseteq x W_n x y w_n(x) y \subseteq w_n(x) y \).

Now, let \( G \) be a \( SR \)-group, where defined families of subsets, \( \varepsilon_n \ (n=0, 1, 2, \cdots) \), which satisfy axioms (A), (B), (α), (β), \((SR_1)\), \((SR_2)\), \((SR_3)\). When we take the totality of \( x V \) for \( V \in \varepsilon_n \) as \( \varepsilon_n(x) \), \((SR_3)\) is obviously fulfilled, and \( G \) becomes a \( SR \)-group. Taking \( \{ V_n \} : V \in \varepsilon_n \) as \( \varepsilon_n(x) \), we may obtain another \( SR \)-group. In any case convergence of sequences coincides.

§ 4. Sufficient conditions for \((SR_1)\), \((SR_2)\).

As sufficient conditions for \((SR_1)\), \((SR_2)\), respectively, we have

\((1)\) there exists a non-negative function \( \phi(\lambda, \mu) \) defined for \( \lambda > 0, \mu > 0 \) such that \( \lim_{\lambda, \mu \to \infty} \phi(\lambda, \mu) = \infty \), and the following hold; if \( U \in \varepsilon_l, V \in \varepsilon_m, W \in \varepsilon_n \) and \( UV \subseteq W \), then there exists a \( n^* \) such that \( UV \subseteq W^* \subseteq W \).

\((2)\) there exists a function \( \phi(\lambda; x) \geq 0 \) defined for \( \lambda > 0, \ x \in G \) such that \( \lim_{\lambda \to \infty} \phi(\lambda; x) \) for any fixed \( x \), and the following holds; if \( U \in \varepsilon_n, V \in \varepsilon_m, x \in G \), and \( x U x^{-1} \subseteq V \), there exists a \( n^* \) such that \( x U x^{-1} \subseteq V^* \subseteq V \).

The proof is similar in [5].

When \( \{ \varepsilon_n \} \) satisfies the condition:

\((***): \) if \( U \in \varepsilon_l, V \in \varepsilon_m \), then \( U \cap V \in \varepsilon_n \), where \( n > \max(l, m) \).

\((1)\), \((2)\) may be replaced by, respectively,

\((1')\) there exists a function \( \phi(\lambda, \mu) \) such as \( \phi \) in \((1)\), and the following hold; for any \( U \in \varepsilon_l, V \in \varepsilon_m \), there exists a \( n^* \) such that \( UV \subseteq W \).

\((2')\) there exists a function \( \phi(\lambda; x) \) such as \( \phi \) in \((2)\), and the following holds; for any \( U \in \varepsilon_n \), and for any \( x \in G \), there exists a \( n^* \) such that \( x U x^{-1} \subseteq V \).
§5. Subgroup, Normal subgroup, Quotient group.

In this section we will define several new notions, i.e. SR-subgroup, R-subgroup, SR-normal subgroup, R-normal subgroup, SR-quotient group, and R-quotient group.

Definition 5. **SR-subgroup, R-subgroup.**

(i) Let \( G \) be a SR-group and \( H \) a subgroup of \( G \). Then \( H \), endowed with the rank induced\(^{12}\) from \( G \), is called a **SR-subgroup**.

(ii) Let \( G \) be an R-group and \( H \) a subgroup of \( G \). Then \( H \), endowed with the rank induced from \( G \), is called an **R-subgroup**.

Definition 6. **SR-normal subgroup, R-normal subgroup.**

(i) If \( G \) is a SR-group and if \( N \) is a normal subgroup of \( G \), then \( N \) is called a **SR-normal subgroup**.

(ii) If \( G \) is an R-group and if \( N \) is a normal subgroup of \( G \), then \( N \) is called an **R-normal subgroup**.

Proposition 2. Every \( r \)-open subgroup \( H \) of a SR-group (hence of a R-group) \( G \) is \( r \)-closed.

Proof. For each \( x \in G \), \( xH \) is \( r \)-open by Corollary 1.

Hence, \( H = G - \bigcup xH \) is \( r \)-closed, because \( \bigcup xH \) is \( r \)-open, where the union is taken over all pairwise disjoint cosets different from \( H \).

(Q.E.D.)

Proposition 3. Let \( U \) be a symmetric\(^{13}\) neighborhood of \( e \) in an R-group \( G \). Then \( H = \bigcup_{n \geq 1} U^n \) is an \( r \)-open and \( r \)-closed subgroup of \( G \).

Proof. Let \( x, y \in H \). Then there exist positive integers \( m, n \) such that \( x \in U^m, y \in U^n \). Hence, \( x = y^{-1} \in U^m (U^*)^{-1} = U^n (U^-)^n = U^n U^n = U^{m+n} \notin H \). Thus, \( H \) is a subgroup of \( G \). Now to show that \( H \) is \( r \)-open, we observe that for each \( y \in H \), \( y \notin U \) where \( U \) is \( r \)-open.

This proves that \( H \) is \( r \)-open and \( r \)-closed by Proposition 2.

(Q.E.D.)

Proposition 4. If \( H \) is an \( r \)-closed R-subgroup of an R-group \( G \), then \( H \) is \( r \)-closed R-normal subgroup of \( G \), so is \( \overline{H} \).

Proof. By using \( \overline{H} = H \), we get this Proposition.

Let \( G \) be a SR-group and \( H \) a subgroup of \( G \). Let \( G/H \) denote the collection of all distinct cosets \( \{xH\}, x \in G \). Let \( f \) be the canonical mapping of \( G \) into \( G/H \) (i.e. \( f: x \mapsto xH \)). Then, for any fundamental sequence of neighborhoods of \( x \in G \), we can consider \( \{f(u_n(x))\} \) a fundamental sequence of neighborhoods with respect to \( \{x \in G|H (x \equiv xH) \} \), thus, we put \( f(u_n(x)) \equiv \tilde{u}_n(x) \). Therefore, \( G/H \) is an R-space (endowed with the rank induced from \( G \)).

Thus, Definition 7. **SR-quotient space, R-quotient space.**

(i) Let \( G \) be a SR-group and \( H \) a subgroup of \( G \). Then \( G/H \), the collection of all distinct cosets \( \{xH\}, x \in G \), is called a **SR-quotient space**.

(ii) If \( G \) is an R-group and if \( H \) is a subgroup of \( G \), then \( G/H \) is called an **R-quotient space**.

Remark 6. \( f \) is an onto and \( r \)-continuous mapping.

Proposition 5. Let \( G \) be a SR-group and \( H \) a subgroup of \( G \), then \( G/H \) is a homogeneous space.

Proof. Let \( x_1, x_2 \in G/H \), then \( x_1 = x_2H \) and \( x_2 = x_2H \). Let \( a \) be in \( G \) such that \( ax_1 = x_2 \). Define the mapping \( f_a: x \mapsto xH \mapsto (axH) = ax \) for \( \forall x \in G/H \). Then \( f_a \) is well-defined and is one-to-one mapping of \( G/H \) onto itself. Also \( f_a^{-1}: x \mapsto (a^{-1}x)H \). Obviously, \( f_a \) is bicontinuous. This \( f_a \) is a homomorphism as is easy to check. Clearly, \( f_a(\tilde{x}_1) = ax_1(ax_1)H = x_1H = x_2 \) shows that \( G/H \) is a homogeneous space.

(Q.E.D.)

Proposition 6. Let \( H \) be a subgroup of a SR-group \( G \), and \( f \) the canonical mapping of \( G \) onto \( G/H \). If \( \{\tilde{e}\} \) is a fundamental system of neighborhoods of \( e \in G \), then \( \{f(\tilde{e})\} \) is a fundamental system of neighborhoods of \( \tilde{e} \in G/H \).

Proof. For each \( e_n, f(\tilde{e}) \) is regarded as a neighborhood of \( \tilde{e} \).


\(^{13}\) A subset \( U \) of a group \( G \) is said to be symmetric if \( U = U^{-1} \).

\(^{14}\) [7], III, pp. 792–793.
Proposition 7. Let $G$ be a $SR$-group (or $R$-group) and $N$ a normal subgroup of $G$. Then
1) The canonical mapping $f : G \rightarrow G/N$ is an $r$-continuous and homomorphism.
2) $G/N$ is a $SR$-group (or $R$-group).

Proof. These are obvious.

Definition 8. $SR$-quotient group, $R$-quotient group.

Let $G$ be a $SR$-group (or $R$-group) and $N$ a normal subgroup of $G$, then the group $G/N$ is called a
$SR$-quotient group (or $R$-quotient group).

Proposition 8. Let $G$ be an $R$-group, $N$ a normal subgroup of $G$, $M$ any $R$-subgroup of $G$, and $f : G \rightarrow G/N$. Then $f(M)$ is an $R$-subgroup of $G/N$, and it is homeomorphic with $MN/N$.

Proof. By an isomorphism theorem of abstract groups.

Proposition 9. (The first law of isomorphism). Let $N$ be a normal subgroup of an $R$-group $G$ and $M$ any $R$-subgroup of $G$. Let $f(m) = m(M \cap N)$, $m \in M$. Then, $f$ endows the rank of $MN/N$ onto $M/M \cap N$.

Proof. By the above arguments.

Moreover,

Proposition 10. (The second law of isomorphism). Let $G$ be an $R$-group, $N$ and $M$ two normal subgroups of $G$ such that $N \subseteq M$. Then, $G/M$ is homeomorphic with $(G/N)/(M/N)$.

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(To be continued)

References


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