On Generalized Continuous Groups

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Synopsis

First, in the ranked spaces, we will define the notion of the R-continuity. Next, using this notion we will define the notion of a generalized continuous group and call it the (Semi-) Ranked Group with indicator \( \omega \). After that, we will define some notions, namely, homeomorphism, homomorphism, isomorphism, subgroup, normal subgroup, quotient group and direct product group. Using these notions, we will attempt to construct a general theory of the Ranked Groups with indicator \( \omega \). Furthermore, in this paper, we will also define some notions, namely, ranked ring, ranked field, linear ranked space, ortho-continuous group, para-continuous group, etc.

Introduction. Prof. K. Kunugi introduced, in 1954, the notion of a non-topological space, namely, the Ranked Space,\(^1\) as an extension of the Metric space and the Normed Space, and introduced further, in 1956, the notion of a ranked integral based on his theory of ranked spaces and called it the (E. R.) integral.\(^2\) After that, in 1968, M. Washihara [6] gave a special definition of the Ranked Group with indicator \( \omega_0 \) (\( \omega_0 \) is the first non-finite cardinal). But the general definition of the Ranked Group with general indicator \( \omega \) has not been given yet. On the other hand, in the Note [12] H. R. Fischer introduced, in 1959, the notion of limit group as a generalization of the notion of the topological groups. In a sense, a ranked group is considered to be a limit group.\(^3\) In this paper we will attempt to give a general definition of the ranked group with general indicator \( \omega \) and construct its general theory.

§ 0. Ranked Spaces. The purpose of this section is to explain some notions of ranked spaces. Let us introduce the notion of ranked spaces according to K. Kunugi [3] and S. Nakanishi [4].

Consider a non-empty set \( S \) (called a space) endowed with such a structure that each point \( p \) of \( S \) has a non-empty family \( \{ V(p) \} \) of subsets of \( S \) satisfying the axioms (A) and (B) of F. Hausdorff.\(^4\) Given a point \( p \) of \( S \), we say that a monotone decreasing sequence of neighbourhoods \( V_\alpha(p) \) is type \( \gamma \), where \( \gamma \) is an ordinal number of Cantor, if \( \alpha \) runs over the set \( 0 \leq \alpha < \gamma \) of all ordinal numbers and if \( V_\alpha(p) \supseteq V_\beta(p) \) for all \( \alpha, \beta \) with \( 0 \leq \alpha \leq \beta \leq \gamma \):

\[
V_\alpha(p) \supseteq V_1(p) \supseteq V_2(p) \supseteq \cdots \supseteq V_\alpha(p) \supseteq \cdots, 0 \leq \alpha < \gamma.
\]

The sequence (\( \gamma \)) which has no neighbourhoods \( V_\alpha(p), \alpha \in \{ V(p) \} \), such that \( \cap V_\alpha(p) \supseteq V_\gamma(p) (0 \leq \alpha \leq \gamma) \)

is said to be \textit{maximal}. We will denote by \( w(p, S) \) the smallest ordinal number of types of maximal

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1) [1]
2) [2]
3) In the case of indicator \( \omega \), we get this statement by the method as is taken in the Note [6] (pp. 4-5).
4) [31], p. 213.
monotone decreasing sequences of neighbourhoods of \( p \). Now, let us consider such a space that there is at least one point having a maximal monotone decreasing sequence of neighbourhoods. Then \( \omega(S), \omega(S) \equiv \inf_{p \in S} \omega(p, S) = \min_{p \in S} \omega(p, S) \), is called the depth of the space \( S \). \( \omega(S) \) is an inaccessible ordinal number.\(^5\)

Let us choose once for all an inaccessible ordinal number \( \omega \) such that \( \omega_0 \leq \omega \leq \omega(S) \). \( \omega \) is called indicator of \( S \). Given an ordinal number \( \alpha \), which runs through the interval \( 0 \leq \alpha < \omega \), suppose that we have a set \( \mathcal{B}_\alpha \) of neighbourhoods, called neighbourhoods of rank \( \alpha \). Then \( S \) is said to be a ranked space if the sequence of sets \( \mathcal{B}_\alpha (0 \leq \alpha < \omega) \) satisfies the following axiom (\( a \)) of K. Kunugi: \(^6\)

(a) For every neighbourhood \( V(p) \) of \( p \) (\( p \in S \)) and for every ordinal number \( \alpha \) such that \( 0 \leq \alpha < \omega \), there exists an ordinal number \( \beta = \beta(p, \alpha, V(p)) \) and a neighbourhood \( U(p) \) of \( p \) such that we have at the same time

\[
\alpha \leq \beta < \omega, \quad U(p) \subseteq V(p), \quad U(p) \in \mathcal{B}_\beta.
\]

We will denote the ranked space by \( \{ S, \mathcal{B}_\alpha \} \). A monotone decreasing sequence of neighbourhoods of points:

\[
V_\alpha(p_\alpha) \supseteq V_1(p_1) \supseteq V_2(p_2) \supseteq \ldots \supseteq V_\alpha(p_\alpha) \supseteq \ldots, \quad 0 \leq \alpha < \omega,
\]

is said to be the fundamental sequence, if there is an ordinal number \( \gamma(\alpha) \) such that \( V_\alpha(p_\alpha) \in \mathcal{B}_{\gamma(\alpha)} \) for all \( \alpha, 0 \leq \alpha < \omega \), and satisfies the following two conditions:

(i) \( \gamma(0) < \gamma(1) < \gamma(2) < \ldots < \gamma(\alpha) < \ldots \) (\( 0 \leq \gamma(\alpha) < \omega \)), sup \( \gamma(\alpha) = \omega \).

(ii) for each \( \alpha, 0 \leq \alpha < \omega \), there is a number \( \lambda = \lambda(\alpha) \) such that \( \alpha \leq \lambda < \omega, p_\lambda = p_{\lambda + 1} \) and \( \gamma(\lambda) < \gamma(\lambda + 1) \) (the equalities).

The ranked space \( S \) is said to be complete, if, for every fundamental sequence \( \{ V_\alpha(p_\alpha); 0 \leq \alpha < \omega \} \) of neighbourhoods, we have \( \bigcap_{\alpha = 0}^\omega V_\alpha(p_\alpha) \neq \emptyset \).

Given a sequence \( \{ p_\alpha; 0 \leq \alpha < \omega \} \) of points of \( S \) and a point \( p \) of \( S \), we say that the sequence \( \{ p_\alpha; 0 \leq \alpha < \omega \} \) ortho-converges to the point \( p \), or that \( p \) is an ortho-limit of \( \{ p_\alpha; 0 \leq \alpha < \omega \} \), if there is a fundamental sequence \( \{ V_\alpha(p); 0 \leq \alpha < \omega \} \) consisting of neighbourhoods of \( p \) such that \( V_\alpha(p) \ni p_\alpha \) for each \( \alpha \). In this case, we shall write

\[
p \in \{ \lim_{\alpha} p_\alpha \}.
\]

\( \{ \lim_{\alpha} p_\alpha \} \) is not a set consisting of one point alone in general.

Let \( R, S \) be two ranked spaces with same indicator \( \omega \). Then we will say that the mapping \( f: R \rightarrow S \) is ortho-continuous at the point \( p \in R \) iff

\[
p \in \{ \lim_{\alpha} p_\alpha \} \Longleftrightarrow f(p) \in \{ \lim_{\alpha} f(p_\alpha) \}.
\]

The mapping \( f \) is said to be ortho-continuous if it is ortho-continuous at each point of \( R \).

Let \( A \) be a subset of \( \{ S, \mathcal{B}_\alpha \} \). For every point \( p \) of \( A \), the neighbourhood of \( p \) in \( A \) is the set of points of \( A \) defined by the relation \( V(p, A) = V(p) \cap A \), where \( V(p) \) is a neighbourhood of \( p \) in \( S \). We also define the family \( \mathcal{B}_\alpha(A) (0 \leq \alpha < \omega) \) of neighbourhoods of rank \( \alpha \) of points of \( A \) as follows:

\[
V(p, A) \in \mathcal{B}_\alpha(A) \text{ iff } V(p) \in \mathcal{B}_\alpha, \text{ where } \mathcal{B}_\alpha \text{ is a family of neighbourhoods of rank } \alpha \text{ in } S.
\]

Then, \( A \) is a ranked space with indicator \( \omega \). We call it a ranked space induced from \( \{ S, \mathcal{B}_\alpha \} \).

Moreover, let us consider a ranked space \( A \) induced from \( S \) such that: for every \( p \in A \) and for every fundamental sequence \( \{ V_\alpha(p, A); 0 \leq \alpha < \omega \} \) of neighbourhoods of \( p \), there is a fundamental sequence \( \{ V_\alpha(p); 0 \leq \alpha < \omega \} \) of neighbourhoods of \( p \) in \( S \) for which we have \( V_\alpha(p, A) = V_\alpha(p) \cap A \) for each \( \alpha \). We call this

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5) A limit number \( \alpha \) is said to be inaccessible, if, for every \( \beta \) with \( \beta < \alpha \) and for every function \( \alpha(\gamma) \) defined for \( \gamma \) with \( 0 < \gamma < \beta \), such that \( 0 < \alpha(\gamma) < \alpha \), we have always \( \sup \alpha(\gamma) < \alpha \).

6) [3], I, p. 319.
A a **ranked subspace** of \( \{S, \mathfrak{B}_a\} \). We will denote the subspace \( A \) of \( \{S, \mathfrak{B}_a\} \) by \( \{A, \mathfrak{B}_a(A)\} \). When \( \{p_\alpha; 0 \leq \alpha < \omega\} \) is a sequence of points of \( A \) and \( p \) is a point of \( A \), we have \( p \in \{\lim_\alpha p_\alpha\} \in A \) iff \( p \in \{\lim_\alpha p_\alpha\} \) in \( S \).\(^7\)

As the trivial examples of ranked spaces, there are metric spaces,\(^9\) M. L. Schwartz's distribution space \( (D), \ldots \), etc.

§ I. **Definition of Ranked Group.** Throughout this paper we suppose that every ranked space \( \{S, \mathfrak{B}_a\} \) satisfies the following axiom (b):

(b) \[ S \in \mathfrak{B}_a. \]

i) **R-continuity.**

**Definition 1.** A mapping \( f: \{S, \mathfrak{B}_a\} \rightarrow \{S', \mathfrak{B}'_a\} \) is said to be **R-continuous** at the point \( p \) if the following condition is fulfilled:

For any fundamental sequence \( \{V_\alpha(p); 0 \leq \alpha < \omega\} \) of any point \( p \) in \( S \), there exists a fundamental sequence \( \{V'_\alpha(p'); 0 \leq \alpha < \omega\} \) of the point \( p' = f(p) \) in \( S' \) such that

\[ f(V_\alpha(p)) \subseteq V'_\alpha(p') \ (V_\alpha; 0 \leq \alpha < \omega). \]

The mapping \( f \) is said to be **R-continuous** if it is R-continuous at each point of \( S \).

**Proposition 1.** Every R-continuous mapping is ortho-continuous. But the converse is not true.

**Definition 2.** \( f: \{S, \mathfrak{B}_a\} \rightarrow \{S', \mathfrak{B}'_a\} \) is called a **homeomorphism** iff \( f \) is bijective and bi-R-continuous. In this case, the spaces \( \{S, \mathfrak{B}_a\} \) and \( \{S', \mathfrak{B}'_a\} \) are said to be **homeomorphic** to each other.

**Proposition 2.** Let \( f \) be a mapping of \( \{S, \mathfrak{B}_a\} \) into \( \{S', \mathfrak{B}'_a\} \) such that \( f: S \ni p \mapsto p' \in S' \). Then, for each \( \alpha \) where \( 0 \leq \alpha < \omega \), we have the followings:

1. if \( f(\mathfrak{B}_a(p)) \subseteq \mathfrak{B}'_a(p') \) then \( f \) is R-continuous at the point \( p \),

   thus,

2. if \( f \) is bijective and if \( f(\mathfrak{B}_a(p)) \subseteq \mathfrak{B}'_a(p') \), \( f^{-1}(\mathfrak{B}'_a(p')) \subseteq \mathfrak{B}_a(p) \), then \( f \) is a homeomorphism.

ii) **Direct Product Ranked Space.**

Let

\[ \{S_\lambda, \mathfrak{B}'_a(p_\lambda)\} \ (p_\lambda \in S_\lambda; \lambda \in \Lambda, \Lambda \text{ is any indexing set}) \]

be ranked spaces with same indicator \( \omega \). And put

\[ \bar{S} = \bigsqcup_{\lambda \in \Lambda} S_\lambda, \quad (\text{direct product set of } S_\lambda) \]

\[ p = (p_\lambda)_{\lambda \in \Lambda} \in \bar{S}, \quad p_\lambda \in S_\lambda \ (\lambda \in \Lambda) \]

\[ \mathfrak{B}_a(p) = \{\bigsqcup_{\lambda \in \Lambda} V_\lambda(p) \mid V_\lambda(p) \in \mathfrak{B}'_a(p_\lambda) \ (a \leq V, a_\lambda, \omega) \ \& \ \text{Min} (a_\lambda; \lambda \in \Lambda) = a \} \ (V_\lambda; 0 \leq \alpha < \omega). \]

Then, \( S \) becomes a ranked space with indicator \( \omega \) by \( \mathfrak{B}_a. \) (c.q.f.d.). We call it the direct product ranked space with indicator \( \omega \) of \( \{S_\lambda, \mathfrak{B}'_a(p_\lambda)\} (\lambda \in \Lambda) \), and denote it by \( \{\bar{S}, \mathfrak{B}_a\}, \{\bigsqcup_{\lambda \in \Lambda} S_\lambda, \bigsqcup_{\lambda \in \Lambda} \mathfrak{B}'_a(p_\lambda)\}, \) etc.

Now, for each \( \lambda \in \Lambda \) and for any fundamental sequence \( \{a_\alpha(p)\}; 0 \leq \alpha < \omega \} \) of any point \( p \) in \( S_\lambda \), a sequence \( \{U_\alpha(p); 0 \leq \alpha < \omega\} \) such that \( p = (p_\lambda)_{\lambda \in \Lambda} \) and \( U_\alpha(p) = (a_\alpha(p_\lambda))_{\lambda \in \Lambda} \ (V_\lambda; 0 \leq \alpha < \omega) \) is considered to be a fundamental sequence in \( \bar{S} \).

Therefore, we define the fundamental sequence in \( \bar{S} \) as follows:

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7) [34], I, p. 619.  
8) [30], [2]  
9) [34], Chap. III, § 1; [3], II, p. 552.
Let \( p = (p_i)_{i \in \mathbb{N}} \) be any point of \( G = \Pi G_i \) and \( U_\alpha (p) (0 \leq \alpha < \omega) \) a system of elements of \( \mathfrak{B} = \bigcup_{\alpha = \delta} \mathfrak{B}_\alpha (\alpha) (0 \leq \alpha < \omega) \).

Then, the sequence \( \{ U_\alpha (p) ; 0 \leq \alpha < \omega \} \) \( (U_\alpha (p))^\Pi (U_i^{(1)} (p))_{i \in \mathbb{N}} \) is said to be a fundamental sequence of \( p \) in \( G \) if, for each \( \lambda \in \Lambda \), \( \{ U_\alpha (p) ; 0 \leq \alpha < \omega \} \) is a fundamental sequence of \( p_i \) in \( G_i \).

**iii) Definition of Ranked Group with indicator \( \omega \).**

**Definition 3.** (i) For \( \{ G, \mathfrak{B}_\alpha \} \) with indicator \( \omega \), if \( G \) is a group and the group operation

\[ f_1 : (G \times G, \mathfrak{B}_\alpha \times \mathfrak{B}_\alpha) \rightarrow G \times \mathfrak{B}_\alpha \times \mathfrak{B}_\alpha \]

is \( R \)-continuous, then \( (G, \mathfrak{B}_\alpha) \) is called the **Semiranked Group with indicator \( \omega \).** We denote this by \( <G, \mathfrak{B}_\alpha>\).

(ii) For \( <G, \mathfrak{B}_\alpha> \), if the group operation

\[ f_2 : (G, \mathfrak{B}_\alpha) \rightarrow G \times \mathfrak{B}_\alpha \times \mathfrak{B}_\alpha \]

is \( R \)-continuous, then \( <G, \mathfrak{B}_\alpha> \) is called the **Ranked Group with indicator \( \omega \).** We denote this by \( (G, \mathfrak{B}_\alpha) \).

**Proposition 3.** For \( (G, \mathfrak{B}_\alpha) \), the conditions (i) and (ii) are equivalent to the following condition (iii):

(iii) \( f_3 : (G \times G, \mathfrak{B}_\alpha \times \mathfrak{B}_\alpha) \rightarrow G \times \mathfrak{B}_\alpha \times \mathfrak{B}_\alpha \)

is \( R \)-continuous.

**Proof.** (c) Let \( \{ V_\alpha (p) ; 0 \leq \alpha < \omega \} \) be any fundamental sequence of any point \( p \in G \times G \) such that \( p = (x, y) \) where \( x, y \in G \). Then, there are two fundamental sequences \( \{ u_\alpha (x) ; 0 \leq \alpha < \omega \} \) and \( \{ v_\alpha (y) ; 0 \leq \alpha < \omega \} \) in \( G \) such that \( V_\alpha (p) = (u_\alpha (x), v_\alpha (y)) (0 \leq \alpha < \omega) \). Since \( f_2 \) is \( R \)-continuous, for the fundamental sequence \( \{ v_\alpha (x) ; 0 \leq \alpha < \omega \} \) in \( G \), there is a fundamental sequence \( \{ V_\alpha (x) \} ; 0 \leq \alpha < \omega \) in \( G \) such that \( V_\alpha (p) \subseteq V_\alpha (x) \) \( (0 \leq \alpha < \omega) \). Moreover, since \( f_1 \) is \( R \)-continuous, for \( \{ u_\alpha (x) ; 0 \leq \alpha < \omega \} \) and \( \{ v_\alpha (y) ; 0 \leq \alpha < \omega \} \), there is a fundamental sequence \( \{ u_\alpha (x, y) ; 0 \leq \alpha < \omega \} \) in \( G \) such that

\[ u_\alpha (x) \cdot v_\alpha (y) \leq u_\alpha (x, y) (0 \leq \alpha < \omega). \]

Hence, we have \( f_3 \) is \( R \)-continuous.

(2) Let \( \{ u_\alpha (x) ; 0 \leq \alpha < \omega \} \), \( \{ v_\alpha (y) ; 0 \leq \alpha < \omega \} \) be any fundamental sequences of \( x, y \) in \( G \). Since \( f_2 \) is \( R \)-continuous, for \( \{ v_\alpha (y) ; 0 \leq \alpha < \omega \} \), there is a fundamental sequence \( \{ V_\alpha (y) \} ; 0 \leq \alpha < \omega \) in \( G \) such that

\[ f_3 : (u_\alpha (x), v_\alpha (y)) \rightarrow (u_\alpha (x) \cdot v_\alpha (y) \leq u_\alpha (x) \cdot v_\alpha (y) (0 \leq \alpha < \omega). \]

But, since \( v_\alpha (y) \leq u_\alpha (x) \cdot v_\alpha (y) \leq v_\alpha (y) (0 \leq \alpha < \omega) \), \( f_2 \) is \( R \)-continuous. Hence, there is a fundamental sequence \( \{ V_\alpha (x, y) \} ; 0 \leq \alpha < \omega \) in \( G \) such that

\[ u_\alpha (x) \cdot v_\alpha (y) = u_\alpha (x) \cdot (v_\alpha (y) \leq u_\alpha (x) \cdot (v_\alpha (y) (0 \leq \alpha < \omega). \]

Moreover, since \( f_3 \) is \( R \)-continuous, there is a fundamental sequence \( \{ u_\alpha (x) ; 0 \leq \alpha < \omega \} \) in \( G \) such that

\[ u_\alpha (x) \cdot (v_\alpha (y) \leq u_\alpha (x, y) (0 \leq \alpha < \omega). \]

Thus, we have

\[ f_1 : (u_\alpha (x), v_\alpha (y)) \rightarrow (u_\alpha (x) \cdot v_\alpha (y) \leq u_\alpha (x, y) (0 \leq \alpha < \omega). \]

Namely, \( f_1 \) is \( R \)-continuous (q.e.d.)

**Proposition 4.** For every ranked group \( G \), the preceding conditions (i), (ii) and (iii) are respectively equivalent to the following conditions:

(i) For \( x, y \in G \) and for any fundamental sequences \( \{ u_\alpha (x) ; 0 \leq \alpha < \omega \} \), \( \{ v_\alpha (y) ; 0 \leq \alpha < \omega \} \) of \( x, y \), there is a fundamental sequence \( \{ u_\alpha (x, y) ; 0 \leq \alpha < \omega \} \) of \( xy \) in \( G \) such that

\[ u_\alpha (x) \cdot v_\alpha (y) \leq u_\alpha (x, y) (0 \leq \alpha < \omega). \]

(ii) For any fundamental sequence \( \{ u_\alpha (x) ; 0 \leq \alpha < \omega \} \) of any point \( x \) in \( G \), there is a fundamental sequence \( \{ u_\alpha (x) ; 0 \leq \alpha < \omega \} \) of \( x \) in \( G \) such that

\[ u_\alpha (x) \leq u_\alpha (x) (0 \leq \alpha < \omega). \]
and

(iii) For \( \forall x, y \in G \) and for any fundamental sequences \( \{u_\alpha(\alpha) ; 0 \leq \alpha < \omega\} \), \( \{v_\alpha(\alpha) ; 0 \leq \alpha < \omega\} \) of \( x, y \), there is a fundamental sequence \( \{w_\alpha(y^{-1}) ; 0 \leq \alpha < \omega\} \) of \( y^{-1} \) in \( G \) such that

\[
u_\alpha(x) \cdot \nu_\alpha(y^{-1}) \subseteq w_\alpha(x y^{-1}) \quad (\forall \alpha ; 0 \leq \alpha < \omega).
\]

In fact, for any \( p \equiv (x, y) \in G \times G \) and for any \( V_\alpha(\alpha) \equiv (u_\alpha(\alpha), v_\alpha(\alpha)) \in \mathfrak{B}_\alpha \times \mathfrak{B}_\alpha \) \( (0 \leq \alpha < \omega) \) such that \( \{u_\alpha(\alpha)\} \), \( \{v_\alpha(\alpha)\} \) are fundamental sequences of \( x, y \) in \( G \), the following statements are true:

(i') \( u_\alpha(x) \cdot v_\alpha(y) = f_1(V_\alpha(\alpha)) \subseteq \alpha \) \( \forall \alpha ; 0 \leq \alpha < \omega \),

(ii') \( u_\alpha(x)^{-1} = f_2(u_\alpha(x)) \subseteq \alpha \) \( \forall \alpha ; 0 \leq \alpha < \omega \), and

(iii') \( u_\alpha(x) \cdot v_\alpha(y)^{-1} \subseteq f_3(V_\alpha(\alpha)) \subseteq \alpha \) \( \forall \alpha ; 0 \leq \alpha < \omega \).

Therefore, this proposition is true.

**Corollary.** If the indicator of the ranked group is \( \omega_0 \), then the notion of ranked group in our sense coincides with the notion of ranked group in the sense of M. Washihara [6].

By the definition of the ranked group, we have the following:

**Proposition 5.** For every \( (G, \mathfrak{B}_\alpha) \), \( f_2 : x \rightarrow x^{-1} \) is a homeomorphism:

**Proposition 6.** Let \( a_1, a_2, \ldots, a_m \) be a finite system of elements of \( (G, \mathfrak{B}_\alpha) \), let \( a_1^{r_1} a_2^{r_2} \ldots a_m^{r_m} = c \) be a product of powers of the \( a's \), where the powers may be positive or negative, and let \( \{U_n(a_1) ; 0 \leq \alpha < \omega\} \), \( \{U_n(a_2) ; 0 \leq \alpha < \omega\} \), \( \ldots \), \( \{U_n(a_m) ; 0 \leq \alpha < \omega\} \) be arbitrary fundamental sequences of the elements \( a_1, a_2, \ldots, a_m \). Then there is a fundamental sequence \( \{W_\alpha(c) ; 0 \leq \alpha < \omega\} \) of the element \( c \) such that

\[
U_n(a_1)^{r_1} U_n(a_2)^{r_2} \ldots U_n(a_m)^{r_m} \subseteq W_\alpha(c) \quad (0 \leq \alpha < \omega),
\]

where \( U_n(a_\alpha) \) is taken equal to \( U_n(a_\alpha) \) if \( a_\alpha = a_j \), the same being true for a greater number of equal elements.

**Neighbourhood systems of the Identity.**

**Translations of the ranked group.**

**Proposition 7.** Let \( a \) be a fixed element of \( (G, \mathfrak{B}_\alpha) \). Then the right and left translations of \( G \), namely,

\[
ra : x \rightarrow xa, \quad la : x \rightarrow ax
\]

are homeomorphisms of \( G \).

In fact, since \( x, a \in \langle G, \mathfrak{B}_\alpha \rangle \), for any fundamental sequences \( \{u_\alpha(\alpha) ; 0 \leq \alpha < \omega\} \), \( \{v_\alpha(\alpha) ; 0 \leq \alpha < \omega\} \) of \( x, a \) in \( G \), there is a fundamental sequence \( \{w_\alpha(\alpha) ; 0 \leq \alpha < \omega\} \) of \( xa \) in \( G \) such that

\[
u_\alpha(x) \cdot v_\alpha(a) \subseteq w_\alpha(xa) \quad (0 \leq \alpha < \omega).
\]

Hence, \( r_\alpha(u_\alpha(x)) = u_\alpha(x) \cdot a \subseteq u_\alpha(x) \cdot v_\alpha(a) \subseteq w_\alpha(xa) = w_\alpha(r_\alpha(x)) \). This shows that \( r_\alpha \) is \( R \)-continuous. Moreover, \( r_\alpha^{-1} : x \rightarrow a^{-1}x \) is \( R \)-continuous by the same argument as above. Hence, \( r_\alpha \) is a homeomorphism. The fact that \( l_\alpha \) is a homeomorphism follows similarly.

**Proposition 8.** Both \( \langle G, \mathfrak{B}_\alpha \rangle \) and \( (G, \mathfrak{B}_\alpha) \) are homogeneous. Namely, for \( \forall p, q \in G \), there exists a homeomorphism \( f \) of \( G \) such that \( f(p) = q \).

In fact, for \( \forall p, q \in G \), consider the mapping \( f : x \rightarrow q p^{-1}x \). Then \( f \) is a homeomorphism by Proposition 7 and \( f(p) = q \). Hence \( \langle G, \mathfrak{B}_\alpha \rangle \) is a homogeneous space. Thus \( (G, \mathfrak{B}_\alpha) \) is also a homogeneous space.

**Neighbourhood systems of the identity.**

**Proposition 9.** Let \( e \) be the identity of group \( G \). And for every \( (G, \mathfrak{B}_\alpha) \), put

\[
\mathfrak{B}_e \equiv \mathfrak{B}_e \equiv \bigcup_{\alpha = 0}^{\omega} \mathfrak{B}_\alpha(e),
\]

for \( \forall a \in G \) and for \( \forall \alpha \) where \( 0 \leq \alpha < \omega \), \( \mathfrak{B}_\alpha(a) = \mathfrak{B}_\alpha(e) \cdot a = a \cdot \mathfrak{B}_\alpha(e) \).
Then, $\mathfrak{B}$ possesses the following properties:

1) for $\forall \ V \in \mathfrak{B}$, $e \in V$,

2) for $\forall \ U, \ V \in \mathfrak{B}$, there is a $W \in \mathfrak{B}$ such that $U \cap V \supseteq W$.

3) for $\forall \ V \in \mathfrak{B}$ and for any $\alpha, 0 \leq \alpha < \omega$, there is a $\beta, \alpha \leq \beta < \omega$, and a $U$ in $\mathfrak{B}$ such that $U \subseteq V$,

4) $G \subseteq \mathfrak{B}$,

5) for $\forall \ U, \ V \in \mathfrak{B}$, there is a $W \in \mathfrak{B}$ such that $UV^{-1} \subseteq W$,

- or 5') for $\forall \ U, \ V \in \mathfrak{B}$ there is a $W \in \mathfrak{B}$ such that $UV \subseteq W$, and for $\forall \ U \in \mathfrak{B}$ there is a $V \in \mathfrak{B}$ such that $U^{-1} \subseteq V$,

and

6) for $\forall a \in G$ and for $\forall \ V \in \mathfrak{B}$, $a \cdot V a^{-1} \subseteq \mathfrak{B}$.

**Proposition 10.** (Sufficient condition)

Let $G$ be an abstract group, $\mathfrak{B}_\alpha (\varepsilon) \ (0 \leq \alpha < \omega)$ a system of families satisfying above conditions 1)—6) of its subsets including the identity $e$ in $G$. If, for any $a \in G$ and for all $a$ (where $0 \leq \alpha < \omega$),

$$(H) \quad \mathfrak{B}_\alpha (a) = \mathfrak{B}_\alpha (e) \cdot a \quad \text{[or $\mathfrak{B}_\alpha (a) = a \cdot \mathfrak{B}_\alpha (e)$]}$$

then

1) $G$ becomes a ranked space with indicator $\omega$

and

2) $\{ G, \mathfrak{B}_\alpha \}$ is a ranked group with indicator $\omega$.

In fact, since every $\mathfrak{B}_\alpha (a) \ (0 \leq \alpha < \omega)$ satisfies the conditions 1)—4), $G$ is a ranked space.

Moreover, for $\forall \ (x, y) \in G \times G$ and for $\forall \ (U, V) \in \mathfrak{B}(x) \times \mathfrak{B}(y)$ (where $U, V \in \mathfrak{B}(e)$), there exist $V'$ and $W$ in $\mathfrak{B}(e)$ such that

$$U \cdot x \cdot V^{-1} = U \cdot x \cdot V^{-1} = U \cdot V^{-1} = W \subseteq \mathfrak{B}(e).$$

Therefore, $\{ G, \mathfrak{B}_\alpha \}$ becomes a ranked group.

**Proposition 11.** If $\langle G, \mathfrak{B}_\alpha \rangle$ (or $\langle G, \mathfrak{B}_\alpha \rangle$) satisfies the following condition

for $\forall \ a \in G$ and for $\forall \ \alpha \geq 0$, $\mathfrak{B}_\alpha (a) = \mathfrak{B}_\alpha (e) \cdot a = a \cdot \mathfrak{B}_\alpha (e)$,

then, for $\forall \ U, V \in G$, there is a mapping $f$ such that $f(\mathfrak{B}_\alpha (a)) = \mathfrak{B}_\alpha (q)$ for each $a$ where $0 \leq \alpha < \omega$.

In fact, for $\forall x \in G$, consider the mapping $f(x) : x \rightarrow x \cdot p \cdot \alpha x$.


§ 2. Subgroup, Normal Subgroup, Quotient Group.

i) Subgroup, Normal subgroup, Quotient group.

**Definition 4.** (1) Let $G$ be a (Semi—) Ranked group with indicator $\omega$. A subset $H$ of $G$ is called a **subgroup** of the (Semi—) Ranked group $G$ if

$\langle i \rangle$ $H$ is a subgroup of the abstract group $G$,

$\langle ii \rangle$ $H$ is a ranked subspace of $\{ G, \mathfrak{B}_\alpha \}$.

We denote this by $\langle H, \mathfrak{B}_\alpha (H) \rangle$ (resp. $\langle H, \mathfrak{B}_\alpha (H) \rangle$).

(II) A subgroup $N$ of a (Semi—) Ranked group $G$ is called a **normal subgroup** of $G$ if $N$ is a normal subgroup of the abstract group $G$.

**Proposition 12.** $\langle H, \mathfrak{B}_\alpha (H) \rangle$ (resp. $\langle H, \mathfrak{B}_\alpha (H) \rangle$) becomes a (Semi—) Ranked group with indicator $\omega$.

**Proof.** To prove this it is sufficient to show that the group operations in $H$ are R-continuous in the ranked space $H$. Let $a, b$ be two elements of the set $H$ and let $\{ u(a) \cap H ; 0 \leq \alpha < \omega \}$, $\{ v(b) \cap H ; 0 \leq \alpha < \omega \}$ any fundamental sequences of $a, b$ in $H$. Since $\{ u(a) ; 0 \leq \alpha < \omega \}$, $\{ v(b) ; 0 \leq \alpha < \omega \}$ are fundamental sequences of $a, b$ in $G$, there exists a fundamental sequence $\{ u_1 (ab^{-1}) ; 0 \leq \alpha < \omega \}$ of $ab^{-1}$ in $G$ such that

$$(u(a) \cap H) \cdot (b \cap H)^{-1} \subseteq u(a) \cdot b^{-1} \subseteq u_1 (ab^{-1}) \cdot (0 \leq \alpha < \omega).$$

On the other hand, since $H$ is a subgroup of $G$ we have the following:
Hence, we have a fundamental sequence \( \{ w_\alpha (ab^{-1}) \cap H ; 0 \leq \alpha \leq \omega \} \) of \( ab^{-1} \) in \( H \) such that

\[
(\alpha \in H) \cap H^{-1} \subseteq \alpha \subseteq H.
\]

Therefore, \( \{ H \times H, \mathfrak{B}_H(H) \times \mathfrak{B}_H(H) \} \) is a normal subgroup of \( \mathfrak{B}_H(H) \).

**Remark** Let \( \{ e \} \) be the identity subgroup of \( G \). Since \( V \cap \{ e \} = \{ e \} \) for \( V \subseteq \mathfrak{B}_H(e) = \bigcup_{\alpha = 0}^{\omega} \mathfrak{B}_H(e), \{ e \} \) becomes a normal subgroup of \( \mathfrak{B}_H(H) \).

**Definition 5.** Let \( H \) be a subgroup of \( \langle G, \mathfrak{B}_H \rangle \) and let \( G \equiv G/H \) be the totality of all left cosets of the subgroup in the group \( G \). If we put

\[
\hat{V} : V(H \cap H^{-1}) \subseteq \mathfrak{B}_H(H) \text{ and } V \subseteq \mathfrak{B}_H(H), \{ \hat{V} (x) \} \subseteq \mathfrak{B}_H(H), \mathfrak{B}_H(x), \{ \hat{V}(x) \} \subseteq \mathfrak{B}_H(H), \mathfrak{B}_H(x) \}
\]

then \( \hat{G} \) becomes a ranked space with indicator \( \omega \). The ranked space \( G/H \) thus obtained we shall call the space of left cosets of the subgroup \( H \) in the group \( G \). Analogously we define the space of right cosets and use the symbol \( G/H \) for it also.

In the cases where there is no danger of ambiguity we shall make no distinction between the spaces of left and right cosets.

We denote the space of cosets by \( \langle \hat{G}, \mathfrak{B}_H \rangle, \langle G/H, \mathfrak{B}_H/H \rangle \), etc.

**Proposition 13.** \( \langle G/H, \mathfrak{B}_H/H \rangle \) is a homogeneous space.

In fact, for \( \forall \hat{a}, \hat{b} \in \hat{G} = G/H \), by the mapping \( f : \hat{G} \ni \hat{x} \rightarrow \hat{b} = \hat{a} \).

**Proposition 14.** Let \( \mathfrak{F} \) be the canonical mapping of the space \( \langle G, \mathfrak{B}_H \rangle \) on the space \( \langle G/H, \mathfrak{B}_H/H \rangle \), i.e.,

\[
\mathfrak{F} : \langle G, \mathfrak{B}_H \rangle \ni \forall x \rightarrow xH \in \langle G/H, \mathfrak{B}_H/H \rangle.
\]

Then, we have \( f(\mathfrak{B}_H(x)) = \mathfrak{B}_H(x) \) for each \( \alpha \) where \( 0 \leq \alpha < \omega \). Thus, \( f \) is an \( R \)-continuous mapping.

**Proposition 15.** If \( N \) is a normal subgroup of \( \langle G, \mathfrak{B}_H \rangle \) (resp. \( \langle G, \mathfrak{B}_H \rangle \)), then \( \langle G/N, \mathfrak{B}_H/N \rangle \) becomes a (semi-)ranked group with indicator \( \omega \).

In fact, since \( G \) is a ranked group, for any fundamental sequences \( \{ a_\alpha (e) \} \subseteq \mathfrak{B}_G(a) \), \( \{ b_\alpha (e) \} \subseteq \mathfrak{B}_G(b) \) in \( G/N \) there exists a fundamental sequence \( \{ a_\alpha (b^{-1}) \} \subseteq \mathfrak{B}_G(a) \), \( \{ b_\alpha (b^{-1}) \} \subseteq \mathfrak{B}_G(b) \) such that \( a_\alpha (b^{-1}) \subseteq a_\alpha (b^{-1}) \subseteq b_\alpha (b^{-1}) \). Thus, we have \( a_\alpha (b^{-1}) \subseteq a_\alpha (b^{-1}) \subseteq b_\alpha (b^{-1}) \subseteq \mathfrak{B}_G(a) \), \( \{ b_\alpha (b^{-1}) \} \subseteq \mathfrak{B}_G(b) \).

**Definition 6.** Let \( N \) be a normal subgroup of \( \langle G, \mathfrak{B}_H \rangle \) (resp. \( \langle G, \mathfrak{B}_H \rangle \)). Then, the (semi-)ranked group \( \langle G/N, \mathfrak{B}_H/N \rangle \) (resp. \( \langle G/N, \mathfrak{B}_H/N \rangle \)) is called the quotient group with indicator \( \omega \) of the (semi-)ranked group \( \langle G, \mathfrak{B}_H \rangle \) (resp. \( \langle G, \mathfrak{B}_H \rangle \)) by the normal subgroup \( N \).

**ii) Isomorphism, Automorphism, Homomorphism.**

Let \( S, S' \) be two ranked spaces with indicator \( \omega \) and let \( H \) a subset of \( S \). We assume that the mapping

\[
\varphi : \{ S, \mathfrak{B}_S \} \ni \forall \psi \rightarrow \psi' \in \{ S', \mathfrak{B}_S \}
\]

is satisfying the following condition (ast):

(i) For any subset \( A \) of \( S \), we have \( \varphi (A \cap H) = \varphi (A) \cap \varphi (H) \),

(ii) For any fundamental sequence \( \{ V_\alpha (p') ; 0 \leq \alpha < \omega \} \) of \( p' \) in \( S \), the sequence \( \{ V_\alpha (p') ; 0 \leq \alpha < \omega \} \) is a fundamental sequence of \( \varphi (p') \) in \( S' \)

and

(iii) For any fundamental sequence \( \{ V_\alpha (p', H') ; 0 \leq \alpha < \omega \} \) of \( p' \) in \( H' \), there exists a fundamental sequence \( \{ V_\alpha (p, H) ; 0 \leq \alpha < \omega \} \) of \( p \) in \( H \) such that \( \varphi (V_\alpha (p, H)) = V_\alpha (p', H') \) for each \( \alpha \) where \( 0 \leq \alpha < \omega \).
Then we have the following:

**Proposition 16.** \( \varphi ([H, \mathcal{B}_a(H)]) = (H', \mathcal{B}'_a(H')). \)

In fact, since \( \varphi \) is satisfying the condition (*), for any fundamental sequence \( \{V_\alpha(p', H'); 0 \leq \alpha < \omega\} \) of \( p' = \varphi(p) \) in \( H' \), there is a \( \{V_\alpha(p, H); 0 \leq \alpha < \omega\} \) such that \( \varphi(V_\alpha(p, H)) = V_\alpha(p', H') \) for each \( \alpha \) where \( 0 \leq \alpha < \omega \).

On the other hand, since \( H \) is a subspace of \( \{S, \mathcal{B}_a\} \), for any fundamental sequence \( \{V_\alpha(p, H); 0 \leq \alpha < \omega\} \) of \( p \) in \( H \), there exists a fundamental sequence \( \{V_\alpha(p); 0 \leq \alpha < \omega\} \) of \( p \) in \( S \) such that \( V_\alpha(p, H) = V_\alpha(p) \cap H \) for each \( \alpha \) where \( 0 \leq \alpha < \omega \). Therefore, we have the following:

\[
V_\alpha(p', H') = \varphi(V_\alpha(p, H)) = \varphi(V_\alpha(p) \cap H) = V_\alpha(p) \cap H'.
\]

And now, since \( \{V_\alpha(p'); 0 \leq \alpha < \omega\} \) is a fundamental sequence of \( p' \) in \( S' \), we have

\[
\varphi ([H, \mathcal{B}_a(H)]) = (H', \mathcal{B}'_a(H')).
\]

**Definition 7.** A mapping \( g \) of a (semi-) ranked group \( G \) into a (semi-) ranked group \( G' \) is called a homomorphism if

(i) \( g \) is a homomorphism of the abstract group \( G \) into the abstract group \( G' \),

(ii) \( g \) satisfies the condition (*).

For instance, the canonical mapping: \( G \ni x \rightarrow xN \in G/N \) is a homomorphism.

**Proposition 17.** If \( (G, \mathcal{B}_a) \sim (G', \mathcal{B}'_a) \) and \( H \) is a subgroup of \( (G, \mathcal{B}_a) \), then \( g(H) \) is a subgroup of \( (G', \mathcal{B}'_a) \).

**Definition 8.** A mapping \( f \) of a (semi-) ranked group \( G \) on a (semi-) ranked group \( G' \) is called an isomorphism if

(i) \( f \) is an isomorphism of the abstract group \( G \) on the abstract group \( G' \),

(ii) \( f \) is a homeomorphism of the ranked space \( G \) on the ranked space \( G' \).

Two (semi-) ranked groups are called **isomorphic** if there exists an isomorphism of one group on the other. In particular, an isomorphism of a (semi-) ranked group \( G \) into itself is called an **automorphism** of the group \( G \).

**Remark** Let \( H \) be a subgroup of \( (G, \mathcal{B}_a) \) and let \( (G, \mathcal{B}_a) \sim (G', \mathcal{B}'_a) \). In general, \( f(H) \) does not form a subspace of \( (G', \mathcal{B}'_a) \). But if \( f \) satisfies the condition (ii) in (*), then \( f(H) \) becomes a subspace of \( \{G', \mathcal{B}'_a\} \).

Thus, in this case, \( f(H) \) is a subgroup of \( (G', \mathcal{B}'_a) \).

**Proposition 18.** If \( (G, \mathcal{B}_a) \sim (G', \mathcal{B}'_a) \) and if \( N \equiv g^{-1}(e') \) is a subspace of \( (G, \mathcal{B}_a) \), then

1. \( N \) is normal in \( (G, \mathcal{B}_a) \),
2. \( (G/N, \mathcal{B}_a/N) \simeq (G', \mathcal{B}'_a) \),
3. This isomorphism satisfies the condition (ii) in (*). Thus, this isomorphism satisfies the condition (*).

**Proof.** As is known from group theory, we have \( G/N \simeq G' \). This algebraic homomorphism \( f \) is \( G/N \ni a \rightarrow g(a) \in G' \). Now, for any fundamental sequence \( \{V_\alpha(a)N; 0 \leq \alpha < \omega\} \) of \( a \) in \( G/N \), we have

\[
f(V_\alpha(a)N) = g(V_\alpha(a)) = V_\alpha(a').
\]

And the sequence \( \{V_\alpha(a'); 0 \leq \alpha < \omega\} \) is a fundamental sequence of \( a' \) in \( G' \). Hence \( f \) is \( R \)-continuous.

Now, let \( \{V_\alpha(a'); 0 \leq \alpha < \omega\} \) be any fundamental sequence of \( a' \) in \( G' \), and put \( H' = G' \). Then, by the condition (iii), there exists a fundamental sequence \( \{V_\alpha(a); 0 \leq \alpha < \omega\} \) of \( a \) in \( G \) such that \( g(V_\alpha(a)) = V_\alpha(a') \) for each \( a \) where \( 0 \leq \alpha < \omega \). Thus, we have the following:

\[
f^{-1}(V_\alpha(a')) = f^{-1}(g(V_\alpha(a))) = V_\alpha(a)N \quad (0 \leq \alpha < \omega).
\]

Since \( \{V_\alpha(a)N; 0 \leq \alpha < \omega\} \) becomes a fundamental sequence of \( a \) in \( G/N \), \( f^{-1} \) is \( R \)-continuous. Therefore, \( f \) is an isomorphism.

**Proposition 19.** Let \( N \) be a normal subgroup of \( (G, \mathcal{B}_a) \), \( M \) any subgroup of \( (G, \mathcal{B}_a) \), and
(G, \mathfrak{G}_a)^< (G/N, \mathfrak{G}_a/N). Then g(M) is a subgroup of (G/N, \mathfrak{G}_a/N). And if MN is a subspace of (G, \mathfrak{G}_a) and N a subspace of MN then MN/N is a subgroup of the ranked group G/N, and g(M) is isomorphic with HN/N.

In fact, this isomorphism is given by the mapping f: mN \mapsto g(m) (m \in M).

**Remark.** Assume that \((G, \mathfrak{G}_a)^< (G', \mathfrak{G}_a)). And let N \supseteq g^{-1}(e') be a subspace of \(G, \mathfrak{G}_a\), A a subgroup including N of \(G, \mathfrak{G}_a\). And suppose that AN becomes a subspace of G. Then since the restriction g|A of g on A is a homomorphism, we have AN/N \cong g(A) in the sense of the ranked group. In particular, if g is the canonical mapping of G onto G/N, then AN/N which is a quotient group of the ranked group A is isomorphic with AN which is a subgroup of the ranked group G/N.

**Proposition 20.** If H is a normal subgroup of \((G, \mathfrak{G}_a)\) and a subgroup N of H is normal in \((G, \mathfrak{G}_a)\) and H forms a subspace of G, then, in the sense of the ranked group, we have \((G/N)/(H/N) \cong G/H\).

In fact, as is known from group theory, we have G/H \cong \(G/N)/(H/N)\). This algebraic isomorphism f is given by the composition mapping h \cdot g of two canonical mappings such that \(G \xrightarrow{g} G/N \xrightarrow{h} (G/N)/(H/N)\). Since h \cdot g is canonical, f is an isomorphism in the sense of the ranked group.

**Remark.** Let M, N be two subgroups of \((G, \mathfrak{G}_a)\). Both MN and M \cap N are not always subgroups of \((G, \mathfrak{G}_a)\). Thus, both MN and M \cap N are not always subgroups of \((G, \mathfrak{G}_a)\). Therefore, the Second Isomorphism Theorem is not always true.

§ 3. Direct Product (semi–) Ranked Group.

i) Direct product (semi–) ranked group.

**Proposition 21.** Let \(\lambda \in \Lambda\), let \((G_i, \mathfrak{G}_i^{(i)})\) (resp. \(\langle G_i, \mathfrak{G}_i^{(i)} \rangle\)) be a (semi–) ranked group with indicator \(\omega\). Then, the direct product ranked space \((\tilde{G}, \tilde{\mathfrak{G}}_a)\), i.e., \(\prod_{i \in \Lambda} \mathfrak{G}_i^{(i)}\) is a (semi–) ranked group with indicator \(\omega\).

**Proof.** We have only to show that the mapping: \(x, y \mapsto xy^{-1}\) of \(\tilde{G} \times \tilde{G}\) onto \(\tilde{G}\) is \(R\)-continuous. Let x, y be two elements of \(\tilde{G}\), and \(\{U_s(x); 0 \leq \alpha < \omega\}\) \(\{V_s(y); 0 \leq \alpha < \omega\}\) two fundamental sequences of x, y in \(\tilde{G}\) such that, for each \(\lambda \in \Lambda\), \(U_i^{(i)}(x) \equiv (u_i^{(i)}(x_i))_{i \in \Lambda}\) and \(V_i^{(i)}(y) \equiv (v_i^{(i)}(y_i))_{i \in \Lambda}\). By the definition of the fundamental sequences in the direct product ranked space \(\tilde{G}\), for each \(\lambda \in \Lambda\), \(\{u_i^{(i)}(x_i); 0 \leq \alpha < \omega\}\) and \(\{v_i^{(i)}(y_i); 0 \leq \alpha < \omega\}\) are respectively fundamental sequences of \(x_i\) and \(y_i\) in \(\{G_i, \mathfrak{G}_i^{(i)}\}\). Since each \(G_i\) is a ranked group with indicator \(\omega_i\), for each \(\lambda \in \Lambda\), there exists a fundamental sequence \(\{a_i^{(i)}(x_iy_i^{-1}); 0 \leq \alpha < \omega\}\) of \(x_iy_i^{-1}\) in \(G_i\) such that \(u_i^{(i)}(x_i) \cdot v_i^{(i)}(y_i)^{-1} \equiv a_i^{(i)}(x_iy_i^{-1})\). Hence, the sequence \(\{W_s(xy^{-1}); 0 \leq \alpha < \omega\}\) such that \(W_s(xy^{-1}) \equiv (a_i^{(i)}(x_iy_i^{-1}))_{i \in \Lambda}\) becomes a fundamental sequence of \(xy^{-1}\) in \(\tilde{G}\). Thus, for each \(\lambda \in \Lambda\), we have the following:

\[U_s(x) \cdot V_s(y)^{-1} = (u_i^{(i)}(x_i)) \cdot (v_i^{(i)}(y_i)^{-1})_{i \in \Lambda} \equiv (a_i^{(i)}(x_iy_i^{-1}))_{i \in \Lambda} = W_s(xy^{-1}).\]

Therefore \((\tilde{G}, \tilde{\mathfrak{G}}_a)\) becomes a ranked group with indicator \(\omega\).

**Definition 9.** The (semi–) ranked group \((\tilde{G}, \tilde{\mathfrak{G}}_a)\) is called the direct product (semi–) ranked group with indicator \(\omega\) of the ranked groups \((G_i, \mathfrak{G}_i^{(i)})\) (resp. \(\langle G_i, \mathfrak{G}_i^{(i)} \rangle\)) (where \(\lambda \in \Lambda\)). We denote this by \((\tilde{G}, \tilde{\mathfrak{G}}_a)\) (resp. \(\langle \tilde{G}, \tilde{\mathfrak{G}}_a \rangle\)) and \(\prod_{\lambda \in \Lambda} \mathfrak{G}_i^{(i)}\) (resp. \(\prod_{\lambda \in \Lambda} \mathfrak{G}_i^{(i)}\)), etc.

**Proposition 22.** Let \(\lambda\) be any indexing set, \(\tilde{G} = \prod_{\lambda \in \Lambda} G_i\), the direct product (semi–) ranked group of the (semi–) ranked groups \(G_i(\lambda \in \Lambda)\), and \(\{V_s(p); 0 \leq \alpha < \omega\} \equiv \{(v_i^{(i)}(p_i))_{i \in \Lambda}\}\) any fundamental sequence of \(p\) in \(\tilde{G}\). Then, by the \(\lambda\)-th projection mapping \(pr_\lambda: \prod_{\lambda \in \Lambda} \mathfrak{G}_i^{(i)} \rightarrow a_i \in G_i, pr_\lambda(V_s(p); 0 \leq \alpha < \omega)\) \(= (v_i^{(i)}(p_i); 0 \leq \alpha < \omega)\) becomes a fundamental sequence of \(p_i\) in \(G_i\). Thus, for each \(\lambda \in \Lambda\), \(pr_\lambda\) is \(R\)-continuous. Moreover, \(pr_\lambda(\lambda \in \Lambda)\) is a homomorphism of \(\tilde{G}\) onto \(G_i\).
Proposition 23. For each \( i = 1, 2, \ldots, m \), let \( N_i \) be a normal subgroup with indicator \( \omega \) of the (semi-) ranked group \( G_i \) with indicator \( \omega \). Then, \( N_1 \times \cdots \times N_m \) is a normal subgroup with indicator \( \omega \) of the direct product (semi-) ranked group \( G_1 \times \cdots \times G_m \) with indicator \( \omega \), and, in the sense of the ranked group, we have

\[
G_1 \times \cdots \times G_m | N_1 \times \cdots \times N_m \cong G_i / N_i \times \cdots \times G_m / N_m.
\]

Proposition 24. Let \( \tilde{G} = \prod_{\lambda \in A} G_i \) be a direct product (semi-) ranked group of some (semi-) ranked groups \( G_i (\lambda \in A) \). Then, \( \tilde{G} \) is complete if \( G_i \) is complete for each \( \lambda \in A \).

ii) Direct product decomposition.

Proposition 25. Let \( N_1, \ldots, N_m \) be a system of ranked groups with same indicator \( \omega \), \( e_i \) the identity of \( N_i (i = 1, 2, \ldots, m) \), \( G' \) the direct product (semi-) ranked group of \( N_1, \ldots, N_m \). And let \( \varphi : N_i \rightarrow \prod_{x \in e_i} (e_i, e_{i-1}, e_i, e_{i+1}, \ldots, e_m) \in G' \) for each \( i = 1, 2, \ldots, m \). Then we have the following statements:

For each \( i = 1, 2, \ldots, m \),

1. \( \varphi \) becomes a homeomorphism of the ranked group \( N_i \) onto an induced space \( N'_{\omega} (\equiv N_i (\omega)) \) in \( G' \) and \( N_i \) is normal in \( G' \).
2. In the sense of ranked groups, we have \( N_i \cong \prod_{x \in e_i} N_i \), \( N_1 \times \cdots \times N_m \cong \prod_{x \in e_i} N_1 \times \cdots \times N_m \).
3. In algebraic sense, \( G' \) decomposes into the direct product of its subgroups \( N_1, \ldots, N_m \).
4. For any \( \{ V_i (x) : 0 \leq a < \omega \} \) (F. S. of \( x \in N_i, 1 \leq i \leq m \)), there is a \( \{ V_x (x) : 0 \leq a < \omega \} \) (F. S. of \( x \in G' \)) such that \( V_i (x) \cap \cdots \cap V_m (x) \subseteq V_x (x) \) \( x = x_1 \cdots x_m \).

Definition 10. Let \( G \) be a (semi-) ranked group with indicator \( \omega \), and \( N_1, \ldots, N_m \) a system of normal subgroups of the (semi-) ranked group \( G \). We say that the (semi-) ranked group \( G \) decomposes into the direct product of its subgroups \( N_1, \ldots, N_m \). If the following conditions are fulfilled:

1. \( \varphi \) is an isomorphism of the ranked group \( G' \) onto the ranked group \( G_i \).
2. \( \varphi \) satisfies the condition (ii) in (\( \varphi \)).
3. \( \varphi \) becomes an \( R \)-continuous identity mapping of \( N_i \) onto itself for each \( i = 1, 2, \ldots, m \).

Proposition 26. Suppose that the ranked group \( G \) can be decomposed into the direct product of its ranked subgroups \( N_1, \ldots, N_m \). And let \( G' \) be the direct product (semi-) ranked group of the ranked groups \( N_1, \ldots, N_m \). Then we have the following statements:

1. \( \varphi : G' \rightarrow \prod_{x \in e_i} N_i \times \cdots \times N_m \) becomes an isomorphism of the ranked group \( G' \) onto the ranked group \( G_i \).
2. \( \varphi \) satisfies the condition (i) in (\( \varphi \)).
3. \( \varphi \) becomes an \( R \)-continuous identity mapping of \( N_i \) onto itself for each \( i = 1, 2, \ldots, m \).

Proposition 27. Let \( G \) be a ranked group, and \( N_1, \ldots, N_m \) a system of normal subgroups of \( G \).

And suppose that \( G \) is decomposed into the direct product of \( N_1, \ldots, N_m \). Then the projection mapping \( \text{pr}_i : G \rightarrow \{ a \in N_i \} \) is continuous for each \( i = 1, 2, \ldots, m \).

Moreover, as is known from the abstract group theory, a ranked group \( (G, \mathfrak{S}_n (G)) \) is isomorphic to the direct product of two subgroups \( (A, \mathfrak{S}_n (A)) \) and \( (B, \mathfrak{S}_n (B)) \) if \( A, \mathfrak{S}_n (A) \) and \( B, \mathfrak{S}_n (B) \) are normal subgroups such that \( A \times B = e \), \( A \cup B = G \), \( \mathfrak{S}_n (A) \times \mathfrak{S}_n (B) = \mathfrak{S}_n (A \times B) \) \( 0 \leq \alpha \leq \omega \). In general, we have the following:

Proposition 28. A ranked group \( (G, \mathfrak{S}_n (G)) \) is isomorphic to the direct product of its subgroups \( (G_i , \mathfrak{S}_n (G_i )) \) \( i = 1, 2, \ldots, m \).

1. \( G_i \) is an abstract normal subgroup of \( G \).
2. \( G_i \cap (UG) = e \) for each \( j = 1, 2, \ldots, m \).
(3) \( G = \bigcup_{i=1}^{m} G_i \) and \( \mathbb{B}_a(G) = \prod_{i=1}^{m} \mathbb{B}_a(G_i) \) for every \( a, 0 \leq a < \omega \).

ii) **Embeddings of any group in a product group.**

If, for each \( \lambda \in \Lambda \) (\( \lambda \in \Lambda \), any indexing set), \( G_\lambda = G \), the product \( \prod_{\lambda \in \Lambda} G_\lambda \) is denoted simply by \( G^\Lambda \). \( G^\Lambda \) is the set of all mappings of \( \Lambda \) into \( G \), and if \( G \) is a group so is \( G^\Lambda \). Now if \( \Lambda = G \), which is a group, then \( G^G \) is also a group.

By the mapping \( \eta_r : G \ni a \mapsto r_a \in G^\Lambda \) (or \( \eta_l : a \mapsto l_a \)), \( G \) is mapped onto \( \eta_r(G) \subseteq G^\Lambda \) (or \( \eta_l(G) \subseteq G^\Lambda \)).

Therefore, we have the following:

**Proposition 29.** Any abstract group \( G \) can be embedded into \( G^\Lambda \), i.e., there exists a one-to-one mapping of \( G \) onto a subset of \( G^\Lambda \).

**Notations.** The mappings \( \eta_r : a \mapsto r_a \) and \( \eta_l : a \mapsto l_a \) of \( G \) into \( G^\Lambda \) will be called the **right** and **left canonical embeddings** of \( G \) into \( G^\Lambda \) respectively. In case \( G \) is an abelian group, \( r_a = l_a \) and thus \( \eta_r = \eta_l \).

**Proposition 30.** Let \( G \) be a (semi-) ranked group. Then \( G \) is isomorphic to \( \eta_r(G) \subseteq G^\Lambda \).

In fact, for every \( (G, \mathbb{B}_a) \) we have the following:

\[
\begin{align*}
\eta_r : a \in G, U &\mapsto r_a = \{ r_a | a \in U \}, \\
\eta_r &\circ \eta_l : a \mapsto r_a \\
\end{align*}
\]

\( (0 \leq a < \omega) \).

§ 4. **Other ranked algebraic systems.**

i) **Ranked Rings.**

**Definition 11.** An abstract ring \( R \) is called a **ranked ring** if the set \( R \) is a ranked space and if the following conditions are fulfilled:

(1) The mapping \( (x, y) \mapsto x - y \) of \( R \times R \) into \( R \) is \( R \)-continuous,

(2) The mapping \( (x, y) \mapsto xy \) of \( R \times R \) into \( R \) is \( R \)-continuous.

These conditions (1) and (2) are respectively equivalent to the following conditions:

(1') For any fundamental sequences \( \{ u_a(x) ; 0 \leq a < \omega \}, \{ v_a(y) ; 0 \leq a < \omega \} \) of any points \( x, y \) in \( R \), there is a fundamental sequence \( \{ w_a(x - y) ; 0 \leq a < \omega \} \) of \( x - y \) in \( R \) such that

\[
\begin{align*}
u_a(x) - v_a(y) &\subseteq w_a(x - y) \quad (\forall a ; 0 \leq a < \omega)
\end{align*}
\]

(2') For any fundamental sequence \( \{ u_a(x) ; 0 \leq a < \omega \}, \{ v_a(y) ; 0 \leq a < \omega \} \) of any points \( x, y \) in \( R \), there is a fundamental sequence \( \{ w_a(xy) ; 0 \leq a < \omega \} \) of \( xy \) in \( R \) such that

\[
\begin{align*}
u_a(x) \cdot v_a(y) &\subseteq w_a(xy) \quad (\forall a ; 0 \leq a < \omega)
\end{align*}
\]

Moreover the condition (1) is equivalent to the following condition (1') and the condition (1'')

(1'') The mappings \( R \times R \ni (x, y) \mapsto x + y \in R \) and \( R \ni x \mapsto x \in R \) are \( R \)-continuous.

(1') For any fundamental sequence \( \{ u_a(x) ; 0 \leq a < \omega \}, \{ v_a(y) ; 0 \leq a < \omega \} \) of any points \( x, y \) in \( R \), there is a fundamental sequence \( \{ w_a(x + y) ; 0 \leq a < \omega \} \) of \( x + y \) in \( R \) such that

\[
\begin{align*}
u_a(x) + v_a(y) &\subseteq w_a(x + y) \quad (\forall a ; 0 \leq a < \omega)
\end{align*}
\]

and

(2) For any fundamental sequence \( \{ u_a(x) ; 0 \leq a < \omega \} \) of any point \( x \) in \( R \), there is a fundamental sequence \( \{ v_a(-x) ; 0 \leq a < \omega \} \) of \( -x \) in \( R \) such that

\[
\begin{align*}-u_a(x) &\subseteq v_a(-x) \quad (\forall a ; 0 \leq a < \omega)
\end{align*}
\]

**Definition 12.** (I) An abstract subring \( H \) of the ranked ring \( R \) is called a **subring** of the ranked ring \( R \) if the set \( H \) is a subspace of the ranked space \( R \).

(II) A subset \( I \) of the ranked ring \( R \) is called an **ideal** of the ranked ring \( R \) if the set \( I \) that is an abstract ideal of the abstract ring \( R \) is also a subspace of the ranked space \( R \).
(III) $R/I$ is called a quotient ring. ($R/I$ becomes a ranked ring.)

(IV) Let $(R_i)_{i \in I}$ be a family of ranked rings with same indicator. Then the direct product $\bar{R} = \prod_{i \in I} \bar{R}_i$ becomes a ranked ring. This ranked ring $\bar{R}$ is called the product ring of the ranked rings $(R_i)$.

In a ranked ring $R$ every right translation $ra_i$ (resp. every left translation $la_i$) is $R$-continuous (and is a homeomorphism if $a_i^{-1}$ exists in $R$).

Let $S$ be a ranked space, and let $f$ and $g$ be two mappings of $S$ into a ranked ring $R$. If $f$ and $g$ are $R$-continuous at a point $p \in S$, then $f + g$, $-f$ and $fg$ are $R$-continuous at this point. (Attend to $f + g : x \mapsto f(x) + g(x)$, $-f : x \mapsto -f(x)$, $fg : x \mapsto f(x)g(x)$.) Thus we have the following:

**Proposition 31.** The $R$-continuous mappings of the ranked space $S$ into the ranked ring $R$ form a subring of the ring $R^S$ of all mappings of $S$ into $R$.

**Definition 13.** (I) A mapping $g$ of a ranked ring $R$ into a ranked ring $R'$ is called a homomorphism if

(i) $g$ is a homomorphism of the abstract ring $R$ into the abstract ring $G'$,

(ii) $g$ satisfies the condition ($\ast$).

(II) A mapping $f$ of a ranked ring $R$ onto a ranked ring $R'$ is called an isomorphism if

(i) $f$ is an isomorphism of the abstract ring $R$ onto the abstract ring $R'$,

(ii) $f$ is a homeomorphism of the ranked space $R$ onto the ranked space $R'$.

Two ranked rings are called isomorphic if there exists an isomorphism of one ring onto the other. In particular, an isomorphism of a ranked ring $R$ into itself is called an automorphism of the ring $R$.

**Proposition 32.** Let $R$ and $R'$ be two ranked rings with same indicator $\omega$ and $H$ a subring of the ranked ring $R$. If $g$ is a homomorphism of $R$ onto $R'$, then $g(H)$ is a subring of the ranked ring $R'$. If $f$ is an isomorphism of $R$ onto $R'$ and if $f$ satisfies the condition (ii) in ($\ast$) then $f(H)$ is a subring of the ranked ring $R'$.

**ii) Ranked Modules.**

**Definition 14.** Given a ranked ring $A$ with an identity element, a set $E$ is called a ranked left $A$-module if the following conditions are fulfilled:

1. $E$ is an abstract left $A$-module
2. $E$ is a ranked additive group,
3. The mapping $(\lambda, x) \mapsto \lambda x$ of $A \times E$ into $E$ is $R$-continuous.

We define similarly the notion of a ranked right $A$-module. Since every right $A$-module can be considered as a left $A^\circ$-module, where $A^\circ$ is the opposite ring of $A$, and since the ranked structure of $A$ is compatible with the ring structure of $A^\circ$, there is no need to distinguish between ranked right $A$-modules and ranked left $A^\circ$-modules.

Let $(E_\lambda)_{\lambda \in \Lambda}$ be an arbitrary family of ranked $A$-modules, and let $E = \Pi_{\lambda \in \Lambda} E_\lambda$ be the $A$-module which is the product of the $E_\lambda$. For each $\mu \in A$ the mapping $(\lambda, x) \mapsto (\lambda, \mu \cdot x)$ satisfies the condition (ii) in ($\ast$), and the mapping $(\lambda, x_\lambda) \mapsto \lambda x_\lambda$ is $R$-continuous. Since the mapping $(\lambda, x) \mapsto \lambda \cdot \mu x$ is the composition of $(\lambda, x_\lambda) \mapsto \lambda x_\lambda$ and $(\lambda, x) \mapsto (\lambda, \mu x)$ for each $\mu \in A$, the mapping $(\lambda, x) \mapsto \lambda \cdot \mu x$ is an $R$-continuous mapping of $A \times E$ into $E_\mu$. Therefore $E$ becomes a ranked $A$-module.

**iii) Ranked Fields.**

**Definition 15.** A set $K$ is called a ranked field if the following conditions are fulfilled:

1. $K$ is an abstract field,
2. $K$ is a ranked space,
3. The algebraic operations operating in $K$ are $R$-continuous in the ranked space $K$.

If $K$ is an abstract divisibility ring we shall denote by $K^*$ the multiplicative group of non-zero elements of $K$. 
If $K$ is a ranked field then the mapping $x \mapsto x^{-1}$ of $K^*$ into $K^*$ is $R$-continuous.

The condition (3) is equivalent to the following condition (3'):

(3') For any fundamental sequences $\{u_a(x); 0 \leq a < \omega\}, \{v_a(y); 0 \leq a < \omega\}$ of any points $x, y$ in $K$ there are two fundamental sequences $\{u_a(\lambda x + y); 0 \leq a < \omega\}, \{w_a(y); 0 \leq a < \omega\}$ of $x + y, xy$ in $K$ such that

$$u_a(x) - v_a(y) \subseteq u_a(x - y), \quad u_a(x) \cdot v_a(y) \subseteq u_a(xy) \quad (\forall a; 0 \leq a < \omega),$$

(2) For any point $x \neq 0 (x \in K)$ and for any fundamental sequence $\{u_a(x); 0 \leq a < \omega\}$, there is a fundamental sequence $\{v_a(x^{-1}); 0 \leq a < \omega\}$ of $x^{-1}$ in $K$ such that

$$u_a(x^{-1}) \subseteq v_a(x^{-1}) \quad (\forall a; 0 \leq a < \omega).$$

If $a \neq 0$, the translations $l_a$ and $r_a$ are homeomorphisms of $K$ onto itself. Thus, the mapping $l: x \mapsto ax + b$ is a one-to-one $R$-continuous mapping for all $a, b \in K$. In fact, for any fundamental sequence $\{u_a(x); 0 \leq a < \omega\}$ of $x$ in $K$, there are respectively four fundamental sequences $\{u_a(a); 0 \leq a < \omega\}, \{u_a'(a); 0 \leq a < \omega\}, \{u_a''(ax); 0 \leq a < \omega\}$ and $\{u_a(\lambda x + b); 0 \leq a < \omega\}$ of $a, b, ax$ and $\lambda x + b$ in $K$ such that

$$l(\{u_a(x)\}) = a \cdot u_a(x) + b \subseteq u_a'(a) \cdot u_a(x) + u_a''(b) \subseteq u_a''(ax) + u_a''(b) \subseteq u_a(ax + b) = w_a(l(x))$$

for all $a$ where $0 < a < \omega$. Hence $l$ is $R$-continuous. Therefore if $a \neq 0$, $l$ is a homeomorphism. Note that the translations $l_a$ and $r_a$ are automorphisms of the (ranked) additive group of $K$ if $a \neq 0$. If $l_a$ (resp. $r_a$) satisfies the condition (ii) in (*), and if $V \subseteq \mathbb{B}(0)$, then $aV \subseteq \mathbb{B}(0)$ (resp. $Va \subseteq \mathbb{B}(0)$) for $a \neq 0$.

**Proposition 33.** In the ranked field $K$ if $\{\lim a\}_a \ni a$ and $\{\lim b\}_a \ni b$ then we have the following properties:

(1) $\{a \in \mathbb{B}(0)\} \ni a + b,$

(2) $\{\lim (-a)\}_a \ni -a,$

(3) $\{\lim a \cdot b\}_a \ni ab,$

(4) if $a \neq 0$ and $a \neq 0$, $\{\lim a^{-1}\}_a \ni a^{-1}.$

In fact, since the algebraic operations operating in $K$ are $R$-continuous we have this proposition.

**iv) Linear Ranked Spaces.**

**Definition 16.** Given a ranked field $K$ with indicator $\omega$, a set $L$ is called a left-linear ranked space (resp. right-linear ranked space) with indicator $\omega$ over $K$ if the following conditions are fulfilled:

(1) $L$ is an abstract left linear space (resp. right linear space),

(2) $L$ is a ranked space with indicator $\omega$,

(3) The mapping $(\lambda, x) \mapsto \lambda x$ of $K \times L$ into $L$ is $R$-continuous.

**Proposition 34.** In view of (H) above condition (3) is equivalent to the conjunction of the following three conditions:

(3') For each $x_0 \in L$, the mapping $\lambda \mapsto \lambda x_0$ is $R$-continuous at the point $\lambda = 0$,

(3'') For each $\lambda_0 \in K$, the mapping $x \mapsto \lambda x$ is $R$-continuous at the point $x = 0$,

(3''') The mapping $(\lambda, x) \mapsto \lambda x$ is $R$-continuous at the point $(0, 0)$.

**Proposition 35.** For every $a \in K$ and every $b \in L$, the mapping $x \mapsto ax + b$ of $L$ into itself is $R$-continuous. And if $a \neq 0$ this mapping becomes a homeomorphism of $L$ onto itself.

An algebraic isomorphism $f$ (resp. algebraic homomorphism $g$) of a linear ranked space $S$ to a linear ranked space $T$ is called an isomorphism (resp. homomorphism) of $S$ to $T$ if $f$ is a one-to-one $R$-continuous mapping (resp. $G$ satisfies the condition (4)).

**EXAMPLE** Let $E$ be a ranked space with indicator $\omega_E$, and also a linear space over the real or complex field $C$. In $C$ we define $S_n$ and $\mathbb{B}_n$ $(0 \leq n < \omega)$ as follows:

$$S_n(\lambda) = \{- \mu \leq |\mu - \lambda| < 1/n, \mu \in \mathbb{C}\} \quad \text{for each} \, \lambda \in \mathbb{C},$$

$$\mathbb{B}_n \ni S_n(\lambda) \quad \text{for all} \, n = 0, 1, 2, \ldots.$$
Then $C$ becomes a ranked space with indicator $\omega_0$ by $\mathfrak{B}_n$. Moreover, if $\{lim \lambda_n\} \ni \lambda$ in $C$ then we have $\lim_{n \to \infty} \lambda_n = \lambda$ in the topological sense. Therefore this linear ranked space $\{E, \mathfrak{B}_n\}$ becomes a linear ranked space in the sense of M. Washihara [5; II].

§ 5. Ortho-continuous group, Para-continuous group. In this section we will give other definitions of ranked groups. They are the ortho-continuous groups and the para-continuous groups.

Given a sequence $\{p_n; 0 \leqslant a < \omega\}$ of points of $\{S, \mathfrak{B}_{\omega}\}$ and a point $p$ of $\{S, \mathfrak{B}_{\omega}\}$. We say that the sequence $\{p_n; 0 \leqslant a < \omega\}$ para-converges$^{10}$ to the point $p$, or that $p$ is a para-limit of $\{p_n; 0 \leqslant a < \omega\}$, if there is a monotone decreasing sequence $\{V_s(p_n); 0 \leqslant a < \omega\}$ consisting of neighbourhoods of $p_n$ and if $V_s(p_n)$ satisfies the following conditions: 

1) $V_s(p_n) \in \mathfrak{B}_{\omega}$. 
2) $\gamma(0) \leqslant \gamma(1) \leqslant \gamma(2) \leqslant \cdots \leqslant \gamma(a) \leqslant \cdots (0 \leqslant \gamma(a) < \omega)$, $\sup_a \gamma(a) = \omega$. 
3) $p \in V_s(p_n)$ for all $a$, $0 \leqslant a < \omega$.

In this case, we shall write 

$p \in \{\text{para-lim} \quad p_n\}$.

Definition 17. Let $R$, $S$ be two ranked spaces with same indicator $\omega$. Then we will say that the mapping $f: R \longrightarrow S$ is para-continuous at the point $p$ in $R$ iff 

$p \in \{\text{para-lim} \quad p_n\} \Rightarrow f(p) \in \{\text{para-lim} \quad f(p_n)\}$.

The mapping $f$ is said to be para-continuous if it is para-continuous at each point of $R$.

Definition 18. For $\{G, \mathfrak{B}_\omega\}$ with indicator $\omega$, if $G$ is an abstract group and the group operations $\{G \times G, \mathfrak{B}_\omega \times \mathfrak{B}_\omega\} \ni (x, y) \longrightarrow xy \in \{G, \mathfrak{B}_\omega\}$, $\{G, \mathfrak{B}_\omega\} \ni x \longrightarrow x^{-1} \in \{G, \mathfrak{B}_\omega\}$ are ortho-continuous (resp. para-continuous), then $\{G, \mathfrak{B}_\omega\}$ is called the ortho-continuous group (resp. para-continuous group) with indicator $\omega$.

Since two notions of ortho-convergence and para-convergence do not necessarily coincide with each other,$^{11}$ two notions of ortho-continuity and para-continuity do not necessarily coincide with each other.$^{12}$ Therefore an ortho-continuous group is not always a para-continuous group and conversely a para-continuous group is not always an ortho-continuous group. On the other hand, from Proposition 1 we have the following statement:

Proposition 36. If $G$ is a ranked group in the sense of Definition 3, $G$ becomes an ortho-continuous group. But the converse is not true. Thus we have

![Diagram]

Remark. If $G$ is a linear ranked space in the sense of M. Yamaguchi [9; II], then $G$ is not always an ortho-continuous additive group. But $G$ becomes a ranked additive group with indicator $\omega_0$ in the sense of Definition 3. (See [5; II])

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10) [17], pp. 23-24.
11) Ibid., pp. 24-25, Propositions 2 and 2'.
12) For the sequence $\{p_n; 0 \leqslant a < \omega\} (p_n \equiv p, 0 \leqslant a < \omega)$, these two notions of continuity coincide with each other.
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(To be continued.)

References


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