Some considerations in the Ranked Spaces

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Synopsis

The purpose of this paper is to study some properties of sets of points in the ranked spaces.

We shall use the same terminology that is introduced in [1] and [2]. And throughout this paper we shall always treat ranked spaces with general indicator \( \omega \).

1. Derived sets, Adherences.

Definition 1. Let \( R \) be a ranked space and let \( E \) any subset of \( R \). A point \( p \) in \( R \) is called an accumulation point\(^1\) of \( E \) if there exists a fundamental sequence \( \{ V_\alpha(p) ; 0 \leq \alpha < \omega \} \) of \( p \) in \( R \) such that

\[
V_\alpha(p) \cap (E - p) \neq \emptyset \quad \text{for all} \quad 0 \leq \alpha < \omega.
\]

The set of all accumulation points of \( E \) is called the derived set of \( E \). We denote this by \( E^d \). \( E^n = \{ p \mid p \in \{ \lim p_\alpha \} \} \) is called the adherence of \( E \) and each point of \( E^n \) is called an adherent point of \( E \).

Proposition 1. \( E \subseteq E^n \). \( E \subseteq F \Rightarrow E^n \subseteq F^n \), \( \phi^n = \phi \), \( R^n = R \), \( E^n \subseteq E^{*n} \).

Proposition 2. \( p \in E^d \Leftrightarrow p \in (E - p)^n \).

Proof. If \( p \) is a point of \( E^d \) then there is a fundamental sequence \( \{ V_\alpha(p) ; 0 \leq \alpha < \omega \} \) of \( p \) in \( R \) such that \( V_\alpha(p) \cap (E - p) \neq \emptyset \) for all \( 0 \leq \alpha < \omega \). Thus there is a sequence \( \{ p_\alpha ; 0 \leq \alpha < \omega \} \) such that

\[
p_\alpha \in V_\alpha(p) \cap (E - p) \quad \text{for all} \quad 0 \leq \alpha < \omega.
\]

Thus we have \( p \in \{ \lim \} p_\alpha \} \subseteq \{ p \in E - p ; 0 \leq \alpha < \omega \} \). Therefore we have \( p \in (E - p)^n \). Conversely, if \( p \in (E - p)^n \) then there is a sequence \( \{ p_\alpha ; 0 \leq \alpha < \omega \} \) such that

\[
p \in \{ \lim \} p_\alpha \} \subseteq \{ p \in E - p ; 0 \leq \alpha < \omega \} \subseteq \{ p \in (E - p)^n \}.
\]

Thus there is a fundamental sequence \( \{ V_\alpha(p) ; 0 \leq \alpha < \omega \} \) of \( p \) such that

\[
V_\alpha(p) \cap (E - p) \neq \emptyset \quad \text{for all} \quad 0 \leq \alpha < \omega.
\]

Hence we have \( V_\alpha(p) \cap (E - p) \neq \emptyset \) for all \( 0 \leq \alpha < \omega \). This shows \( p \in E^d \).

Proposition 3. \( E^d = E \cup E^d \), \( (E \cup F)^n \subseteq E^n \cup F^n \), \( (E \cup F)^n = E^n \cup F^n \).

Proof. From \( p \in E^d \cup p \in (E - p)^n \subseteq E^n \) we get \( E^d \subseteq E^n \). Hence \( E \cup E^d \subseteq E^n \). Conversely, let \( p \) be an element of \( E^n \). In the case of \( p \in E \) we get \( E^n \subseteq E \subseteq E \cup E^d \). In the case of \( p \in E^d \) then we have \( E^n \subseteq E \cup E^d \) \( \Longleftrightarrow \) \( p \in E^n \). This is a contradiction. Hence we have \( p \in E^d \). Thus we have

\[
E^d = E \cup E^d.
\]

Proposition 4. For any subset \( E \) of the ranked space \( R \), the following two conditions are equivalent:

1) \( p \in E^n \).

2) There is a fundamental sequence \( \{ V_\alpha(p) ; 0 \leq \alpha < \omega \} \) of \( p \) in \( R \) such that \( V_\alpha(p) \cap E \neq \emptyset \) for all \( \alpha \), \( 0 \leq \alpha < \omega \).

Proof. Let \( p \) be a point of \( E^n \). Then we have \( p \in E \) or \( p \in E^d \). And there is a fundamental sequence \( \{ V_\alpha(p) ; 0 \leq \alpha < \omega \} \) of \( p \) in \( R \). In this time, if \( p \in E \) then \( V_\alpha(p) \cap E \neq \emptyset \) for all \( \alpha \), \( 0 \leq \alpha < \omega \), and if \( p \in E^d \) then \( V_\alpha(p) \cap E \neq \emptyset \) \( \Longleftrightarrow \) \( V_\alpha(p) \cap (E - p) \neq \emptyset \) for all \( \alpha \), \( 0 \leq \alpha < \omega \). Hence if \( p \in E^n \) there is a fundamental sequence

\[
\text{[Footnote]}\quad [4], \text{II, p. 788.}
\]

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\( \{ V_n(p); 0 \leq \alpha < \omega \} \) of \( p \) in \( R \) such that \( V_n(p) \cap E = \emptyset \) for all \( \alpha, 0 \leq \alpha < \omega \). Conversely, suppose that there is a fundamental sequence \( \{ V_n(p); 0 \leq \alpha < \omega \} \) of a point \( p \) in \( R \) such that \( V_n(p) \cap E = \emptyset \) for all \( \alpha, 0 \leq \alpha < \omega \). Then if \( p \in E \) we have \( p \in E^2 \) and if \( p \notin E \), since \( E - p = E \), we have \( V_n(p) \cap (E - p) = V_n(p) \cap E = \emptyset \) for all \( \alpha, 0 \leq \alpha < \omega \). Thus we get \( p \in E^2 \subseteq E^2 \), i.e., \( p \in E^2 \).

Since \( \phi \equiv \phi^* \) & \( E \subseteq F \Rightarrow E^2 \subseteq F^2 \), \( p \in E^2 \Rightarrow p \in (E - p)^2 \) are true, we have the following statement:

**Proposition 5.** The ranked space \( R \) becomes a space \((V)\) in the sense of M. Fréchet.2)

## 2. Open sets, Closed sets.

**Definition 2.** Let \( E^e \equiv R - E \) be the complementary set of a subset \( E \) of the ranked space \( R \). Then \( E^e \equiv R - (E^e)^e \), each point of \( E^e \), \( E^e \equiv (E^e)^e \), each point of \( E^e \), \( E^e \equiv R - (E^e^e \cup E^e^e) \) and each point of \( E^e \) are respectively called the **interior** of \( E \), an **inner point** of \( E \), the **exterior** of \( E \), an **outer point** of \( E \), the **frontier** of \( E \) and a **boundary point** of \( E \).

From these definitions we have the following statement:

**Proposition 6.** \( E \subseteq E \subseteq E^e \), \( \phi^* = \phi \), \( E^e \subseteq E \), \( E^e = E - E^e \), \( E^e = E^e \), \( E^e = E^e \), \( E^e = E^e \), \( R^e = E^e \cup E^e \), \( E^e = E^e \), \( E^e = E^e \) (direct sum) = \( E^e \cup E^e \) (direct sum) = \( E^e \cup E^e \). Therefore \( E \) is open in \( R \).

**Definition 3.** Let \( E \) be a subset of the ranked space \( R \). If \( E \subseteq E \), i.e., \( E^e = E \) then \( E \) is called a **closed set** in \( R \). And if \( E^e \) is a closed set in \( R \) then \( E \) is called an **open set** in \( R \). (These definitions coincide with the definitions in the Note [3].)

**Proposition 7.** A subset \( E \) of the ranked space \( R \) is an open set in \( R \) iff \( E^e = E \).

**Proof.** If \( E \) is open in \( R \) then \( E^e = E \). From \( E^e = E - E^e \) we get \( E = E^e \). Conversely, if \( E^e = E \) then \( E \cap (E^e)^d = E \cap (E^e)^d = (E - (E^e)^d) \cap (E^e)^d = \emptyset \). Hence we have \( (E^e)^d \subseteq E^e \). Thus we get \( E^e = E^e \cup (E^e)^d = E^e \). Therefore \( E \) is open in \( R \).

**Proposition 8.** For any subset \( E \) of the ranked space \( R \), the following two conditions are equivalent:

1. \( p \in E^e \),
2. For any fundamental sequence \( \{ V_n(p); 0 \leq \alpha < \omega \} \) (\( F. \ S. \) in \( R \)) of \( p \in E \), there is an \( \alpha', 0 \leq \alpha' < \omega \), such that \( E \supseteq V_{\alpha'}(p) \).

**Proof.** If \( p \in E^e \) then we have \( p \subseteq E - (E^e)^d \), i.e., \( p \notin E \) & \( p \subseteq (E^e)^d \). Thus, for any fundamental sequence \( \{ V_n(p); 0 \leq \alpha < \omega \} \) of \( p \in E \) there is an \( \alpha', 0 \leq \alpha' < \omega \), such that \( V_{\alpha'}(p) \cap (E^e - p) = \emptyset \). From \( E - p = E \) we have \( V_{\alpha'}(p) \cap E = \emptyset \). Hence, we have \( E \supseteq V_{\alpha'}(p) \). Conversely, if, for any fundamental sequence \( \{ V_n(p); 0 \leq \alpha < \omega \} \) of \( p \in E \), there is an \( \alpha', 0 \leq \alpha' < \omega \), such that \( E \supseteq V_{\alpha'}(p) \) then we have \( p \subseteq E \) & \( p \subseteq (E^e)^d \). Thus we have \( V_{\alpha'}(p) \cap (E^e - p) = \emptyset \), i.e., \( p \subseteq (E^e)^d \). Hence we have \( p \subseteq E - (E^e)^d = E^e \).

**Corollary 1.** \( E \subseteq F \Rightarrow E^2 \subseteq F^2 \).

**Corollary 2.** A subset \( E \) of the ranked space \( R \) is open in \( R \) iff, for \( \forall \ p \in E \) and for \( \forall \{ V_n(p); 0 \leq \alpha < \omega \} \) (\( F. \ S. \) of \( p \in R \)), there is an \( \alpha', 0 \leq \alpha' < \omega \), such that \( E \supseteq V_{\alpha'}(p) \). Therefore, our notion of open sets coincides with the notion of open sets in the sense of the Note [4].

In fact, if \( E \) is open in \( R \) we get \( E \) by Proposition 8. Conversely, if, for \( \forall p \in E \) and for \( \forall \{ V_n(p); 0 \leq \alpha < \omega \} \) (\( F. \ S. \) of \( p \in R \)), there is an \( \alpha', 0 \leq \alpha' < \omega \), such that \( E \supseteq V_{\alpha'}(p) \) then we have \( p \in E^e \), i.e., \( E \subseteq E^e \). Thus we have \( E = E^e \).

From Corollary 2 we have the followings:

**Proposition 9.** If both \( E \) and \( F \) are open (resp. closed) in the ranked space \( R \), then \( E \cup F \) (resp. \( E \cap F \)) is open (resp. closed) in \( R \), but \( E \cap F \) (resp. \( E \cup F \)) is not always open (resp. closed) in \( R \).

In fact, let \( E \) and \( F \) be two closed sets in \( R \). From \( (E \cap F)^e = (E \cap F)^e \) & \( (E \cup F)^e = (E \cup F)^e = E \cap F \) it follows that \( E \cap F \) is closed in \( R \).

**Corollary.** Any union (resp. intersection) of open sets (resp. closed sets) in \( R \) is open (resp. closed in \( R \)).

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3. Continuous mappings.

Prof. K. Kunugi introduced the notion of ortho-continuity in the Note [2]. We introduced another notion of continuity ([7]).

Definition 4. Let $R$, $S$ be two ranked spaces with same indicator $\omega$. Then we will say that the mapping $f: R \rightarrow S$ is $R$-continuous at the point $p$ in $R$ if the following condition is fulfilled:

For any fundamental sequence $\{V_a(p); 0 \leq \alpha < \omega\}$ of any point $p$ in $R$, there is a fundamental sequence $\{U_a(q); 0 \leq \alpha < \omega\}$ of the point $q = f(p)$ in $S$ such that $f(V_a(p)) \subseteq U_a(q)$ for all $\alpha, 0 \leq \alpha < \omega$.

The mapping $f$ is said to be $R$-continuous if it is $R$-continuous at each point of $R$.

From the definition of $R$-continuity we get the following statement:

Proposition 10. Every $R$-continuous mapping is ortho-continuous, but the converse is not always true.

Proposition 11. Let $R$, $S$ be two ranked spaces with same indicator and let $f: R \rightarrow S$ a mapping of $R$ into $S$. Then the following three conditions are equivalent.

1. $f$ is $R$-continuous at a point $p$ in $R$.
2. Let $E$ be a subset of $R$ and let $p \in E^a$, then we have $f(p) \in f(E)^a$, i.e., $f(E^a) \subseteq f(E)^a$.
3. Let $F$ be a subset of $S$ and let $p \in f^{-1}(F)^a$, then we have $f(p) \in F^a$.

Proof. (1)$\Rightarrow$(2); Let $f$ be an $R$-continuous mapping at the point $p$, then for any fundamental sequence $\{V_a(p); 0 \leq \alpha < \omega\}$ of $p$ in $R$ there is a fundamental sequence $\{U_a(f(p)); 0 \leq \alpha < \omega\}$ of $f(p)$ in $S$ such that $f(V_a(p)) \subseteq U_a(f(p))$, $\forall \alpha, 0 \leq \alpha < \omega$.

Since $p$ is belonging to $E^a$ there is a fundamental sequence $\{V_a(p); 0 \leq \alpha < \omega\}$ of $p$ in $R$ such that $V_a(p) \cap E^a \neq \emptyset$, $\forall \alpha, 0 \leq \alpha < \omega$. Thus, we have $f(V_a(p)) \cap f(E)^a \neq \emptyset$, $\forall \alpha, 0 \leq \alpha < \omega$. Therefore, we get $f(V_a(p)) \subseteq U_a(f(p))$, $\forall \alpha, 0 \leq \alpha < \omega$, i.e., $f(p) \in f(E)^a$.

(2)$\Rightarrow$(1); Now we suppose that the mapping $f$ is not $R$-continuous at the point $p$ in $R$. Then, for any fundamental sequence $\{U_a(f(p)); 0 \leq \alpha < \omega\}$ of $f(p)$ in $S$ there is an $\alpha'$, $0 \leq \alpha' < \omega$, such that $f(V_{alpha}(p)) \nsubseteq U_{alpha}(f(p))$. Thus, we have the following fact:

$$p_{\alpha'} \in V_{\alpha'}(p) \& f(p_{\alpha'}) \notin U_{\alpha'}(f(p)).$$

Let $E$ be the set of all points $p$ satisfying the above condition. Then we have $p \in E^a$ and $f(E) \cap U_a(f(p)) \neq \emptyset$, i.e., $f(p) \notin f(E)^a$. Therefore, if $f(p) \in f(E)^a$ then $f$ is $R$-continuous at the point $p$ in $R$.

(2)$\Leftrightarrow$(3); Put $E = f^{-1}(F)$. From $f(E) = f^{-1}(F)$, using the condition (2), it follows that $p \in E^a = f^{-1}(F)^a \subseteq f(E)^a$.

(3)$\Leftrightarrow$(2); Put $F = f(E)$ for the set $E$ such that $p \in E^a$. From $E \subseteq f^{-1}(F)^a = f^{-1}(F)$ it follows that $E^a \subseteq f^{-1}(F)^a$. Thus, we have $p \in f^{-1}(F)^a$. Therefore, by the condition (3), we have $f(p) \in F^a = f(E)^a$.

Proposition 12. Let $R$, $S$ and $T$ be three ranked spaces with same indicator and let $R \rightarrow S \rightarrow T$. If $f$ is $R$-continuous at the point $p \in R$ and if $g$ is $R$-continuous at the point $q = f(p) \in S$ then the composed mapping $g \circ f$ is $R$-continuous at the point $p$ in $R$.

In fact, we have the following fact:

$$g(f(U_a(p))) \subseteq g(U_a(f(p))) \subseteq W_a(g(f(p))) (\forall \alpha, 0 \leq \alpha < \omega).$$

Let $\mathcal{G}$ be the set of all ranked spaces with same indicator $\omega$ and let $M \left( R, S \right)$ be the set of all $R$-continuous mappings of $R \left( \in \mathcal{G} \right)$ into $S \left( \in \mathcal{G} \right)$. Furthermore, we shall denote by the form $V_a(p) \rightarrow V_{alpha}(p)$ the fact that for any fundamental sequence $\{V_a(p); 0 \leq \alpha < \omega\}$ of $p$ in $R \in \mathcal{G}$ there exists a fundamental sequence $\{V_{alpha}(p'); 0 \leq \alpha < \omega\}$ of $p' = f(p)$ in $R' \in \mathcal{G}$.

Now we have the following three statements.
(i) For \( \forall f \in M(R, S), \forall g \in M(S, T) \) and \( \forall h \in M(T, U) \), \( h \cdot (g \cdot f) = (h \cdot g) \cdot f \) is satisfied.
(ii) For each \( R \in \mathcal{C} \), there exists the identity morphism \( 1_R \in M(R, R) \) and \( f \cdot 1_R = f \) is satisfied for \( \forall f \in M(R, S) \)

and

(iii) \( 1_R \cdot g = g \) is satisfied for \( \forall g \in M(S, R) \).

In fact, from the form \( V \xrightarrow{f} V' \xrightarrow{g} V'' \xrightarrow{h} V''' \) it follows that the form \( V \xrightarrow{h \cdot (g \cdot f)} V''' \). On the other hand, we have the form \( V \xrightarrow{f} V' \xrightarrow{h^{-1} \cdot g} V'' \). Therefore \( h \cdot (g \cdot f) = (h \cdot g) \cdot f \) is true. Moreover the conditions (ii) and (iii) are clearly true. Therefore we get following statement:

**Proposition 13.** We have the category \( \mathcal{C} \) of ranked spaces. Objects, all ranked spaces with same indicator \( \omega \); morphisms, all \( R \)-continuous mappings \( f : R \rightarrow S \) of one space \( R \in \mathcal{C} \) into a second one \( S \in \mathcal{C} \).

### 4. Homeomorphisms.

Let \( R, S \) be two ranked spaces with same indicator and let \( f : R \rightarrow S \) be a one-to-one \( R \)-continuous mapping of \( R \) onto \( S \).

**Proposition 14.** Let \( E \) be a subset of the ranked space \( R \). If \( p \in (E - p)^a \) then we have \( f(p) \in (f(E) - f(p))^a \).

**Proof.** Since \( f \) is one-to-one we have \( f(E - p) = f(E) - f(p) \). And since \( f(p) \) is \( R \)-continuous, if \( p \in (E - p)^a \) then we have \( f(p) \in (E - p)^a \). On the other hand we have \( f(E - p)^a = (f(E) - f(p))^a \). Thus we have \( f(p) \in (f(E) - f(p))^a \).

**Definition 5.** If \( f : R \rightarrow S \) is bijective and \( R \)-continuous then it is called the **homeomorphism**. If \( f \) is a homeomorphism then its inverse mapping \( f^{-1} \) is also a homeomorphism.

**Proposition 15.** Let \( f : R \rightarrow S \) be a homeomorphism and \( E \) a subset of \( R \). Then we have the following statement:

\[ p \in (E - p)^a \iff f(p) \in (f(E) - f(p))^a. \]

**Proof.** Put \( q = f(p) \). And let \( F \) be a subset of \( S \). Since \( f^{-1}(q) \) becomes a homeomorphism we have the following fact:

\[ q \in (F - q)^a \iff f^{-1}(q) \in (f^{-1}(F) - f^{-1}(q))^a. \]

Now put \( f^{-1}(F) = E \). Then we have the following statement:

\[ f(p) \in (f(E) - f(p))^a \iff p \in (E - p)^a. \]

**Proposition 16.** Let \( f : R \rightarrow S \) be a bijection. Then the following three conditions are equivalent:

1. \( f \) is a homeomorphism.
2. \( f(E^a) = f(E)^a \) is satisfied for any subset \( E \) of \( R \).
3. \( f^{-1}(F)^a = f^{-1}(F)^a \) is satisfied for any subset \( F \) of \( S \).

**Proof.** (1) \( \Rightarrow \) (2); By Proposition 15 we get \( p \in (E - p)^a \iff f(p) \in f(E - p)^a \). Hereupon replace \( E - p \) by \( E \). Then we get \( f(E^a) = f(E)^a \).

(2) \( \Rightarrow \) (1): Since \( f(E^a) \subseteq f(E)^a \) is satisfied for any subset \( E \) of \( R \), \( f \) is \( R \)-continuous on \( R \) and \( f^{-1}(f(E)^a) \subseteq E^a \) is satisfied for any subset \( E \) of \( R \). Put \( F = E \). Then we have \( E = f^{-1}(F) \). Thus, we have \( f^{-1}(F)^a \subseteq f^{-1}(F)^a \). Therefore, \( f^{-1} \) is \( R \)-continuous on \( S \).

(2) \( \Rightarrow \) (3); Put \( f^{-1}(F) = E \). Then we have \( F = f(E) \). From \( f(E^a) = f(E)^a \) we have \( f(f^{-1}(F)^a) = F^a \). Thus we have \( f^{-1}(F)^a = f^{-1}(F)^a \).

(3) \( \Rightarrow \) (2); This is clear.

**Proposition 17.** Let \( f : R \rightarrow S \) be a homeomorphism. Then we have \( f(E^a) = f(E)^a \) for any subset \( E \) of \( R \).
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Proof. If we replace $E$ by $E^\alpha$ in the equality $f(E^\alpha)=f(E)^\alpha$ then we have $f((E^\alpha)^\alpha)=f((E)^\alpha)^\alpha$. On the other hand we have $f((E)^\alpha)^\alpha=f((E)^\alpha)^\alpha$ & $f((E)^\alpha)^\alpha=f((E)^\alpha)^\alpha$. Hence we have $f(E)^\alpha=f(E)^\alpha$.

Let $R$, $S$ be two ranked spaces with same indicator. Let $E$ be a subset of $R$ and $F$ a subset of $S$. If there is a homeomorphism $f$ such that $F=f(E)$ then the set $F$ is said to be homeomorphic with the set $E$. This homeomorphic relation is denoted by $E\sim F$.

Proposition 18. $E\sim E$, $E\sim F \Rightarrow F \sim E$, $E\sim F$ & $F\sim G \Rightarrow E\sim G$.

5. Open mappings, closed mappings.

Definition 6. A mapping $f$ on a ranked space with indicator $\omega$ to another ranked space with same indicator $\omega$ is open (resp. closed) iff the image of each open set (resp. closed set) is open (resp. closed).

Proposition 19. Let $f: R \rightarrow S$ be a bijection of the ranked space $R$ to the ranked space $S$. Then the following conditions are coincide with each other.

(i) $f$ is open. (2) $f$ is closed.

Moreover we get the following propositions.

Proposition 20. Let $R$, $S$, $T$ be three ranked spaces with same indicator and let $R \xrightarrow{f} S \xrightarrow{g} T$. If both $f$ and $g$ are open (resp. closed) then the composed mapping $g \circ f$ is open (resp. closed).

Proposition 21. $f: R \rightarrow S$ is a homeomorphism iff $f$ is a one-to-one $R$-continuous open (or closed) mapping.

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References


