On Generalized Continuous Groups II

By

Toshitada SHINTANI*

Tomakomai Technical College
(Received on January 10, 1972)

Synopsis.

In this paper we report on some results on the ranked groups and on the linear ranked spaces.

§ 1. Direct Product Decomposition**.

We shall give a better definition of Direct Product Decomposition than proceeding one (14).

Let \((N_i, \mathfrak{B}_i^{(0)})\) \((1 \leq i \leq m)\) be ranked groups with same indicator \(\omega\) and \(\{G', \mathfrak{B}'\}\) a direct product space of the ranked spaces \(\{N_i, \mathfrak{B}_i^{(0)}\}\) \((1 \leq i \leq m)\) defined as follows:

\[
G' = \bigotimes_{i=1}^{m} N_i \otimes \cdots \otimes N_m \quad \text{(direct product group)};
\]

\[
\mathfrak{B}'(x') = \{ \bigotimes_{i=1}^{m} V_i(x_i) : V_i(x_i) \in \mathfrak{B}_i(x_i) \text{ (} \alpha \leq \gamma \leq \omega) \text{ and Min } (\alpha_1, \ldots, \alpha_m) = \alpha \}
\]

for any \(x' = (x_1, \ldots, x_m)\) \((x_i \in N_i)\) and any \(\alpha\) such that \(0 \leq \alpha < \omega\).

And we define the fundamental sequence in \(\{G', \mathfrak{B}'\}\) as follows:

Let \(x' = (x_1, \ldots, x_m)\) be any point of \(\{G', \mathfrak{B}'\}\) and \(V_{\alpha}(x')\) \((0 \leq \alpha < \omega)\) an element of \(\mathfrak{B}' = \bigcup_{\alpha=0}^{\omega} \mathfrak{B}_{\alpha}(x')\). Then the sequence \(\{V_{\alpha}(x') : 0 \leq \alpha < \omega\}\) such that \(V_{\alpha}(x') \equiv (V_{\alpha}(x_1)_{1 \leq i \leq m}\) is said to be a fundamental sequence of \(x'\) in \(G'\) if, for each \(i, 1 \leq i \leq m\), \(V_{\alpha}(x_i)_{1 \leq i \leq m}\) \(0 \leq \alpha < \omega\) is a fundamental sequence of \(x_i \in G_i\).

Definition. Let \((N_i, \mathfrak{B}_i^{(0)})\) \((1 \leq i \leq m)\) be normal in \((G, \mathfrak{B})\). We say that \((G, \mathfrak{B})\) decomposes into the direct product of its subgroups \((N_1, \mathfrak{B}_1^{(0)}), \ldots, (N_m, \mathfrak{B}_m^{(0)})\) if the following two conditions are fulfilled:

(i) \(G\) can be decomposed into the algebraically direct product of its subgroups \(N_1, \ldots, N_m\);

(ii) For any fundamental sequence \(\{V_{\alpha}(x)\}\) of \(x \in G\) and for each \(i = 1, 2, \ldots, m\), there exists a fundamental sequence \(\{V_{\alpha}(x_i)_{1 \leq i \leq m}\}\) of \(x_i \in N_i\) such that \(V_{\alpha}(x) = V_{\alpha}(x_1) \cdots V_{\alpha}(x_m)\) \((x = x_1 \cdots x_m)\).

Theorem A. We have the followings.

(1) There is a mapping \(\varphi_i\) of \(N_i\) into \(G'\) and \(N'_i \equiv \varphi_i(N_i)\) becomes a normal subgroup of \(G'\).

And in the sense of the ranked groups, we have

\[
N_i \equiv N'_i, \quad \bigotimes_{i=1}^{m} N_i \equiv \bigotimes_{i=1}^{m} N'_i.
\]

(2) \(G'\) can be decomposed into the algebraically direct product of its subgroups \(N'_1, \ldots, N'_m\).

(3) For each \(i = 1, 2, \ldots, m\), and any fundamental sequence \(\{V_{\alpha}(x_i^j)_{0 \leq \alpha < \omega}\}\) of \(x_i^j \in N'_i\), there exists a fundamental sequence \(\{V_{\alpha}(x')_{0 \leq \alpha < \omega}\}\) of \(x' = x'_1 \cdots x'_m\) in \(G'\) such that \(V_{\alpha}(x')_{0 \leq \alpha < \omega}\) for all \(\alpha\), \(0 \leq \alpha < \omega\).

Proof.

(1) \(\varphi_i : N_i \ni x_i \longmapsto (e_1, \ldots, e_{i-1}, x_i, e_{i+1}, \ldots, e_m) \in G'\) and \(N'_i \equiv \varphi_i(N_i)\) \((1 \leq i \leq m)\).

Then \(N'_i\) is a normal subgroups of the abstract group \(G'\) and each \(\varphi_i\) is an algebraic isomorphism.

* 数学, 講師, 一般教科
** 1971年 4月, 日本数学学会年会(於東京都立大学)にて一部講演.
1) RG = ranked group.
2) IS = induced ranked space.
By \( \mathfrak{B}_a(N_i') = \mathfrak{B}_a(N_i') \cap N_i' \) \((0 \leq a < \omega)\), \(N_i'\) becomes an induced space of \(G'\). We will show that \(N_i'\) becomes a subspace of \(G'\). For any \(V'(x_i', N_i') \in \mathfrak{B}_a(N_i')\) we have

\[
\begin{align*}
V'(x_i', N_i') & = N_i' \cap (\cup_{0 \leq i < \omega} V_i(\ell_1), \ldots, V_i(\ell_{i-1}), V_i(\ell_i), V_i(\ell_{i+1}), \ldots, V_i(\ell_m), \{\ell_m\}(\{\ell_m\})) \\
& = (e_1, \ldots, e_{i-1}, N_i, e_{i+1}, \ldots, e_m) \cap (V_i(\ell_1), \ldots, V_i(\ell_{i-1}), V_i(\ell_i), V_i(\ell_{i+1}), \ldots, V_i(\ell_m), \{\ell_m\}(\{\ell_m\})) \\
& = (e_i \cap V_i(\ell_i), \ldots, e_{i-1} \cap V_i(\ell_{i-1}), N_i \cap V_i(\ell_i), e_{i+1} \cap V_i(\ell_{i+1}), \ldots, e_m \cap V_i(\ell_m), \{\ell_m\}(\{\ell_m\})) \\
& \equiv V'(x_i') \in \mathfrak{B}_a(x_i') 
\end{align*}
\]

Therefore, for every fundamental sequence \(\{V'_i(x_i', N_i')\}\) of \(x_i'' \in N_i''\), \(\{V'_i(x_i')\}\) becomes a fundamental sequence of \(x_i'' \in G'\).

Now, since \(V'_i(x_i', N_i') \subseteq N_i''\), for any fundamental sequence \(\{V'_i(x_i', N_i')\}\), there exists a fundamental sequence \(\{V'_i(x_i')\}\) of \(x_i'' \in G'\) such that \(V'_i(x_i', N_i') = N_i'' \cap V'_i(x_i', N_i') = N_i'' \cap V'_i(x_i'')\).

Thus \(N_i''\) becomes a ranked subspace of \(G'\). Hence \(N_i''\) is normal in \(G'\).

Next we will show that \(N_i'' \cong N_i''\) and \(\bigotimes_{i=1}^\omega N_i'' \cong \bigotimes_{i=1}^\omega N_i''\). It is clear that \(\varphi_i\) is a bijection.

Let \(V(x_i)\) be any element of \(\mathfrak{B}_a(\ell)(x_i)\). Since \(V(x_i) \subseteq N_i\) we have \(N_i \cap V(x_i) = V(x_i)\).

Therefore we get \(\varphi_i(N_i \cap V(x_i)) = (e_1, \ldots, e_{i-1}, N_i \cap V(x_i), e_{i+1}, \ldots, e_m) \in \mathfrak{B}_a(N_i) \equiv \mathfrak{B}_a(N_i')\) and \(\varphi_i(N_i \cap V(x_i)) = (e_1, \ldots, e_{i-1}, V(x_i), e_{i+1}, \ldots, e_m) \in \mathfrak{B}_a(x_i')\) when \(x_i'' = (e_1, \ldots, e_{i-1}, x_i, e_{i+1}, \ldots, e_m)\).

Thus, for any fundamental sequence \(\{V_i(x_i)\}\) of \(x_i'' \in N_i''\), \(\varphi_i(V_i(x_i)) = \{e_1, \ldots, e_{i-1}, V_i(x_i), e_{i+1}, \ldots, e_m\}\) becomes a fundamental sequence of \(x_i'' \in N_i''\). Thus \(\varphi_i\) is a rank preserving R-continuous mapping of \(N_i''\) onto \(N_i''\).

Now, let \(\{V_i(x_i', N_i')\}\) be a fundamental sequence of \(x_i'' = (e_1, e_{i-1}, x_i, e_{i+1}, \ldots, e_m)\) in \(N_i''\).

Since \(N_i''\) is a subspace of \(G'\) there is a fundamental sequence of \(x_i'' \in G'\), \(\{V_i(x_i')\}\), such that \(V_i(x_i') \cap N_i = N_i'' \cap V_i(x_i') \in \mathfrak{B}_\tau(\omega)(N_i')\) for some \(\tau(\omega), 0 \leq \tau(\omega) < \omega\).

From \(V_i(x_i') = (V_i(\ell_1), \ldots, V_i(\ell_{i-1}), V_i(\ell_i), V_i(\ell_{i+1}), \ldots, V_i(\ell_m), \{\ell_m\}(\{\ell_m\}))\) and \(V_i(x_i'') \subseteq N_i\) we have \(V_i(x_i'', N_i') = (e_1, \ldots, e_{i-1}, N_i, e_{i+1}, \ldots, e_m) \cap (V_i(\ell_1), \ldots, V_i(\ell_{i-1}), V_i(\ell_i), V_i(\ell_{i+1}), \ldots, V_i(\ell_m), \{\ell_m\}(\{\ell_m\}))\)

\[
= (e_1, \ldots, e_{i-1}, N_i \cap V_i(\ell_i), e_{i+1}, \ldots, e_m) \\
= (e_1, \ldots, e_{i-1}, V_i(x_i'), e_{i+1}, \ldots, e_m) 
\]

Thus we get \(\varphi^{-1}(V_i(x_i', N_i')) = V_i(x_i')\).

Hence if \(\{V_i(x_i', N_i')\}\) is a fundamental sequence of \(x_i'' \in N_i''\) then \(\varphi^{-1}(V_i(x_i', N_i')) = V_i(x_i')\) becomes a fundamental sequence of \(x_i'' = \varphi_i(x_i')\) in \(N_i''\).

Therefore \(\varphi^{-1}\) is an R-continuous mapping. Thus \(\varphi_i\) is an isomorphism in the sense of ranked groups.

Thus we have \(N_i'' \cong N_i''\).

\[
\begin{array}{cccc}
\varphi_1 & : & N_i' & \ni x_i' & \rightarrow x_i'' \in N_i'' \\
\varphi & : & N_i & \ni x_i & \rightarrow x_i'' \in N_i'' \\
\varphi_m & : & N_m & \ni x_m & \rightarrow x_m'' \in N_m''
\end{array}
\]

Then there exists a fundamental sequence of \(x_i'' \in N_i''\), \(\{V_i(\ell)(x_i')\}\), such that \(\varphi_i(V_i(\ell)(x_i')) \subseteq V_i(\ell)(x_i'')\) \((0 \leq a < \omega)\) for each \(V_i(\ell)(x_i')\) (i.e., F.S. of \(x_i'' \in N_i''\)). Thus there is a fundamental sequence of \((x_1'', x_m'')\) in \(N_1'' \otimes \cdots \otimes N_m''\), \(\{V_i(\ell)(x_i'), V_i(\ell)(x_m')\}\) such that \(\varphi_i(V_i(\ell)(x_i'), V_i(\ell)(x_m'))\) \(\subseteq V_i(\ell)(x_i'')\) \((0 \leq a < \omega)\) for all \(V_i(\ell)(x_i'), V_i(\ell)(x_m')\) (i.e., F.S. of \((x_1'', x_m'') \in N_1'' \otimes \cdots \otimes N_m''\).

Namely \(\varphi_i\) is R-continuous.
Analogously $\varphi^{-1}$ becomes an $R$-continuous mapping. Thus $\varphi$ is an isomorphism in the sense of ranked groups. Therefore we have $\prod_{t=1}^{m} N_t \cong \prod_{t=1}^{m} N'_t$.

(2) From abstract group theory we get this statement.

(3) Since there exists $\{V_{a^{(i)}}(x_i')\}$ (i.e., F.S. of $x'_t$ in $G'$) such that $V_{a^{(i)}}(x'_t) = V_{a^{(i)}}(x'_t) \cap N_t$ for each $\{x'_t\} \in \{G'_t\}$, there exists $\{V_{a^{(i)}}(x_i')\}$ (i.e., F.S. of $x'_t = x'_1 \cdots x'_m$ in $G'$) such that $V_{a^{(i)}}(x'_1) \cdots V_{a^{(i)}}(x'_m) = (V_{a^{(i)}}(x'_1) \cap N_1) \cdots (V_{a^{(i)}}(x'_m) \cap N_m \leq V_{a^{(i)}}(x'_1) \cdots V_{a^{(i)}}(x'_m) \leq V_{a^{(i)}}(x'_1) \cdots x'_m) \leq G'$. (Q.E.D.)

**Theorem B.** Let $G$ be the direct product decomposition of $N_1, \ldots, N_m$ and $G'$ the direct product group of $N'_1, \ldots, N'_m$. Then there exists an isomorphism of the ranked group $G'$ onto the ranked group $G$. And there exists an $R$-continuous identity mapping of the ranked group $N_t$ onto itself for each $i = 1, \ldots, m$.

**Proof.** Let $\{V_{a^{(i)}}(x'_i)\} = \{\{V_{a^{(i)}}(x_i), \ldots, V_{a^{(i)}}(x_i), \cdots \} (x_i') \in \mathbb{P}(x')$ for some $\gamma$, $0 < \gamma < \omega$ be any fundamental sequence of $x'_i$ in $G'$.

Since $G'$ is a direct product ranked group we have $V_{a^{(i)}}(x_i) \in \mathbb{P}(x_i')$ for some $\gamma$, $0 < \gamma < \omega$, and $\{V_{a^{(i)}}(x_i)\}$ becomes a fundamental sequence of $x_i$ in $N_i$.

Since $G$ is the direct product decomposition of $N_1, \ldots, N_m, \ldots$ for every fundamental sequence $\{V_{a^{(i)}}(x_i)\}, \ldots, \{V_{a^{(i)}}(x_i)\}$ in the groups $N_1, \ldots, N_m$, there exists a fundamental sequence of $x'_1 \cdots x'_m \leq U_a(x_1)$, such that $V_{a^{(i)}}(x'_1) \cdots V_{a^{(i)}}(x'_m) \leq U_a(x'_1) \cdots x'_m$, $x'_1 \cdots x'_m (0 < \alpha < \omega)$. Thus we have $\varphi(V_{a^{(i)}}(x'_i)) = V_{a^{(i)}}(x'_i) \cdots V_{a^{(i)}}(x'_m) \leq U_a(x'_1) \cdots x'_m$. Therefore $\varphi$ is $R$-continuous.

Conversely, let $\{V_a(x)\}$ be any fundamental sequence of $x = x_1 \cdots x_m (x_i \in N_i)$ in $G$. Since $G$ is decomposed into $N_1, \ldots, N_m$, there is a fundamental sequence of $x_i$ in $N_i$, $\{V_{a^{(i)}}(x_i)\}$, such that $V_a(x) = V_{a^{(i)}}(x_i) \cdots V_{a^{(i)}}(x_m) = V_{a^{(i)}}(x_i) \cdots \cdots V_{a^{(i)}}(x_m)$.

Thus we have $\varphi^{-1}(V_a(x)) = (V_{a^{(i)}}(x_i), \ldots, V_{a^{(i)}}(x_m)) \equiv V_{a^{(i)}}(x'_i), x'_i = (x_1, \ldots, x_m)$.

Since $\{V_{a^{(i)}}(x'_i)\}$ becomes a fundamental sequence of $x'_i$ in $G'$, $\varphi^{-1}$ is $R$-continuous.

Next, it is clear that $\varphi \circ \varphi^{-1}$ is the identity mapping of $N_t$ onto itself.

Moreover we have $\varphi \circ \varphi^{-1}(V_{a^{(i)}}(x_i)) = \varphi((e_{1,1}, e_{1,1}, V_{a^{(i)}}(x_i), e_{1,1}, e_{1,1}, e_{1,1})) = V_{a^{(i)}}(x_i)$.

Thus $\varphi \circ \varphi^{-1}$ is $R$-continuous. (Q.E.D.)

**Theorem C.** If $G$ is the direct product decomposition of its subgroups $(N_1, \mathbb{R}_a^{(1)})$ and $(N_2, \mathbb{R}_a^{(2)})$ then we have $(G/N_1, \mathbb{R}_a/N_1) = (N_2, \mathbb{R}_a^{(2)})$.

**Proof.** Let $\forall (x_1, x_2) \in G$ and $\varphi : N_2 \supset x_2 \mapsto x_1 N_1 \in G/N_1$. It is clear that $\varphi$ becomes an algebraic isomorphism of $N_2$ onto $G/N_1$.

Next, for $\forall V^{(2)}(x_2) \in \mathbb{R}_a^{(2)}$, we have $V^{(2)}(x_2) \cdot N_1 = V^{(2)}(x_2) \cdot V^{(1)}(x_1) N_1 \equiv V^{(1)}(x_1) \cdot V^{(2)}(x_2) N_1 \equiv V^{(1)}(x_1) \cdot V^{(2)}(x_2) \cdot N_1 \in \mathbb{R}_a N_1 (\because \gamma = \text{Min}(\alpha, \beta))$.

Because abstract group $G$ is the direct product of $N_1$ and $N_2$.

Thus $\{V^{(2)}(x_2)\}$ becomes a fundamental sequence of $\varphi(x) = x_2 N_1$ in $G/N_1$ for any fundamental sequence $\{V^{(2)}(x_2)\}$ of $x_2$ in $N_2$.

Therefore $\varphi$ is $R$-continuous.

Conversely, we have $\varphi^{-1}(V^{(2)}(x_1) \cdot V^{(2)}(x_2) \cdot N_1) = \varphi^{-1}(V^{(2)}(x_2) \cdot N_1) = V^{(2)}(x_2) (0 < \alpha < \omega)$ for any fundamental sequence $\{x_1 \cdot N_1\} = \{V^{(2)}(x_1) \cdot V^{(2)}(x_2) \cdot N_1\}$ of $x = x_1 x_2$ in $G/N_1$.

Namely, $\varphi^{-1}$ is $R$-continuous.

Therefore $\varphi$ becomes an isomorphism in the sense of ranked groups. (Q.E.D.)
§ 2. Convergences in the Ranked Group.

Let us consider two convergences, i.e., ortho-convergence and para-convergence in the ranked group.

Theorem. Let \((G, \mathfrak{G}_0)\) be a ranked group with indicator \(\omega \supseteq \omega_0\). Suppose that
\[
\mathfrak{G}_0(a) = a \cdot \mathfrak{G}_0(e) = \mathfrak{G}_0(e) \cdot a \quad (0 \leq V_a < \omega, \forall a \in G)
\]
and \(\{\mathfrak{G}_0(V_a(e))\} (\mathfrak{G}_0 \in G)\) is monotone decreasing iff \(\{V_a(e)\}\) is monotone decreasing.

Then we have
\[
(\text{ortho-lim } a \mathfrak{G}_0) \ni p \text{ in } G \Leftrightarrow (\text{para-lim } a \mathfrak{G}_0) \ni p \text{ in } G
\]
for any sequence \(\mathfrak{G}_0(a)\) in \((G, \mathfrak{G}_0)\).

Remark. If \((G, \mathfrak{G}_0)\) is commutative we have always \(a \cdot \mathfrak{G}_0(e) = \mathfrak{G}_0(a) \cdot e\) for each \(a, 0 \leq a < \omega\), and any \(a \in G\).

Proof of the theorem. Since \((\text{ortho-lim } a \mathfrak{G}_0) \ni p \text{ there exists a fundamental sequence } \{V_a(p)\}\)
such that \(V_a(p) \ni \mathfrak{G}_0\) for each \(a, 0 \leq a < \omega\). As \((G, \mathfrak{G}_0)\) is a ranked group there exists a fundamental sequence \(\{V_a'(p^{-1})\}\) such that \(p^{-1} \in V_a(p^{-1}) \subseteq V_a'(p^{-1})\) for each \(a, 0 \leq a < \omega\).

On the other hand, there exist a fundamental sequence \(\{U_a(e)\}\) and a monotone decreasing sequence \(\{U_a'(p^{-1})\}\) such that
\[
e \in \mathfrak{G}_0(U_a(p)) = p \cdot \mathfrak{G}_0(U_a'(p))) \text{ (from } V_a(p) \ni \mathfrak{G}_0 \text{ and } \mathfrak{G}_0(p) = p \cdot \mathfrak{G}_0(e))
\]
\[
= U_a'(p^{-1}) \cdot p \cdot (U_a'(p^{-1}))' \in \mathfrak{G}_0(U_a'(p^{-1})), \forall (a \uparrow \omega) \text{ as } a \uparrow \omega \text{ (from } \mathfrak{G}_0(p) = p \cdot \mathfrak{G}_0(e)).
\]
Thus we get \(U_a'(p^{-1}) \ni p^{-1}\) for each \(a, 0 \leq a < \omega\).

Therefore we have
\[
p^{-1} \in U_a'(p^{-1}) \subseteq U_a''(p) \text{ (} 0 \leq a < \omega\)
\]
for some monotone decreasing sequence \(\{U_a''(p)\}\) such that \(U_a''(p) \in \mathfrak{G}_0(U_a(p)), \forall (a \uparrow \omega)\) as \(a \uparrow \omega\).

Namely, we have \((\text{para-lim } a \mathfrak{G}_0) \ni p\).

Conversely suppose that \((\text{para-lim } a \mathfrak{G}_0) \ni p\). Since there exists a monotone decreasing sequence \(\{V_a(p)\}\) such that \(V_a(p) \ni p \text{ \& } V_a'(p) \in \mathfrak{G}_0(V_a(p))\) for each \(a, 0 \leq a < \omega\), and \(e \uparrow \omega\) as \(a \uparrow \omega\), we get
\[
p \in V_a(p) = p \cdot V_a'(e) \quad (0 \leq a < \omega)
\]
for some fundamental sequence \(\{V_a'(e)\}\) in \((G, \mathfrak{G}_0)\). Thus there is a point \(p \in V_a'(e)\) such that \(p \cdot p^{-1} = p\) for each \(a, 0 \leq a < \omega\).

Thus there exist two fundamental sequences \(\{U_a'(e)\}\) and \(\{U_a(p)\}\) such that
\[
p_a = p \cdot p^{-1} \in p \cdot V_a'(e) \subseteq p \cdot U_a'(e) = U_a(p)
\]
for each \(a, 0 \leq a < \omega\). Therefore we get \((\text{ortho-lim } a \mathfrak{G}_0) \ni p\).

This completes the proof.

§ 3. Linear Ranked Spaces.

We shall introduce linear ranked spaces as certain generalized normed linear spaces.

Let \(E\) be a linear space over real or complex field \(K\) and also a ranked space with indicator \(\omega_0\).

We now introduce following notations:
\[
E \equiv \{E, \mathfrak{G}_0\} \text{ (i.e., a ranked space)};
\]
\[
\mathfrak{G}_0 \equiv \bigcup_{a \in \mathfrak{G}_0} \mathfrak{G}_0;
\]
\[
\| x \| \equiv \text{ the rank of } V(x) \in \mathfrak{G}_0;
\]
\[
\{u_n(x)\} \equiv \text{ a fundamental sequence of } x \in E;
\]
\[
\mathfrak{G}(x) \equiv \text{ all of fundamental sequences with respect to } x \in E.
\]

Suppose that \(E\) satisfies the following condition (I) or (II):

(1) (i) For \(\forall x \in K\) and \(\forall V \in \mathfrak{G}(x)\), there is a \(W \in \mathfrak{G}(lx)\) such that
\[
\| x \| \leq W \quad \| lx \| = \left\lfloor \frac{\| x \|}{\| l \|} \right\rfloor
\]

(above \(\lfloor \cdots \rfloor\) is the Gaussian symbol);
For $V U \in \mathbb{V}(x)$ and $V V \in \mathbb{V}(y)$, there is a $W \in \mathbb{V}(x+y)$ such that

$$U + V \subseteq W \quad \& \quad \| x + y \|_W \leq \text{Min. } \left\{ \left( \frac{\| x \|_V}{2} \right), \left( \frac{\| y \|_V}{2} \right) \right\}$$

(II) \quad (1^\circ) \quad \text{For } V \mathcal{A} \in \mathcal{K} \text{ and } V \{u_n(x)\} \in \mathcal{K}(x), \text{ there is } \{v_n(\lambda x)\} \in \mathcal{K}(\lambda x) \text{ such that}

$$\lambda \cdot u_n(x) \subseteq v_n(\lambda x) \quad \& \quad \| \lambda x \|_{v_n} = \left\lfloor \frac{x}{\lambda} \right\rfloor$$

for all $n$, $0 \leq n < \omega_0$.

(2\textsuperscript{a}) \quad \text{For } V \{u_n(x)\} \in \mathcal{K}(x) \text{ and } V \{v_n(y)\} \in \mathcal{K}(y), \text{ there is } \{w_n(x+y)\} \in \mathcal{K}(x+y) \text{ such that}

$$u_n(x) + v_n(y) \subseteq w_n(x+y) \quad \& \quad \| x + y \|_{w_n} \leq \text{Min. } \left\{ \left( \frac{\| x \|_{u_n}}{2} \right), \left( \frac{\| y \|_{v_n}}{2} \right) \right\}$$

for all $n$, $0 \leq n < \omega_0$.

**Definition.** We call above $E$ a linear ranked space over $K$.

**Remark 1.** Above $E$ becomes a linear ranked space in the sense of [14, p. 58].

**Remark 2.** From the axiom (ii) we have the following statements:

1. For $V U \in \mathbb{V}(x)$ there exists $U \supseteq V \subseteq \mathbb{V}(x)$ such that $U \cap \mathbb{V}$ and $\| x \|_U \leq \| x \|_V < \omega_0$.

2. For $V \{u_n(x)\} \in \mathcal{K}(x)$ there exists $\{v_n(x)\} \in \mathcal{K}(x)$ such that $u_n(x) \supseteq v_n(x)$ and $\| x \|_{u_n} \leq \| x \|_{v_n} < \omega_0$ for all $n$, $0 \leq n < \omega_0$.

**Remark 3.** $\| x \| \geq 0$ for all $V \in \mathbb{V}(x)$. $\| x \| \leq \sup_{\{x \in \mathbb{V}(x) \mid \| x \| < \omega_0 \}} \| x \|_{v_0}$.

**Examples of above spaces.**

**Type (I);** (i) \quad \text{(Semi-) Normed space } (E, \| \cdot \|) \text{.}

Let $v(n; 0) = \{x \in E \mid x \| < \frac{1}{n} \}$, $\mathcal{B}_0(0) = \{v(n; 0)\}$ (one set family) and let $\mathcal{B}_0 = \{E\}$.

(2) \quad \text{Countably normed space } (\mathcal{V}, \| \cdot \|_\mathcal{V}) \text{.}

Let $v(n; 0) = \{p \in \mathcal{V} \mid \| p \| < \frac{1}{n} \}$, $\mathcal{B}_0(0) = \{v(n; 0)\}$ (one set family) and let $\mathcal{B}_0 = \{\mathcal{V}\}$.

(3) \quad \text{Countably normed space as linear Metric Space.}

I.M. Gelf'fand [8, p. 21] introduced a metric in a countably normed space. A metric space is considered as a ranked space with depth $\omega_0$.

(4) \quad \text{Perfect space. (See Gelf'fand [8, p. 54]).}

(5) \quad \text{Dual space of countably normed space } (\Phi, \| \cdot \|_\Phi) \text{.}

Let $v(n; p; 0) = \{x \in \Phi_k \mid x \|_p < \frac{1}{n} \}$, $\mathcal{B}_0(0) = \{v(n; p; 0)\}$ and let $\mathcal{B}_0 = \{\Phi\}$.

(6) \quad \text{L. Schwartz’s distribution space } D.

Let $v(n; k; 0) = \{p \in D \mid \| p \| < \frac{1}{n} \}$ and let $\mathcal{B}_0(0) = \{v(n; k; 0)\}$.

(7) \quad \text{Dual space } D' \text{ of space } D.

Let $v(n; 0) \equiv v(n; 0)$, $(m_1, \ldots, m_n)$ is a countable set. Let $U_{m_1} \ldots U_{m_n} \subseteq \frac{1}{n}$ and let $\mathcal{B}_0(0) = \{v(n; 0)\}$.

**Type (II);** (i) \quad \text{All examples in type (I).}

(ii) \quad \text{Union space of Countably normed spaces. } \Phi^{(m)}.

Let $\Phi^{(m)} = \bigcup_{m=1}^{\infty} \Phi^{(m)}$ be the union space of countably normed spaces $\Phi^{(m)}(m = 1, 2, \ldots)$ where $\Phi^{(m)} \subset \Phi^{(m+1)} \subset \cdots \subset \Phi^{(m)} \subset \cdots$ & the systems $\{ \| \cdot \|_{\Phi^{(m)}} \}$ and $\{ \| \cdot \|_{\Phi^{(m+1)}} \}$ are equivalent in $\Phi^{(m)}$.

And put $v(n; m; 0) = \{p \in \Phi^{(m)} \mid \| p \|^{(m)} < \frac{1}{n} \}$, $\mathcal{B}_0(0) = \{v(n; m; 0)\}$ for $n \geq 1$, and $\mathcal{B}_0 = \{\Phi^{(m)}\}$. 


(iii) Conjugate space $\mathcal{V}'$.
Let $\mathcal{V}'$ be the conjugate space to a countably normed space $\mathcal{V}$. Then we have $\mathcal{V}' = \bigcup_{p=1}^{\infty} \mathcal{V}'_{(p)}$ by Gelfand [8, p. 36].

(iv) Nuclear Space in the sense of Y. Nagakura. See [7; II].


1. We consider again linear ranked space in the sense of [14, p. 58].

**Theorem.** Locally convex linear topological space $S$ becomes a linear ranked space.

**Corollary 1.** Conjugate space $S'$ to a locally convex linear topological space $S$ is a linear ranked space.

**Corollary 2.** Let $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_k \subseteq \cdots$ be an increasing sequence of locally convex linear topological spaces $S_k (k=1,2,\ldots)$. Then the inductive limit space $S = \bigcup_{k=1}^{\infty} S_k$ becomes a linear ranked space. (Thus the conjugate space to $S$ is so.)

**Proof.** Suppose that the locally convex topology on above linear space $S$ is defined by a system of semi-norms $\{p_a(x)\}_{a \in A}$ on $S$. Let $v(n, B; 0) = \{x ; a \in B \Rightarrow p_a(x) < \frac{1}{n}\}$ for any finite subset $B$ of $A$ and let $\mathfrak{B}(0) = \{v(n, B; 0) ; \forall B \subseteq A\}$.

Then we get above theorem.

Since $S'$ is a locally convex space, we get corollary 1.

Finally, we shall prove corollary 2. Let $p_a(x)$ be a semi-norm on $S$ such that, by the topology on $S$, $p_a(x)$ is continuous on $S_k$ for each $k=1,2,\ldots$. Since all of above semi-norms $\{p_a(x)\}_{a \in A}$ defines the locally convex topology on $S, S$ is a locally convex linear topological space. Thus we get corollary 2. (Q.E.D.)

2. Examples of such linear ranked spaces.

(1) $D$ and its Fourier transformation $D$.

(2) Fréchet spaces (thus Banach spaces).

(3) LF-spaces (i.e., the inductive limit space of Fréchet spaces).

(4) Bornological spaces and the inductive limit space of Bornological spaces.

(5) Barreled spaces and the inductive limit space of Barreled spaces.

(6) Montel spaces and the inductive limit space of Montel spaces.

(7) The conjugate spaces to above spaces.

(8) L. Hörmander's space $\mathcal{C}(\Omega) = \bigcap_{\mu} L^{p,\mu}(\Omega)$.

§ 5. Linear Forms and Extension Theorem.

Let $R$ and $S$ be two linear spaces over the same field $\Phi$ (of real or complex numbers). The mapping $f$ of $R$ into $S$ is called **linear** if

$$f(x+y) = f(x) + f(y), \quad f(\lambda x) = \lambda f(x),$$

for all $x \in R, y \in R$ and $\lambda \in \Phi$. The linear mapping $f$ is one-to-one iff $f^{-1}(0) = \{0\}$; in general $f^{-1}(0)$ is a linear subspace of $R$. Moreover in the set $L$ of all linear mappings of $R$ into $S$, addition and multiplication by scalars can be defined by

$$(f+g)(x) = f(x) + g(x), \quad (\lambda f)(x) = \lambda f(x);$$

then $L$ becomes a linear space over $\Phi$.

**Proposition 1.** When $R$ and $S$ are both linear ranked spaces, the $R$-continuous linear mappings

---

3) [11], p. 27.
4) [12], p. 77.
of $R$ into $S$ form a linear subspace of $L$.

Because the $R$-continuity of $f$ and $g$ implies the $R$-continuity of $f+g$ and $\lambda f$.

**Proposition 2.** If $R$ and $S$ are (homogeneous) linear ranked spaces and $f$ is a linear mapping of $R$ into $S$, then $f$ is $R$-continuous on $R$ iff $f$ is $R$-continuous at the origin.

**Proof.** If $f$ is $R$-continuous at $0$, and $\{U_n(0)\}$ is any fundamental sequence of $0$ in $R$, there is a fundamental sequence $\{V_n(0)\}$ of $0$ in $S$ such that $f(U_n(0)) \subseteq V_n(0)$ for every $n$, $0 \leq n < \omega$. Then for each point $a$ of $R$, $f(a + U_n(0)) = f(a) + f(U_n(0)) \subseteq f(a) + V_n(0)$, and so $f$ is $R$-continuous at $a$.

**Definition.** If $R$ is a linear space over $\Phi$, a linear mapping of $R$ into the scalar field $\Phi$ itself is called a linear form (or linear functional) on $R$. A linear form $f$ on a linear ranked space $R$ is called continuous at $x \in R$ if

$$\lim_{n \to \infty} f(x_n) = x \text{ in } R \iff f(x) = \lim_{n \to \infty} f(x_n) = x \text{ in } \Phi.$$

**Remark.** If $R$ is a normed linear space then we have

$$\lim_{n \to \infty} x_n \in R \iff \lim_{n \to \infty} \|x_n - x\| = 0 \text{ in } R.$$

**Proposition 3.** If a linear form $f$ on a (homogeneous) linear ranked space $R$ is continuous at $0$, then $f$ is continuous on the whole of $R$.

**Proof.** If $\{\lim_{n \to \infty} x_n\} \subseteq R$ then there is a fundamental sequence $\{\nu_n(0) + x\} = \{x_n\}$ of $x$ such that $\nu_n(0) + x \in x_n$ for each $n$, $0 \leq n < \omega$. Then $\{\nu_n(0)\}$ becomes a fundamental sequence of $0$ and we have $\nu_n(0) \subseteq x_n - x$ for each $n$, $0 \leq n < \omega$. Thus

$$f(x_n) - f(x) = f(x_n - x) \to 0.$$

**Proposition 4.** Let $f$ and $g$ be two continuous linear forms on $R$ and $X$ an $r$-dense subset of $R$. When $f(x) = g(x)$ for any $x \in X$, we have $f = g$ on $R$.

**Proof.** For any $x \in R$ there is a sequence $\{x_n\}$ such that $\lim_{n \to \infty} x_n \subseteq x$ and $x_n \in X$.

Since $f$ and $g$ are continuous we get $f(x) = g(x)$ from $f(x_n) = g(x_n)$.

From linear topological space theory\(^a\), we have the **Hahn-Banach theorem**, i.e.,

**Theorem** (Hahn-Banach extension theorem). Suppose that $p(x)$ is a positive homogeneous subadditive function on a real linear space $R$. If a linear form $q(x)$, defined on a linear subspace $X$, satisfies

$$q(x) \leq p(x)$$

then $q(x)$ can be extended to a linear form $\ell$, defined on the whole of $R$, which satisfies

$$\ell x \leq p(x)$$

for $x \in R$.

If $R$ is a linear ranked space and $p(x)$ is continuous at $0$, then $\ell$ is also continuous.

**References**


5) [9]. p. 190
Ibid., 43 (1967), 590–593.


