Some structures on the Ranked Spaces

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Synopsis

The aim of this paper is to study the notion of sheaves treated by the method of ranked spaces.

Throughout this paper we shall treat only ranked spaces with same indicator \( \omega \leq \omega_0 \). And let these ranked spaces be always satisfying axioms (A), (B), (a) and (b)\(^1\).

§ 1. Sheaves.

1°. Sheaf of sets.

Let \( F \) and \( M \) be two ranked spaces and \( \pi : F \rightarrow M \) a unique mapping of \( F \) into \( M \).

**Definition 1.** The mapping \( \pi \) is called a local R-homeomorphism iff any \( f \in F \) has an \( r \)-open neighborhood of \( f \) in \( F \), \( \bar{U}(f) \), such that \( \pi : \bar{U} \rightarrow \pi(\bar{U}) \) is an R-homeomorphism and \( \pi(\bar{U}) \) becomes an \( r \)-open neighborhood of \( \pi(f) \) in \( M \).

**Definition 2.** A sheaf (of sets) on \( M \) is a triple \((F, \pi, M)\) where

(i) \( F \) is a ranked space.

(ii) \( \pi : F \rightarrow M \) is a local R-homeomorphism onto \( M \).

This \( \pi \) is called the projection of \((F, \pi, M)\).

**Definition 3.** The stalk over \( x \in M \) is the subset \( F_x = \pi^{-1}(x) \) of \( F \).

**Definition 4.** A section \( s \) of \( F \) over a subset \( X \) of \( M \) is an R-continuous mapping from \( X \) into \( F \) such that \( s \circ \pi \) is the identity \( 1 \).

For any \( y \in F \), there exists a section over some \( V \subset M \) passing through \( y \). Take \( V \) to be a homeomorph under \( \pi \) of some neighborhood \( W \) of \( y \) and let \( s = (\pi|_W)^{-1} \).

The set of all sections over \( X \) is denoted by \( \Gamma(X, F) \) and \( F(X) \). Especially, for every point \( x \in M \), \( F_x = \Gamma(x, F) \) holds.

For two subsets \( X \) and \( Y \) of \( M \), \( X \subset Y \) decides the mapping \( \rho_X^Y \) such that

\[ \rho_X^Y : \Gamma(Y, F) \rightarrow \Gamma(X, F), \rho_X^Y(s) = s|_X \text{ where } i : X \rightarrow Y \text{ is an inclusion map.} \]

This \( \rho_X^Y \) is called a restriction of the section \( s \).

**Proposition 1.** For a sheaf \((F, \pi, M)\), we have

(i) The stalk over \( x \in M \) is discrete.

(ii) The projection \( \pi \) is an \( r \)-open mapping.

(iii) Every section of \( F \) over \( X \subset M \) is an \( r \)-open mapping.

(iv) If any two sections agree at a point \( x_0 \in M \) then they agree in an \( r \)-open neighborhood of \( x_0 \).

**Proof.** (i) and (ii) are clear. (iii); Let \( s : V \rightarrow F \) be a section over an \( r \)-open set \( V \) in \( M \).

For any point \( x \in V \) there is an \( r \)-open neighborhood \( \bar{U} \) of \( s(x) \in F_x \). Then the set \( W = V \cap \pi(\bar{U}) \) becomes

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1) [8], I, pp 46-47.
an r-open neighborhood of x. Since the projection \( \pi : \tilde{U} \rightarrow \pi(\tilde{U}) \) is an R-homeomorphism and \( \pi = s(W) = W \) holds for W, the set s(W) is an r-open neighborhood of s(x). That is, s is an r-open mapping. (iv); Let s : U → F and t : V → F be two sections over r-open sets (of M) U and V respectively. And suppose that s(x₀) = t(x₀) at a point x₀ ∈ U ∩ V. From W = \( \pi(s(U) \cap t(V)) \) follows s(x) = t(x), x ∈ W. Moreover s, t and \( \pi \) are r-open mappings. Thus W is an r-open neighborhood of x₀. (Q.E.D.)

2°. Sheaf of groups.

Definition 5. A sheaf of (resp. abelian) groups on M is a triple (\( F, \pi, M \)) where

(i) (\( F, \pi, M \)) is a sheaf of sets on M.

(ii) The stalk \( F_x = \pi^{-1}(x) \) of F at any point \( x \in M \) is a (resp. an abelian) group.

(iii) The group operations are R-continuous.

Similarly, a sheaf of A-modules is defined.

We shall now list elementary consequences of above definition.

Proposition 2. If (\( F, \pi, M \)) is a sheaf of groups on M then the set \( \Gamma(X, F) \) forms a group and its group operations are given by following relations:

\[
(s \circ t)(x) = s(x) \circ t(x), \quad \forall x \in \Gamma(X, F), \quad x \in X.
\]

For each point \( x \in M \), let \( e_x \) be the unit of group \( F_x \). Then \( e_x \) assigns a section over M such that \( e : M \rightarrow F, \quad e(x) = e_x, \quad x \in M \).

Proposition 3. If (\( F, \pi, M \)) is a sheaf of groups we have the followings:

(a) The restrictions of its section \( \rho : \Gamma(Y, F) \rightarrow \Gamma(X, F), \quad \rho \circ \pi = \gamma \circ \pi \) become group homomorphisms.

(b) \( \rho_x = 1 \).

(c) \( \rho_x \circ \rho_y = \rho_{xy} \) when \( X \subseteq Y \subseteq Z \).

Example: Constant sheaf. Let F be a set (or an A-module) and let F be discrete. Make the direct product \( F = M \times F \) for a ranked space M. Then F becomes a sheaf of sets on M by the projection \( \pi \) such that \( \pi : F \rightarrow M, \quad \pi(x, a) = x, \quad x \in M, \quad a \in F \). F is called the constant sheaf on M.

3°. Presheaf.

Definition 6. Let M be a ranked space. A presheaf on M is a system F where

(i) Each r-open set U in M assigns an A-module F(U).

(ii) For any two r-open sets U and V such that U ⊆ V, there is a group homomorphism \( \rho_U^V : F(V) \rightarrow F(U) \) such that

(a) \( \rho_U^V \circ \rho_V^W = \rho_U^W \) when U ⊆ V ⊆ W.

§ 2. Homomorphisms, subsheaves direct sum of sheaves and quotient sheaves.

Definition 7. (a) Let (\( F, \pi, M \)) and (\( F', \pi', M \)) be two sheaves of sets on space M.

A homomorphism of sheaves (of sets) \( h : F \rightarrow F' \) is an R-continuous mapping such that \( \pi = \pi' \circ h \), i.e.,

\( h(F_x) \subseteq F'_x \) for all \( x \in M \).

(b) Let (\( F, \pi, M \)) and (\( F', \pi', M \)) be two sheaves of groups (resp. A-modules) on M.

A homomorphism of sheaves of groups (resp. A-modules), \( h : F \rightarrow F' \), is an R-continuous mapping such that \( \pi = \pi' \circ h \) and the restriction \( h_x : F_x \rightarrow F'_x \) of h to stalk is a group (resp. A-module) homomorphism for all x.

A homomorphism \ h \ becomes an r-open mapping and a local R-homeomorphism.

Let \( F_i = (F_i, \pi_i, M) \) (\( i = 1, 2 \)) be two sheaves of sets. Suppose that

\( F_1 \oplus F_2 = \{ (f_1, f_2) : f_1 \in F_1, \pi_1(f_1) = \pi_2(f_2) \} \) is the direct product ranked space of \( F_1 \) and \( F_2 \),

and let \( \pi (f_1, f_2) = \pi_1(f_1) = \pi_2(f_2) \in M \). Then \( (F_1 \oplus F_2, \pi, M) \) is a sheaf (of sets) on M.
Definition 8. \( (F_1 \oplus F_2, \mathbb{Z}, M) \) is called the direct sum of \( F_1 \) and \( F_2 \).

Definition 9. A subsheaf \( H \) of a sheaf of sets (resp. groups, \( A \)-modules), \( \mathcal{F} = (F, \pi, M) \), is a sheaf of sets (resp. groups, \( A \)-modules), \( \mathcal{H} = (H, \pi, M) \), such that \( H \subseteq F \) and the identity mapping \( 1 : H \rightarrow F \) is a homomorphism of sheaves.

Let \( E \) be a ranked space, \( R \) an equivalence relation on \( E \), and let \( \hat{E} \equiv E/R \) the quotient set. Let \( \hat{V}(\hat{x}) \) be a neighborhood of \( \hat{x} \in \hat{E} \) and let \( V(x) \) a neighborhood of some \( x \in E \) such that \( \hat{V}(\hat{x}) = V(x)/R \). By letting the rank of \( \hat{V}(\hat{x}) \) equal the rank of \( V(x) \), the quotient set \( \hat{E} \) becomes a ranked space. This \( \hat{E} \equiv E/R \) is called a ranked quotient space.

Definition 10. Let \( \mathcal{H} \) be a subsheaf of \( \mathcal{F} \). Then a ranked quotient space \( \bigcup_{x \in M} F_x/\mathcal{H}_x \) becomes a sheaf on \( M \). This sheaf is called a quotient sheaf and denoted by \( \mathcal{F}/\mathcal{H} \).

If a homomorphism of sheaves (of groups), \( h : \mathcal{F} \rightarrow \mathcal{G} \) is bijective then \( h \) is called an isomorphism of sheaves (of groups) and denoted by \( h : \mathcal{F} \cong \mathcal{G} \). In this case \( h \) is an \( R \)-homeomorphism.

From topological sheaf theory, we get following proposition.

Proposition 4. If \( \varphi : \mathcal{F} \rightarrow \mathcal{G} \) is a homomorphism of sheaves of groups and if \( \varphi \) satisfies the condition (*) in [8; I, p. 51] then \( \mathcal{G}' \equiv \varphi(\mathcal{F}) \) is a subsheaf of \( \mathcal{G} \) and the kernel of \( \varphi \), \( \mathcal{F}' \equiv \varphi^{-1}(0) \), is a subsheaf of \( \mathcal{F} \). Moreover we have \( \mathcal{F}/\mathcal{F}' \cong \mathcal{G}' \).

References
