On Some Structures of the Ranked Spaces

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Synopsis.
In this paper we construct the completion of the ranked spaces. This problem has not been solved yet. The completion is essential and very important notion in the theory of functional analysis. Moreover we give a few important examples of ranked spaces.

1. Introduction.** K. Kunugi ([5]) studied the Baire's category theorem in a ranked space. He showed then that in a topological space which is at the same time a complete ranked space following generalized Baire's theorem holds.

Theorem (K. Kunugi ([5])). If a topological space $R$ is a complete ranked space with indicator $\omega \geq \omega_0$, then, for any well-ordered sequence

$$G_\alpha, G_{1\alpha}, \ldots, G_n, \ldots; 0 \leq \alpha < \omega$$

of topologically open and topologically everywhere dense subsets in $R, \bigcap \alpha G_\alpha$ is also topologically everywhere dense in $R$.

This theorem is a generalization of Baire's theorem which states that every complete metric space or every locally compact regular space is a Baire space.

After that, K. Kunugi ([5]) and H. Okano ([12]) studied a completion of a ranked space on a topological space. Their theory has been applying to extend the Lebesgue integrals or the Denjoy integrals by K. Kunugi ([6]), H. Okano ([13]), S. Nakanishi ([11]) and many mathematicians. Their generalized integrals are called the (E. R.) integrals.

Recently, the notion of the ranked space is being studied as a generalization of the topological space by using new general topological methods ([7]). But, by such standpoint, the general construction of the completion of the ranked spaces has not been given yet.

We shall grasp the notion of the ranked spaces as a generalization of the notion of the metric spaces or the uniform spaces ([18]) or the extended uniform spaces ([2]).

From this standpoint, in 3 and 4, we shall construct the completion of the ranked spaces without assuming the uniformity property for the ranked spaces. And such construction is purely done by the new methods of ranked spaces.

In 5, we shall give a simple example of the ranked space with non-indicator $\omega_0$. In 6, we shall show that every $\Sigma$-space has the structure as a ranked space with indicator $\omega_0$.

2. Preliminaries.
Throughout this paper we assume that the ranked spaces $\{R, \mathcal{E}_R\}$ satisfy the axioms (A) and (B) of

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** Throughout this paper we shall use the same terminology that is introduced in [7] and [14].
1) See p. 84.
F. Hausdorff, the axioms (a) and (b) of K. Kunugi ([7]) and that they have the indicator \( \omega \), \( \omega_0 \leq \alpha < \omega(\omega) \) (\( \omega(\omega) \) is the depth of \( \omega \) and \( \omega \) is an inaccessible ordinal number). New, let us recall some basic concepts in the general ranked spaces.

A monotone decreasing sequence of neighborhoods of points in \( R \):

\[
V_0(\alpha) \supseteq V_1(\beta) \supseteq V_2(\gamma) \supseteq \cdots \supseteq V_\omega(\rho_0) \supseteq \cdots , \quad 0 \leq \alpha < \omega,
\]

is said to be a fundamental sequence, if there is an ordinal number \( \gamma(\alpha) \) such that \( V_\alpha(\rho_0) \in \mathfrak{U}(\omega) \) for all \( \alpha \), \( 0 \leq \alpha < \omega \), and satisfies the following two conditions:

(i) \( \gamma(0) \leq \gamma(1) \leq \gamma(2) \leq \cdots \leq \gamma(\alpha) \leq \cdots \) (\( 0 \leq \gamma(\alpha) < \omega \)), \( \sup \gamma(\alpha) = \omega \).

(ii) For each \( \alpha \), \( 0 \leq \alpha < \omega \), there is a number \( \kappa = \lambda(\alpha) \) such that \( \alpha \leq \lambda < \omega \), \( \rho_1 = \rho_{\lambda - 1} \) and \( \gamma(\lambda) \leq \gamma(\lambda + 1) \) (except the equality).

The ranked space \( R \) is said to be complete, if, for every fundamental sequence \( \{ V_\alpha(\rho_0) ; 0 \leq \alpha < \omega \} \) of neighborhoods, we have \( \bigcap \alpha V_\alpha(\rho_0) = \emptyset \).

Given a sequence \( \{ \rho_0 ; 0 \leq \alpha < \omega \} \) of points of \( R \) and a point \( p \) of \( R \), we say that the sequence \( \{ \rho_0 ; 0 \leq \alpha < \omega \} \) \( r \)-converges or \( \omega \)-converges to the point \( p \), or that \( p \) is an \( r \)-limit or \( \omega \)-limit of \( \{ \rho_0 ; 0 \leq \alpha < \omega \} \), if there exists a fundamental sequence \( \{ V_\alpha(\rho_0) ; 0 \leq \alpha < \omega \} \) consisting of neighborhoods of \( p \) such that \( V_\alpha(\rho_0) \ni p \) for each \( \alpha \). In this case, we write \( \{ \lim_\alpha \rho_0 \} \ni p \). \( \lim_\alpha \rho_0 \) is not a set consisting of one point alone in general.

By \( \text{cl}_r(E) \), we show the set of all \( r \)-limit points of a subset \( E \) of the ranked space \( R \). We say that a set \( E \) is \( r \)-dense in the ranked space \( R \) if \( \text{cl}_r(E) = R \). A set \( E \) \((\subseteq R) \) is called \( r \)-closed if \( \text{cl}_r(E) = E \).

Let \( R \), \( S \) be two ranked spaces with same indicator \( \omega \). We say that a mapping \( f : R \rightarrow S \) is \( r \)-continuous at a point \( p \) in \( R \) if \( \lim_\alpha \rho_0 \ni p \ni \{ \lim_\alpha f(\rho_0) \} \ni f(p) \). A mapping \( f \) is said to be \( r \)-continuous if it is \( r \)-continuous at each point in \( R \).

A mapping \( f : (R, \mathfrak{U}_R) \rightarrow (R', \mathfrak{U}_{R'}) \) is called an \( r \)-homeomorphism if \( f \) is bijective and \( bi-r \)-continuous.

In this case, the spaces \( R \) and \( R' \) are said to be \( r \)-homeomorphic or \( r \)-equivalent.

Let \( A \) be a subset of \( (R, \mathfrak{U}_R) \). Put \( V(p, A) = V(p) \cap A \) for each \( V(p) \in \mathfrak{U}_R(p) \) \((0 \leq \alpha < \omega) \) and \( \forall p \in A \).

By the relation \( \mathfrak{U}_A(A) = \mathfrak{U}_R(A) \cap A \), \( A \) becomes a ranked space with indicator \( \omega \). Then \( A \) is called the ranked space induced from \( R \) or the \( r \)-subspace of \( R \).

Given a sequence \( \{ \rho_0 ; 0 \leq \alpha < \omega \} \) of points of \( (R, \mathfrak{U}_R) \) and a point \( p \) of \( (R, \mathfrak{U}_R) \), we say that the sequence \( \{ \rho_0 ; 0 \leq \alpha < \omega \} \) \( para \)-converges to the point \( p \), or that \( p \) is a para-limit of \( \{ \rho_0 ; 0 \leq \alpha < \omega \} \), if there is a monotone decreasing sequence \( \{ V_\alpha(\rho_0) ; 0 \leq \alpha < \omega \} \) consisting of neighborhoods of \( \rho_0 \) \((0 \leq \alpha < \omega) \) and if \( V_\alpha(\rho_0) \) satisfies the following three conditions:

1. \( V_\alpha(\rho_0) \in \mathfrak{U}(\omega) \) \((\forall \gamma(\alpha) \leq \omega) \).
2. \( \gamma(0) \leq \gamma(1) \leq \gamma(2) \leq \cdots \leq \gamma(\alpha) \leq \cdots \), \( \sup \gamma(\alpha) = \omega \).
3. \( p \in V_\alpha(\rho_0) \) for all \( \alpha \), \( 0 \leq \alpha < \omega \).

In this case, we write \( \{ \text{para-lim} \rho_0 \} \ni p \).

**Proposition.** Every sequence of points \( \{ \rho_0 ; 0 \leq \alpha < \omega \} \) in \( R \) is \( r \) and \( para \)-convergence at a point \( \rho_0 \in R \) if and only if there exist two monotone decreasing sequences of neighborhoods \( \{ U_\alpha(\rho_0) ; 0 \leq \alpha < \omega \} \) and \( \{ V_\alpha(\rho_0) ; 0 \leq \alpha < \omega \} \) such that

1. \( U_\alpha(\rho_0) \in \mathfrak{U}(\omega) \cap \mathfrak{U}(\omega) \) and \( \gamma(\alpha) \alpha \ni \omega \) as \( \alpha \ni \omega \)

and

2. \( U_\alpha(\rho_0) \cap V_\alpha(\rho_0) \ni \{ \rho_0, p \} \) for all \( \alpha \), \( 0 \leq \alpha < \omega \).

3. Completion of Ranked Spaces.

We will consider the problem of completion: construction of a complete ranked space containing...
a given ranked space as an \( r \)-dense subspace.

Let \( (R, \mathfrak{R}_\omega) \) be a ranked space, \( \omega(R) \geq \omega_0 \) the depth of \( R \), \( \omega(\omega_0 \leq \omega_0 \leq \omega(\omega)) \) the indicator of \( R \) and let \( \mathcal{F} \) the set of all fundamental sequences of neighborhoods in \( R \).

In general, we have not always \( \bigcap \{ V_\alpha(p_\alpha) : 0 \leq \alpha < \omega \} \in \mathcal{F} \).

We shall, hereafter, assume the following postulates for \( (R, \mathfrak{R}_\omega) \):

1. \( \mathbf{(T_\alpha^\mathfrak{R})} \) For a point \( p \) and any point \( q \) such that \( p \neq q \), there exist \( (u_\alpha(p) ; 0 \leq \alpha < \omega) \in \mathcal{F} \) and \( \mu, 0 \leq \mu < \omega \), such that \( u_\alpha(p) \not\subseteq q \).

2. \( \mathbf{(C^*)} \) If \( \{P_\alpha(q_\alpha)\}, \{u_\alpha(q_\alpha)\} \in \mathcal{F} \), then, for each index \( \alpha \), \( 0 \leq \alpha < \omega \), there are

\[
\lambda_\alpha \equiv \lambda_\alpha = \lambda_\alpha, V_\alpha(p_\alpha), U_\alpha(q_\alpha) \supseteq \alpha \quad \text{and} \quad \delta_\alpha \equiv \delta_\alpha = \delta_\alpha \quad (0 \leq \lambda_\alpha, \delta_\alpha < \omega)
\]

such that if

\[
U_\alpha(p_\alpha) \cap V_\alpha(q_\alpha) \not\subseteq \phi
\]

then

\[
U_\alpha(p_\alpha) \subseteq V_\alpha(p_\alpha).
\]

3. \( \mathbf{(r)} \) If \( U(p) \supseteq V(q) \), \( U(p) \in \mathfrak{R}_\omega, V(q) \in \mathfrak{R}_\omega \), and \( \gamma' \subseteq \gamma'' \), then for any \( \gamma, \gamma' \subseteq \gamma'' \), there exist a point \( r \in R \) and a neighborhood \( W(r) \in \mathfrak{R}_\omega \) such that \( U(p) \supseteq W(r) \supseteq V(q) \).

4. \( \mathbf{(r') \text{ Under the assumption in (r'), there exist a rank } \gamma \text{ and a neighborhood } W(p) \in \mathfrak{R}_\omega \text{ such that } \gamma' \subseteq \gamma \text{ and } U(p) \supseteq W(p) \supseteq V(q).} \)

These postulates hold always for every metric space. And these postulates are weaker than the assumptions in the Notes \([5]\) and \([12]\).

Then we will prove that the following theorem holds:

**Completion Theorem** (T. Shintani \([15]\)). Let \( R \) be a given ranked space of depth \( \omega(R) \geq \omega_0 \) and let \( \omega, \omega_0 \leq \omega \leq \omega(R) \), be any indicator of the space \( R \). If \( R \) satisfies the postulates \( (T_\alpha^\mathfrak{R}), (C^*), (r) \), then there exists the completion \( \hat{R} \) of \( R \) such that \( \hat{R} \) is a ranked space with indicator \( \omega \) and it is uniquely determined by \( R \). Namely, for a given ranked space \( R \) with indicator \( \omega \), we can always construct the ranked space \( \hat{R} \) with indicator \( \omega \). Moreover \( \hat{R} \) fulfills the following properties:

1. In \( \hat{R} \), for any fundamental sequence \( \{V_\alpha(p_\alpha) ; 0 \leq \alpha < \omega \} \), \( \bigcap \{V_\alpha(p_\alpha) ; 0 \leq \alpha < \omega \} \not\subseteq \phi \) holds always. Namely \( R \) is complete.

2. There exists an \( r \)-homeomorphism of \( R \) onto an \( r \)-subspace \( R' \equiv f(R) \) of \( \hat{R} \).

3. \( \hat{R} = \text{cl}_R(f(R)), i.e., f(R) \) is \( r \)-dense in \( \hat{R} \).

**Corollary 1.** For any two indicators \( \omega_1 \) and \( \omega_\alpha (\mu \leq \omega) \), if \( \hat{R} \) is \( \omega_1 \)-completion of \( R \) then \( \hat{R} \) is \( \omega_\alpha \)-completion of \( R \).

**Corollary 2.** If \( R \) satisfies the postulates \( (T_\alpha^\mathfrak{R}) \) and \( (C^\alpha) \), then, essentially, there exists the completion \( \hat{R} \) of \( R \). Thus, for each fundamental sequence \( \{V_\alpha(p_\alpha) ; 0 \leq \alpha < \omega \} \) in \( \hat{R} \), the sequence \( \{p_\alpha ; 0 \leq \alpha < \omega \} \) of points in \( \hat{R} \) is always para-convergent in \( \hat{R} \).

**Remark.** The completion in the ordinary sense of the metric space \( R \) is a completion of the ranked space \( R \) with indicator \( \omega_0 \) in above sense.

4. **Proof of Completion Theorem.**

**Definition.** We will call two members \( l' = \{V_\alpha'(p_\alpha')\} \) and \( l'' = \{V_\alpha''(p_\alpha'')\} \) of \( \mathcal{F} \) equivalent and write \( l' \equiv l'' \) if for every \( \alpha, 0 \leq \alpha < \omega \), there are \( p_\alpha \in R \) and a member \( \{V_\alpha(p_\alpha)\} \in \mathcal{F} \) which have the following property: for every \( \delta, 0 \leq \delta < \omega \), there exists \( \theta, 0 \leq \theta < \omega \), such that

\[
V_\alpha'(p_\alpha') \cup V_\alpha''(p_\alpha'') \subseteq V_\theta(p_\delta).
\]

Using the axioms \( (b), (r) \) and the transfinite induction, we get the following lemma.
Lemma 1. For each $\Pi = \{ V_\alpha (p_\alpha) \} \in \mathcal{F}$ there are a sequence of points $\{ q_\alpha ; 0 \leq \alpha < \omega \}$ and a fundamental sequence $\Pi^* = \{ U_\gamma (q_\gamma) ; 0 \leq \gamma < \omega \}$ such that

1) $\{ p_\alpha ; 0 \leq \alpha < \omega \}$ is a subsequence of $\{ q_\alpha ; 0 \leq \alpha < \omega \}$,

2) $\Pi$ is a subsequence of $\Pi^*$,

3) $U_\gamma (q_\gamma) \subseteq U_\gamma (V_\gamma) ; 0 \leq \gamma < \omega$ and

4) $\Pi^* \supseteq \Pi$.

Lemma 2. If $\Pi' = \{ V_\nu (p_\nu') \}$ and $\Pi'' = \{ V_\nu (p_\nu'') \}$ are such that for every $\theta , 0 \leq \theta < \omega$, $V_\theta (p_\theta') \cap V_\theta (p_\theta'') \neq \emptyset$, then $\Pi' \sim \Pi''$.

Proof. Since $V_\theta (p_\theta') \cap V_\theta (p_\theta'') \neq \emptyset$ (for each index $\nu (0 \leq \nu < \omega)$ of $\Pi'$, there are two indices $\lambda = \lambda (\nu)$ and $\delta = \delta (\nu, \lambda)$ of the axiom $(\text{C})$. Therefore we get

$V_\theta (p_\theta'') \cap V_\delta (p_\delta') \supseteq V_\theta (p_\theta'') \cap V_\delta (p_\delta') \neq \emptyset$ when $\theta = \theta \max \{ \lambda, \delta \}$.

From $(\text{C})$ we have $V_\nu (p_\nu') (p_\nu'') \subseteq V_\nu (p_\nu')$ and from $\lambda (\nu) \leq \nu$ we have $V_\lambda (p_\lambda) \subseteq V_\nu (p_\nu')$. Hence

$V_\theta (p_\theta'') \cap V_\delta (p_\delta') \subseteq V_\delta (p_\delta') \subseteq V_\lambda (p_\lambda) \subseteq V_\nu (p_\nu')$.

and $\Pi' \sim \Pi''$. (Q.E.D.)

Corollary. For any $\{ V_\alpha (p_\alpha) \}$ and $\{ V_\alpha (p_\alpha) \}$ in $\mathcal{F}$, we have $\{ V_\alpha (p_\alpha) \} \sim \{ V_\alpha (p_\alpha) \}$.

In fact, for every $\theta , 0 \leq \theta < \omega$, $U_\theta (p_\theta) \cap V_\theta (p_\theta) = \emptyset$ holds.

Lemma 3. $\Pi \sim \Pi$.

Proof. The first two properties of the equivalence relation are clear. Suppose that $\Pi' \sim \Pi'' \sim \Pi'''$. There are $\Pi' = \{ V_\alpha (p_\alpha) \}, \Pi'' = \{ V_\alpha (p_\alpha) \}$ such that, by assumption, for every $\alpha , 0 \leq \alpha < \omega$, there are two indices $\beta$ and $\gamma$ (so $\beta < \gamma < \omega$) such that

i) $V_\beta (p_\beta) \cup V_\gamma (p_\gamma) \subseteq V_\gamma (p_\gamma) \subseteq V_\alpha (p_\alpha)$,

ii) $V_\beta (p_\beta) \cup V_\gamma (p_\gamma) \subseteq V_\gamma (p_\gamma) \subseteq V_\alpha (p_\alpha)$.

Hence $\cap_{i=0}^{\text{Max} (\beta, \gamma)} V_\alpha (p_\alpha) \cap V_\beta (p_\beta \cap V_\gamma (p_\gamma) \subseteq V_\alpha (p_\alpha) \cap V_\beta (p_\beta \cap V_\gamma (p_\gamma)$ and $\Pi' \sim \Pi''$ by Lemma 2. Hence, for any $\nu , 0 \leq \nu < \omega$, we have $p_\nu \in R \subseteq \{ V_\nu (p_\nu) \} \subseteq \mathcal{F}$ and $\alpha = \alpha (\nu) (0 \leq \alpha < \omega)$ such that

i) $V_\alpha (p_\alpha \cap V_\beta (p_\beta) \subseteq V_\gamma (p_\gamma)$

and $\Pi' \sim \Pi''$. (Q.E.D.)

Lemma 4. If $\Pi = \{ V_\alpha (p_\alpha) \} \sim \Pi = \{ V_\alpha (p_\alpha) \}$, then there is an index $\beta = \beta (\alpha) , 0 \leq \beta < \omega$, such that

i) $V_\beta (p_\beta) \subseteq \{ V_\alpha (p_\alpha) \} \subseteq \{ V_\alpha (p_\alpha) \}$.

From i) and ii) we have $V_\beta (p_\beta) \subseteq \{ V_\alpha (p_\alpha) \} \subseteq \{ V_\alpha (p_\alpha) \}$ and $\Pi' \sim \Pi''$. (Q.E.D.)

Proof. Since $\Pi \sim \Pi'$, there are $\{ V_\alpha (p_\alpha) \} \subseteq \mathcal{F}$, $\delta = \delta (\alpha)$ and $\beta = \beta (\alpha)$ such that

i) $V_\beta (p_\beta) \subseteq \{ V_\alpha (p_\alpha) \} \subseteq \{ V_\alpha (p_\alpha) \}$.

Now,

ii) $V_\beta (p_\beta) \subseteq \{ V_\alpha (p_\alpha) \} \subseteq \{ V_\alpha (p_\alpha) \}$

Hence, we have $\{ V_\alpha (p_\alpha) \} \subseteq \{ V_\delta (p_\delta) \}$

Hence, by the axiom $(\text{C})$, we get $\{ V_\delta (p_\delta) \} \subseteq \{ V_\alpha (p_\alpha) \}$

Hence, from i), ii) and iv), we have
\[ V_b'(p_a') \subseteq \bigcup_{\alpha<\omega} V_{b^{(\alpha)}}(p_{b^{(\alpha)}}) \subseteq V_a(p_a). \] (Q.E.D.)

**Remark.** Given \( u = \{ U_a(p_a) \}, \ v = \{ V_a(p_a) \} \in \mathcal{F} \), we denote by \( u \prec v \) the relation between \( u \) and \( v \) such that for every \( U_a(p_a) \) there is a \( V_b(p_b) \) contained in \( V_a(p_a) \). If we define \( u \succeq v \) by \( u \prec v \) \& \( u \not\prec v \) \( [11] \), then \( u \sim v \iff u \succeq v \). This shows that our completion concludes all completions by \([5], [12], [6], [13], \) and \([11] \).

By Lemma 3, we can construct the quotient set \( \hat{\mathcal{R}} \equiv \mathcal{R}/\sim \).

Now, we will give a rank to \( \hat{\mathcal{R}} \) by the following method.

Let \( \forall P \in \mathcal{R} \). Then \( \exists P \exists \{ V_a(p_a) \} \) such that \( V_a(p_a) \in \mathcal{R} \).

By the relations
\[ \hat{V}(P) \equiv \{ Q \in \hat{\mathcal{R}} \mid \exists \exists \{ U_a(x_a) \} \ ; \ \exists \theta' (0 \leq \theta' < \omega) \text{ such that } U_0'(x_0') \subseteq V_0(p_0') \}
\]
and
the rank of \( \hat{V}(P) \equiv \text{the rank of } V_0(p_0) \),

we define a neighborhood \( \hat{V}(P) \) with rank of \( P \) in \( \hat{\mathcal{R}} \). And we denote by \( \hat{\mathcal{S}}_a(P) \) the set of all neighborhoods with rank \( a \) of the point \( P \). Every subset which contains a neighborhood with rank of \( P \) is called a neighborhood of \( P \) in \( \hat{\mathcal{R}} \).

**Lemma 5.** \( \hat{\mathcal{R}} \) is a ranked space, of depth \( \omega(R) \geq \omega \), which satisfies the axioms \((A), (B), (a) \) and \((b) \).

**Proof.** Axiom \((a) \): For any neighborhood \( \hat{V}(P) \) of \( P \) in \( \hat{\mathcal{R}} \) there are a neighborhood \( \hat{V}'(P) \) of \( P \) and an ordinal number \( \gamma \), \( 0 \leq \gamma < \omega \), such that \( \hat{V}(P) \supseteq \hat{V}'(P) \in \hat{\mathcal{S}}_\gamma \).

If \( a \leq \gamma \) then let \( \beta = \gamma \). If \( a > \gamma \) then by the axiom \((r) \) we have
\[ V_\gamma'(p_\gamma) \supseteq \exists \exists V_\alpha'(p_\alpha) \supseteq V_\alpha'(p_\alpha) \ (a < \beta < \omega). \]
From
\[ \hat{U}(P) \equiv \{ Q \in \hat{\mathcal{R}} \mid \exists \theta' \ ; \ U_\theta(x_\theta) \subseteq V_\theta'(p_\theta) \}, \]
we get the followings:
\( (a) \).
\( \forall \hat{V}(P), \forall \alpha < \beta < \omega \) \& \( \hat{V}(P) \subseteq \hat{U}(P) \in \hat{\mathcal{S}}_\beta \).

( \( A \).) By the definition of neighborhoods in \( \hat{\mathcal{R}} \), we get \( \hat{V}(P) \subseteq \hat{\mathcal{P}} \).

(B). If \( \hat{V}(P) \) \( (i=1,2) \) are any neighborhoods of \( P \), then \( \hat{V}(P) \supseteq \exists \hat{U}(P) \in \hat{\mathcal{S}}_\beta \ (i=1,2) \).

Now, let \( \hat{U}(P) \equiv \{ Q \in \hat{\mathcal{R}} \mid \exists \theta' \ ; \ U_\theta'(x_\theta') \subseteq V_\theta'(p_\theta') \} \ (i=1,2) \), then
\[ \{ V_\alpha'(p_\alpha)(i=1) \} \supseteq \{ V_\beta'(p_\beta)(i=2) \}. \]
Hence there is an index \( \theta \), \( 0 \leq \theta < \omega \), such that \( V_\alpha'(p_\alpha)(i=1) \supseteq V_\theta'(p_\theta)(i=2) \).

Thus we have
\[ V_\alpha'(p_\alpha)(i=1) \cap V_\beta'(p_\beta)(i=2) \supseteq V_\theta'(p_\theta)(i=2) \] when \( \theta = \text{Max} \{ \theta_1, \theta_2 \} \).

Therefore if we put
\[ \hat{W}(P) \equiv \{ Q \in \hat{\mathcal{R}} \mid \exists \exists \{ U_a(p_a) \} \ ; \ U_a(p_a) \subseteq V_\alpha'(p_\alpha) \} \]
then we have
\[ \hat{U}(P) \cap \hat{W}(P) \supseteq \hat{W}(P) \in \hat{\mathcal{S}}_\beta. \] (Q.E.D.)

**Notation.** Let \( \hat{\mathcal{F}} \) be the set of all fundamental sequences of neighborhoods in \( \hat{\mathcal{R}} \).

**Lemma 6.** For every \( \{ \hat{V}_a(p_a) \ ; \ 0 \leq a < \omega \} \in \hat{\mathcal{F}} \) in \( [\hat{R}, \hat{\mathcal{S}}_\alpha] \), we have always \( \bigcap \alpha \hat{V}_a(p_a) = \hat{\phi} \).

Hence \( \hat{\mathcal{R}} \) is complete.

**Proof.** Let \( \hat{V}_a(p_a) \equiv \{ Q \in \hat{\mathcal{R}} \mid U_a'(p_a) \subseteq V_{b^{(\alpha)}}(p_{b^{(\alpha)}}) \} \in \hat{\mathcal{S}}_{(\alpha+1)} \) and \( \forall \forall \{ U_{a+1}(p_{a+1}) \} \in \hat{\mathcal{S}}_{(\alpha+1)} \).

From \( (C^\phi) \), \( \exists \exists \{ u_y, (y) \} \) \& \( \in \hat{V}_{a+1}(p_{a+1}) \) holds. On the other hand, since \( \hat{V}_a(p_a) \supseteq \hat{V}_{a+1}(p_{a+1}) \), there are \( \{ v_y(y) \} \in \hat{Q} \) \& \( 0 \leq \theta' < \omega \), and then \( V_\alpha'(p_\alpha)(i=1) \supseteq V_\beta'(p_\beta)(i=2) \) holds.

Since \( \{ \hat{u}_y(y) \} \sim \{ v_y(y) \} \), there exists an index \( \theta' \) such that \( v_\theta'(y_\theta') \supseteq u_\theta'(y_\theta') \).

Hence we have
\[ V_0^{(\alpha)}(p_\alpha) \supseteq V_0^{(\alpha+1)}(p_{\alpha+1}) \quad (\forall \alpha, 0 \leq \alpha < \omega). \]

Now, by the axiom \((\rho')\) and the transfinite induction, there exists a fundamental sequence such that \(\{V_0^{(\alpha)}(p_\alpha) \mid 0 \leq \alpha < \omega\}\) is its subsequence.

Let \(P\) be the class which concludes such fundamental sequence, then we have \(\hat{V}_\alpha(P_\alpha) \ni P\) for every \(\alpha, 0 \leq \alpha < \omega\). Therefore we have always \(\bigcap_{\alpha \in \omega} \hat{V}_\alpha(P_\alpha) \ni \phi\). \((Q.E.D.)\)

Lemma 7. The mapping \(f : R \ni f(p) \mapsto P \ni \{V_\alpha(p)\}\) of \(R, \mathbb{S}_\alpha\) onto a ranked space \(\{f(\hat{R}), \hat{\mathbb{S}}_\alpha \cap f(R)\}\) induced from \(\{\hat{R}, \mathbb{S}_\alpha\}\) is one-to-one and bi-\(r\)-continuous.

Proof. (1). \(f\) is one-to-one. In fact, let \(x, y\) be two distinct point of \(R\) and let \(\{u_a(x), \{v_a(y)\}\}\) any two fundamental sequence in \(R\). If \(\{u_a(x)\} \sim \{v_a(y)\}\), then by \((T_0)\) there are \(\{U_a(x)\} \ni f(x)\) and \(p, 0 \leq p < \omega\), such that \(U_a(x) \ni P_y\) and \(y\). From \(\{U_a(x)\} \sim \{v_a(y)\}\), we get \(U_a(x) \sim \{v_a(y)\}\). Hence there is an index \(z\), \(0 \leq z < \omega\), such that \(U_a(x) \ni v_z(y)\). This is a contradiction.

Thus if \(x \sim y\), then, for every \(\{u_a(x)\}\) and \(\{v_a(y)\}\), we have \(\{u_a(x)\} \sim \{v_a(y)\}\). Hence, for two points \(X\) and \(Y\) in \(\hat{R}\) such that \(X \ni Y\{u_a(x)\}\) and \(Y \ni Y\{v_a(y)\}\), we have \(X = f(x) \ni f(y) = Y\). Thus \(f\) is one-to-one.

(2). For every \(V(p) \ni \mathbb{S}_\alpha(p)\) we have \(f(V(p)) = \hat{V}(P, f(R)) \ni \mathbb{S}_\alpha(P, f(R)) \ni f(P)\). Thus \(f\) is \(r\)-continuous.

In fact, from the axiom \((a)\) there exists \(\{V_\alpha(p)\}\) \(\ni f\) such that \(V(p)\) is its member.

Let \(P\) be the equivalence class which concludes \(\{V_\alpha(p)\}\), then \(f(P) = P\). Now, let \(\hat{V}(P) = \{q \in \hat{R} \mid q \ni \{u_\alpha(q)\}, \forall \alpha, U_\alpha(q) \subseteq \mathbb{V}(p)\}\).

Then, from \((C)\), there is an index \(q, 0 \leq q < \omega\), such that \(u_\alpha(q) \ni p\). Therefore we get \(X \ni \hat{V}(P)\). On the other hand, as \(x = X \ni f(R)\) we get \(X \ni f(R) \ni \hat{V}(P)\).

Thus, \(V(p) = \hat{V}(P) \ni f(R) \ni \hat{V}(P, f(R)), P = f(P)\).

Now, let \(\forall Q \ni \hat{V}(P) \ni f(R)\), then \(Q \ni f(R)\). Hence, \(Q \ni \{u_\alpha(q)\}\). Moreover, from \(Q \ni \hat{V}(P)\), there exists an index \(q', 0 \leq q' < \omega\), such that \(g \ni v_{q'}(q) \subseteq \mathbb{V}(p)\). Thus, it follows \(V(p) = \hat{V}(P, f(R))\).

Hence if \(\{\lim \alpha p_\alpha\} \ni p, \alpha, \{V_\alpha(p)\} \ni \mathbb{V}(P)\) such that \(V_\alpha(p) \ni \forall \alpha, 0 \leq \alpha < \omega\), then we have \(f(V_\alpha(p)) = \hat{V}(P, f(R)) = f(V(p)) \ni f(R)\). Thus \(f(V_\alpha(p)) \ni f(R)\).\((Q.E.D.)\)

Lemma 8. The mapping \(f : R \ni f(p) \mapsto P \ni \{V_\alpha(p)\}\) of \(R, \mathbb{S}_\alpha\) onto a ranked space \(\{f(\hat{R}), \hat{\mathbb{S}}_\alpha \cap f(R)\}\) induced from \(\{\hat{R}, \mathbb{S}_\alpha\}\) is one-to-one and bi-\(r\)-continuous.

Proof. (1). \(f\) is one-to-one. In fact, let \(x, y\) be two distinct point of \(R\) and let \(\{u_a(x), \{v_a(y)\}\}\) any two fundamental sequence in \(R\). If \(\{u_a(x)\} \sim \{v_a(y)\}\), then by \((T_0)\) there are \(\{U_a(x)\} \ni f(x)\) and \(p, 0 \leq p < \omega\), such that \(U_a(x) \ni P_y\) and \(y\). From \(\{U_a(x)\} \sim \{v_a(y)\}\), we get \(U_a(x) \sim \{v_a(y)\}\). Hence there is an index \(z\), \(0 \leq z < \omega\), such that \(U_a(x) \ni v_z(y)\). This is a contradiction.

Thus if \(x \sim y\), then, for every \(\{u_a(x)\}\) and \(\{v_a(y)\}\), we have \(\{u_a(x)\} \sim \{v_a(y)\}\). Hence, for two points \(X\) and \(Y\) in \(\hat{R}\) such that \(X \ni Y\{u_a(x)\}\) and \(Y \ni Y\{v_a(y)\}\), we have \(X = f(x) \ni f(y) = Y\). Thus \(f\) is one-to-one.

(2). For every \(V(p) \ni \mathbb{S}_\alpha(p)\) we have \(f(V(p)) = \hat{V}(P, f(R)) \ni \mathbb{S}_\alpha(P, f(R)) \ni f(p)\). Thus \(f\) is \(r\)-continuous.

In fact, from the axiom \((a)\) there exists \(\{V_\alpha(p)\}\) \(\ni f\) such that \(V(p)\) is its member.

Let \(P\) be the equivalence class which concludes \(\{V_\alpha(p)\}\), then \(f(P) = P\). Now, let \(\hat{V}(P) = \{q \in \hat{R} \mid q \ni \{u_\alpha(q)\}, \forall \alpha, U_\alpha(q) \subseteq \mathbb{V}(p)\}\).

Then, from \((C)\), there is an index \(q, 0 \leq q < \omega\), such that \(u_\alpha(q) \ni p\). Therefore we get \(X \ni \hat{V}(P)\). On the other hand, as \(x = X \ni f(R)\) we get \(X \ni f(R) \ni \hat{V}(P)\).

Thus, \(V(p) = \hat{V}(P) \ni f(R) \ni \hat{V}(P, f(R)), P = f(P)\).

Now, let \(\forall Q \ni \hat{V}(P) \ni f(R)\), then \(Q \ni f(R)\). Hence, \(Q \ni \{u_\alpha(q)\}\). Moreover, from \(Q \ni \hat{V}(P)\), there exists an index \(q', 0 \leq q' < \omega\), such that \(g \ni v_{q'}(q) \subseteq \mathbb{V}(p)\). Thus, it follows \(V(p) = \hat{V}(P, f(R))\).

Hence if \(\{\lim \alpha p_\alpha\} \ni p, \alpha, \{V_\alpha(p)\} \ni \mathbb{V}(P)\) such that \(V_\alpha(p) \ni \forall \alpha, 0 \leq \alpha < \omega\), then we have \(f(V_\alpha(p)) = \hat{V}(P, f(R)) = f(V(p)) \ni f(R)\). Thus \(f(V_\alpha(p)) \ni f(R)\).\((Q.E.D.)\)

Thus \(f\) is an \(r\)-continuous mapping of \(R\) onto \(f(R)\).

(3). \(f^{-1}\) is \(r\)-continuous.

Let \(\{\hat{V}_\alpha(P, f(R))\}\) be a fundamental sequence in \(f(R)\) such that \(\hat{V}_\alpha(P, f(R)) \ni \forall \alpha\). In fact, \(\{\lim \alpha p_\alpha\} \ni f\)

In fact, \(f^{-1}\) is \(r\)-continuous.

Let \(\{\hat{V}_\alpha(P, f(R))\}\) be a fundamental sequence in \(f(R)\) such that \(\hat{V}_\alpha(P, f(R)) \ni \forall \alpha\). In fact, \(\{\lim \alpha p_\alpha\} \ni f\)

Thus, \(f^{-1}\) is \(r\)-continuous.
Lemma 8. $cl_r(f(R)) = \tilde{R}$, i.e., $f(R)$ is $r$-dense in $\tilde{R}$.

Proof. $cl_r(f(R)) \subseteq \tilde{R}$ is clear. We will show that $cl_r(f(R)) \supseteq \tilde{R}$. Let $\forall P \in \tilde{R}$, $\forall V(P) \in \tilde{R}$ and let $\tilde{V}(P) = \{ Q \in \tilde{R} \mid Q \supseteq U_{\theta}(q), Q \supseteq \bar{U}_{\theta}(q) \subseteq V_\theta(p_\theta) \}$. Thus there is a point $q \in R$ such that $q \in U_{\theta}(p_\theta) \subseteq V_\theta(p_\theta)$.

As the indicator of $R$ is $\omega$, for an index $\theta$ of $\{ V_\theta(p_\theta) \}$, $0 \leq \nu < \omega$ and for any $\theta$, $0 \leq \theta < \omega$, we have $V_{\theta}(p_\theta) \supseteq V_\theta(p_\theta) \supseteq V_\theta(p_\theta)$. By $(C^*)$, for some $Q \in f(R)$ and for every $U_{\theta}(q) \in \tilde{Q}$ there exist two indices $\lambda(\theta) = \lambda(q, V_\theta(p_\theta)) \geq 0$ and $\delta(\theta) = \delta(q, \lambda)$. Hence we have $U_{\theta}(\delta(q, \lambda) \subseteq V_\theta(p_\theta)$. Hence we have $Q = f(q) \subseteq \tilde{V}(P)$. Thus, for every $V_\theta(p_\theta) \subseteq \tilde{V}$ there are points $Q_\alpha \in \tilde{R}$ and $q_\alpha \in R$ such that $Q_\alpha = f(q) \subseteq \tilde{V}_\theta(P \subseteq V_\theta(p_\theta))$. Therefore we have $\{ \lim Q_\alpha \} \subseteq \tilde{R}$, i.e., $P \subseteq cl_r(f(R))$. Thus we have $\tilde{R} \subseteq cl_r(f(R))$. Hence $\tilde{R} = cl_r(f(R))$. (Q.E.D.)

Every neighborhood with rank of $P$ in $\tilde{R}$ had been defined by an element $\tilde{N} = \{ V_\theta^{(1)}(p_\theta) \}$ of $P$. Now, let $\tilde{R}$ be a ranked space defined by another element $\tilde{N} = \{ V_\theta^{(2)}(p_\theta) \}$ of $P$. Then two spaces $\tilde{R}$ and $\tilde{R}$ are $r$-equivalent. Namely, we have the following lemma:

Lemma 9. (Uniqueness of complete extension).

$\tilde{R} \in cl_r(f(R)) = cl_r(f(R)) = \tilde{R}$.

Proof.

Let

$\begin{align*}
\forall Q \in cl_r(f(R)), & \exists (\tilde{V}_\alpha(P)) \in \tilde{R}, \exists (\tilde{U}_\alpha(P)) \in \tilde{R}.
\tilde{V}_\alpha(P) \supseteq Q_\alpha \supseteq \bar{U}_\alpha(P) \supseteq V_\alpha(p_\alpha) \supseteq V_\alpha(p_\alpha) \subseteq V_\alpha(p_\alpha).
\tilde{U}_\alpha(P) \supseteq Q_\alpha \supseteq \bar{U}_\alpha(P) \supseteq V_\alpha(p_\alpha) \supseteq V_\alpha(p_\alpha) \subseteq V_\alpha(p_\alpha).
\end{align*}$

As $P \subseteq cl_r(f(R))$, there is a sequence $\{ Q_\alpha \}$ such that $Q_\alpha \subseteq V_\alpha(p_\alpha) \subseteq V_\alpha(p_\alpha) \subseteq V_\alpha(p_\alpha)$. From $\tilde{N} \subseteq \tilde{N}$ and Lemma 4, for a given index $\alpha$, $0 \leq \alpha < \omega$, there exists an index $\alpha''$, $0 \leq \alpha'' < \omega$, such that $V_\alpha(p_\alpha) \subseteq V_\alpha(p_\alpha) \subseteq V_\alpha(p_\alpha)$. Now, let $\tau_\alpha$ be the minimum of such $\alpha''$. Then we have $\tilde{V}_\alpha(P) \subseteq \tilde{U}_\alpha(P)$ for every $\alpha$, $0 \leq \alpha < \omega$.

Hence we get

$\tilde{U}_\alpha(P) \subseteq \tilde{V}_\alpha(P)$ \quad (\forall \alpha, 0 \leq \alpha < \omega).

Set $Q_\alpha^{(2)} = Q_\alpha^{(1)}(\forall \alpha, 0 \leq \alpha < \omega)$. Then we have $f(R) \exists Q_\alpha^{(2)} \in \tilde{V}_\alpha(P)$ and therefore $\{ \lim Q_\alpha^{(2)} \} \exists P$ in $f(R)$, i.e., $P \subseteq cl_r(f(R))$.

Thus we have

$\tilde{r}_r(f(R)) \subseteq \tilde{r}_r(f(R))$.

By the similar way we get

$\tilde{r}_r(f(R)) \subseteq \tilde{r}_r(f(R))$.

Hence we have

$\tilde{R} = \tilde{r}_r(f(R)) = \tilde{r}_r(f(R)) = \tilde{R}$.

Thus we get the uniqueness of completion. (Q.E.D.)

And we complete the proof of the Completion Theorem.

5. An example of the ranked space with non-indicator $\omega_0$.

Such example has not been given yet. We will give a simple of the ranked space with indicator $\Omega$ ($\Omega$ is the ordinal of the second number class). Thus we can construct the ranked spaces with any indicator $\omega_0$.

Proposition. $\{ \xi : 0 \leq \xi < \Omega \}$ is a ranked space with indicator $\Omega$.

3) The set $R$ is compact in the sense of M. Fréchet and also non-compact. (cf. K. Kunugi: Kaiseki-gaku Yôron, p. 223, (1933), Japan.)
Proof. We define a neighborhood $V_{a'}(a)$ with rank $a' \ (0 \leq a' < \Omega)$ of a point $a$ in $R$ by the following relation:

$$V_{a'}(a) := \{ \xi \mid a' \leq \xi < \Omega \} \ if \ a' \leq a < \Omega,$$

$$= \{ a \} \cup \{ \xi \mid a' \leq \xi < \Omega \} \ if \ a < a' < \Omega.$$  

Then $R$ is a ranked space with indicator $\Omega$ and it satisfies the axioms (A), (B), (a) and (b). This is shown as follows:

Case. $a' \leq a < \Omega$; (A) and (b) are clear. (B) For every $a'$ and $a''$ such that $a', a'' \leq a < \Omega$ we have

$$V_{a'}(a) \cap V_{a''}(a) = V_{max(a', a'')} (a).$$

(a) \ ; \ \forall a'', \ a' \leq a'' \leq a < \Omega \Rightarrow V_{a'}(a) \supseteq V_{a''}(a).$$  

Case. $a < a' < \Omega$; (A) and (b) are clear. (B) For every $a'$ and $a''$ such that $a < a', a'' \leq a < \Omega$ we have

$$V_{a'}(a) \cap V_{a''}(a) \supseteq V_{max(a', a'')} (a).$$

(a) \ ; \ \forall a'', \ a < a' \leq a'' \leq a < \Omega \Rightarrow V_{a'}(a) \supseteq V_{a''}(a).$$

Case. $a' \leq a < a'' < \Omega$; (A) and (b) are clear. (B) $V_{a'}(a) \cap V_{a''}(a) = V_{a''}(a).$

(a) \ ; \ \forall \beta, \ a' \leq \beta \leq a'' \leq a < \Omega \Rightarrow V_{a'}(a) \supseteq V_{\beta}(a). \ (Q.E.D.)$

6. On $\Sigma$-spaces.

K. Nagami ([9]) introduced the notion of $\Sigma$-spaces as a generalization of $M$-spaces by K. Morita ([8]).

A perfectly normal space and a countably compact space are examples of $P$-spaces ([8]).

Every $M$-space or $M'$-space or $\omega$-space is a $\Sigma$-space ([9]). Every $\Sigma$-space is a $P$-space ([9]).

Definition. Let $\mathcal{F}$ be a covering of a Hausdorff space $E$ and $x$ a point of $E$.

Then we set

$$C(x, \mathcal{F}) = \bigcap \{ F \mid x \in F \in \mathcal{F} \}.$$

A $\Sigma$-net of a space $E$ is a sequence $\{ \mathcal{F}_n \}_{n \in \mathcal{N}}$ of locally finite closed coverings satisfying the following condition:

If $K_1 \supseteq K_2 \supseteq \cdots$ is a sequence of non-empty closed sets of $E$ such that $K_n \subseteq C(x, \mathcal{F}_n)$ for some point $x$ in $E$ and for each $n \in \mathcal{N}$, then $\bigcap \mathcal{K}_n \neq \emptyset$. A space $E$ is called a $\Sigma$-space if it has a $\Sigma$-net.

Now we have the following statement:

Proposition. Every $\Sigma$-space $E$ has a structure as a ranked space with indicator $\omega_0$.

In fact, for $\forall \beta \in E$ we set

$$v(n ; \beta) = K_n \cup \{ \beta \} \ (n \in \mathcal{N}), \ \mathcal{V}_n (\beta) = \{ \text{the set of every } u(n ; \beta) \} \ \text{and} \ \mathcal{V}_0 (\beta) = \{ E \}.$$  

Then $\{ E, \mathcal{V}_n \}$ is a ranked space with indicator $\omega_0$. (Q.E.D.)

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References


