On Generalized Continuous Groups III

By

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Synopsis.

In this paper we will study an open problem of the closed-graph theorem offered by Alexander Grothendieck. We will show that the Linear Ranked Space satisfies his conjecture. A purpose of such study is to develop more general and unified functional analysis on wide spaces without metric or norms, We shall be able to develop such analysis on a Linear Ranked Space as a generalization of Banach space theory. For instance, we can extend the most part of results in ([1]), ([2]), or ([20]) by such methods of ranked spaces. Moreover every space considered for the conjecture has a structure as a linear ranked space.

§ 1. Introduction.*

Historically, the Linear Ranked Space has been introduced independently by H. Okano ([7]), M. Washi-hara ([16]), M. Yamaguchi ([19]), and T. Shintani ([12]). The author's studies ([12],[13]) are most general. These studies are applying to the theories of generalized integrals, of partial differential equations as an extension of L. Hörmander's theory, of nuclear spaces as an extension of I. M. Gel'fand's theory, and of the others by many mathematicians ([6],[18],[9], etc.).

Now, let $E$ and $F$ be two topological vector spaces. Let us consider a linear mapping $f$ from a departure space $E$ into an arrival space $F$. Then if the graph of $f$ is closed then $f$ is continuous under convenient assumption on $E$ and $F$.

Conjecture (A. Grothendieck ([3])). Let $E$ be a Fréchet space. Then is there a class $\mathcal{F}$ of arrival topological vector spaces with the following good properties?

(i) Every space $F \in \mathcal{F}$ is wider than every Fréchet space.
(ii) The class $\mathcal{F}$ is closed by the following operations:

1. The image of $F \in \mathcal{F}$ by a continuous linear mapping belongs to $\mathcal{F}$.
2. The closed subspace of $F \in \mathcal{F}$ belongs to $\mathcal{F}$.
3. The product space $F=\prod_{n=1}^{\infty} F_n$ of spaces $\{F_n\} \subset \mathcal{F}$ belongs to $\mathcal{F}$.
4. The inductive limit of spaces $\{F_n\} \subset \mathcal{F}$ belongs to $\mathcal{F}$.
(iii) For $E$ and $F \in \mathcal{F}$ the closed-graph theorem holds.

These properties are fundamental and essential in the theory of Hilbert spaces or Banach spaces.

Recently, this conjecture is studying on wider spaces $E$ and $F$ by A. Martinet ([8]), L. Schwartz ([11]), W. Słowiskowski ([14]), M. De Wilde–H. G. Garnir ([13]), M. Nakamura ([10]) and many mathematicians.

§ 2. Notations and Some properties of Linear Ranked Spaces.***

For each point $x$ of a ranked space $\mathcal{B}_a(x)$ denotes the system of all neighborhoods of $x$ with rank $a$, $F(x)$

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** Throughout this paper we shall use the same terminology that is introduced in [5] and [12].
*** 1972年8月26日、1972年開位空間論シンポジウムにて一部講演.
the set of all fundamental sequences of \( x \).

**Definition 1.** ([5], [12]). Let \( E \) and \( F \) be two ranked spaces with same indicator \( w \geq w_0 \).

1. A mapping \( f : E \rightarrow F \) is said to be \( r \)-**continuous** at a point \( x \) in \( E \) if
   \[
   \lim_{x_n \to x} f(x_n) \exists x \in E \iff \lim_{x_n \to x} f(x) \exists f(x) \in F.
   \]

2. A mapping \( f : E \rightarrow F \) is said to be \( R \)-**continuous** at a point \( x \) in \( E \) if for any fundamental sequence \( \{V_n(x)\} \) of \( x \) in \( E \) there is a fundamental sequence \( \{U_n(f(x))\} \) of \( f(x) \) in \( F \) such that
   \[
   f(V_n(x)) \subseteq U_n(f(x)) \quad \text{for all } \alpha, 0 \leq \alpha < w.
   \]

Every \( R \)-continuous mapping is \( r \)-continuous, but the converse is not always true.

**Definition 2.** ([12], [16]). A linear space \( E \) over the real or complex field \( \Phi \) is called a \textit{linear ranked space} over \( \Phi \) if this set \( E \) is a ranked space with indicator \( w_0 \) and if the following conditions are fulfilled:

1. The mapping \( (x, y) \mapsto x+y \) of \( E \times E \) into \( E \) is \( R \)-continuous.
2. For any \( (\{u_n(x)\} \in F(x) \) and any sequence \( \{\lambda_n\} \subseteq \Phi \) with limit \( \lambda_n = \lambda \), there is a \( \{V_n(\lambda x)\} \subseteq F(\lambda x) \) such that
   \[
   \lambda_n \cdot u_n(x) \subseteq V_n(\lambda x) \quad \text{for all } n, 0 \leq n < w_0.
   \]

Hereafter we assume that every linear ranked space \( \{E, \mathcal{R}_E\} \) is homogeneous, namely, every neighborhood of \( x \) in \( E \) is written in the form of \( \forall \mathbb{U} \exists \mathbb{U}_n = \{x, y \} \quad (\forall n, 0 \leq n < w_0) \) holds for each point \( x \) in \( E \).

**Definition 3.** ([12]). (i) A subset \( H \) of \( E \) becomes a ranked space by the induced rank from \( E \), i.e., by the relations \( \mathcal{R}_H(x, y) \Rightarrow \mathcal{R}_E(x, y) \cap H \) and
   \[
   \{U_n(x)\} \in F(x) ; H \iff U_n(x) \cap H \quad (\forall n, 0 \leq n < w_0).
   \]

Such \( H \) is called a \textit{subspace} of \( \text{RS}^E \).

(ii) A subset \( M \) of \( E \) is called a \textit{subspace} (resp. a \textit{closed subspace}) of \( E \) if \( M \) is a vector subspace (resp. an \( r \)-closed set and also a vector subspace) of \( E \).

**Proposition 1.** Let \( E \) be a linear ranked space and \( H \) an its vector subspace. Then we have the following:

1. \( H \) is a subspace of \( E \).
2. \( \text{cl}_r(\mathcal{H}) \) is a subspace of \( H \). Hence a closed subspace of \( E \) is a linear ranked space.

**Proof.** (1). \( H \subseteq E \) is a ranked space by the induced rank from \( E \). For any \( \{U_n(x)\}, \{V_n(y)\} \subseteq F(H) \), we have
   \[
   U_n(x) + V_n(y) = u_n(x) \cap H + v_n(y) \cap H \quad \{u_n(x)\}, \{v_n(y)\} \subseteq F \}
   \]

Moreover for any \( \{\lambda_n\} \subseteq \Phi \) and any \( \{U_n(x)\} \subseteq F(H) \) there exists \( \{W_n(\lambda x)\} \subseteq F(H) \) such that
   \[
   \lambda_n \cdot u_n(x) \subseteq \lambda_n \cdot (u_n(x) \cap H) \subseteq \lambda_n \cdot u_n(x) \cap H \subseteq \lambda_n \cdot \mathbb{U}_n(x) \cap H \Rightarrow V_n(\lambda x). \quad \text{Hence } H \subseteq E \text{ is a linear ranked space.}
   \]

2. \( \text{cl}_r(\mathcal{H}) \) is a ranked space by the induced rank from \( E \). Moreover we have the following facts:
   \[
   \forall x, y \in \text{cl}_r(\mathcal{H}) \Rightarrow x + y \in \text{cl}_r(\mathcal{H}).
   \]

Thus \( \text{cl}_r(\mathcal{H}) \) is a vector subspace of \( E \). Hence \( \text{cl}_r(\mathcal{H}) \) is a linear ranked space. (Q.E.D.)

**Proposition 2.** Let \( E \) and \( F \) be two linear ranked spaces over \( \Phi \) and let a mapping

1. See [12].
2. RS=Ranked Space.
3. LRS=Linear Ranked Space.
4. \( M(\subseteq E) \) is called an \( r \)-closed set in \( E \) if \( \text{cl}_r(M) = M \).
Let $f : E \rightarrow F$ be $r$-continuous. Then the image $f(E)$ of $E$ is a linear ranked space.

Because $f(E)$ is a ranked space by the induced rank from $F$ and also $f(E)$ is a vector space, hence $f(E)$ is a linear ranked space by Proposition 1. (1).

**Proposition 3.** Let $\{H_\alpha, \Psi_\alpha\}$ ($\alpha \in \Lambda$) be a system of (vector) subspaces of $E$. Then we have the followings:

1. $\prod_{\alpha \in \Lambda} H_\alpha$ is a subspace of $E$.

2. If $H = \bigcup_{\alpha \in \Lambda} H_\alpha$ is a vector subspace of $E$ then $H$ is a linear ranked space.

In fact, $H$ becomes a linear ranked space by the induced rank of $E$ or by the relation $\Psi_n \equiv \prod_{\alpha \in \Lambda} \Psi_\alpha^{(n)}$, $n \in N$. $H$ is a linear ranked space.

**Definition 4.** ([12]) Let $\{E_\alpha, \Psi_\alpha^{(n)}(x_\alpha)\}$ ($\alpha \in A$, $0 \leq \alpha < \omega$) be a system of ranked spaces with same indicator $\omega \geq \omega_0$. Then the direct product set $E = \prod_{\alpha \in A} E_\alpha$ is a ranked space by the following relations:

$$\Psi_n(x) \equiv \{\prod_{\alpha \in A} \Psi_\alpha^{(n)}(x) \mid \Psi_\alpha^{(n)}(x_\alpha) \in \Psi_\alpha^{(n)}(x_\alpha) \text{ and } \sup_{\alpha \in A}(x_\alpha) = \alpha\} (x = (x_\alpha) \in E, 0 \leq \alpha < \omega)$$

and

$$\{V_n(x) \in F(x ; E) \equiv \prod_{\alpha \in A} V_\alpha(x) \equiv \prod_{\alpha \in A} \Psi_\alpha^{(n)}(x_\alpha) \text{ and } \{V_\alpha^{(n)}(x_\alpha) \in F(x_\alpha ; E) \} (\alpha \in A)\}.$$  

Such space $E$ is called the direct product ranked space of ranked spaces $\{E_\alpha\}$ ($\alpha \in A$).

**Proposition 4.** The direct product space $E = \prod_{n=1}^{\infty} E_n$ of linear ranked spaces $\{E_n\}$ ($n \in N$) over $\Phi$ is a linear ranked space over $\Phi$.

**Proof.** Since every $E_n$ is a vector space, $\prod_{n=1}^{\infty} E_n$ is a vector space. And $E$ is also a ranked space by the product rank,

1. For $\forall \{u_m^{(n)}(0)\}_m, \forall \{v_m^{(n)}(0)\}_m \in F(0 ; E_n)$ there exists $\{u_m^{(n)}(0)\}_m \in F(0 ; E_n)$ such that $u_m^{(n)}(0) + v_m^{(n)}(0) \subseteq w_m^{(n)}(0)$ for $V_m, V_n \in N$.

Hence we have

$$\prod_{n=1}^{\infty} \{u_m^{(n)}(0) + v_m^{(n)}(0)\} \subseteq \prod_{n=1}^{\infty} w_m^{(n)}(0) (\forall m \in N).$$

Thus the addition in $E$ is $R$-continuous.

2. For $\forall \{\lambda_m^{(n)}(0)\}_m \in \Phi(\lambda_m^{(n)}(0), 0)$ and $\forall \{u_m^{(n)}(0)\}_m \in F(0 ; E_n)$ there exists $\{v_m^{(n)}(0)\}_m \in F(0 ; E_n)$ such that $\lambda_m^{(n)}(0) \cdot u_m^{(n)}(0) \subseteq v_m^{(n)}(0)$ for $V_m, V_n \in N$.

Hence we have

$$\prod_{n=1}^{\infty} \{\lambda_m^{(n)}(0) \cdot u_m^{(n)}(0)\} \subseteq \prod_{n=1}^{\infty} v_m^{(n)}(0) (\forall m \in N).$$

Therefore $E = \prod_{n=1}^{\infty} E_n$ is a linear ranked space over $\Phi$. (Q.E.D.)

**Definition 5.** In the direct product $E = \prod_{k \in N} E_k$ of vector spaces $E_k$ ($k \in N$) over $\Phi$ if a vector space $E$ ($\subseteq E$) is generated by $\bigcup_{k \in N} E_k$ then $E$ is called the direct sum of vector spaces $E_k$ ($k \in N$) and it is denoted by $\sum_{k \in N} E_k$. $\bigcup_{k \in N} E_k$ is called the direct sum of $E_k$ ($k \in N$) if every $E_k$ ($k \in N$) is a linear ranked space over $\Phi$.

**Proposition 5.** If $E_k$ ($k \in N$) is a system of linear ranked spaces over $\Phi$ then $\sum_{k \in N} E_k$ is a linear ranked space over $\Phi$.

**Proof.** $E_\alpha (\subseteq E)$ is a vector space and a subspace of $E$.

Let $E \in \sum_{k \in N} E_k$ and $\tilde{V}_n(x) \equiv (V_n^{(k)}(x))_{k \in N} \in F(x ; E_k), \{\tilde{V}_n(x) \} \in F(x ; \tilde{E})$.

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5) $N$ denotes the set of all natural numbers.
For \( \mathbb{V}(\bar{u}_n(\bar{x})), \mathbb{V}(\bar{v}_n(\bar{y})) \in \mathbb{F}(\mathbb{E}) \) we have
\[
\bar{u}_n(\bar{x}) + \bar{v}_n(\bar{y}) = (u_n(k)(x_k) + v_n(k)(y_k)) \cap \mathbb{E} \\
= (u_n(k)(x_k) + v_n(k)(y_k)) \cap \mathbb{E} \\
\subseteq (\mathbb{F}u_n(k)(x_k + y_k)) \cap \mathbb{E} \\
( \{ u_n(k)(x_k + y_k) \} \in \mathbb{F}(x_k + y_k ; E_k)).
\]
Moreover for any \( \{ \lambda_n \} \subseteq \Phi \) \( (\lambda_n \to \lambda) \) and any \( \{ \bar{u}_n(\bar{x}) \} \in \mathbb{F}(\bar{x} ; \mathbb{E}) \) we have
\[
\lambda_n \cdot \bar{u}_n(\bar{x}) = (\lambda_n \cdot u_n(k)(x_k)) \cap \mathbb{E} \subseteq (\mathbb{F}u_n(k)(\lambda \cdot x_k)) \cap \mathbb{E} \equiv \bar{u}_n(\lambda \bar{x}) \in \mathbb{F}(\mathbb{E}) \left( \{ u_n(k)(\lambda x_k) \} \in \mathbb{F}(E_k) \right).
\]
Hence \( \mathbb{E} \) is a linear ranked space over \( \Phi \). (Q.E.D.)

**Definition 6.** Let \( \mathbb{E} = \mathbb{E} / M \) be the quotient vector space over \( \Phi \) by a vector subspace \( M \) of a given vector space \( \mathbb{E} \) over \( \Phi \). \( \mathbb{E} \) is called the **quotient linear ranked space** over \( \Phi \) by \( M \) if \( \mathbb{E} \) is a linear ranked space over \( \Phi \).

**Proposition 6.** Let \( \mathbb{E} \) be a linear ranked space over \( \Phi \) and \( M \) a vector subspace of \( \mathbb{E} \). Then the quotient vector space \( \mathbb{E} = \mathbb{E} / M \) is a linear ranked space over \( \Phi \).

**Proof.** By the following relations:
\[
\mathcal{N}_n(\bar{x}) \equiv \mathcal{N}(\bar{x}) \cap \mathbb{E} \quad \text{def.} \equiv \{ a + M \mid a \in \mathcal{V}(x) \} \quad (x \in \mathbb{E})
\]
and
\[
\mathcal{V}_n(\bar{x}) = \text{the set of all neighborhoods of } \bar{x} \text{ with rank } a \quad (0 < a < a_0)
\]
\( \mathbb{E} \) is a linear ranked space over \( \Phi \). (Q.E.D.)

**Definition 7.** Let \( \{ E_n \} \) be a sequence of ranked spaces with same indicator \( a_0 \geq a_0 \). Suppose that it satisfies the following condition: \( E_n \) is a closed subspace of \( E_{n-1} \) for each \( n \), \( n \in N \). If \( E = \bigcup_{n=1}^{\infty} E_n \) is a ranked space then such ranked space \( E \) is called the **inductive limit** of ranked spaces \( E_n \quad (n=1,2,3,...) \) and it is denoted by \( E = \lim_{n \to \infty} E_n \).

**Proposition 7.** The inductive limit \( E = \lim_{n \to \infty} E_n \) of linear ranked spaces over \( \Phi \) \( E_1 \subseteq E_2 \subseteq \cdots \subseteq E_n \subseteq \cdots \) is a linear ranked space over \( \Phi \).

**Proof.** We set \( E_n = \{ E_n, \mathcal{V}_n(\bar{x}) \} \quad (n \in N) \) and \( U^{(n)}(0) \in \mathcal{V}_n(0) \) \( (\mathcal{M}, 0 \leq a < a_0) \). Since \( E_n \) is a closed subspace of \( E_{n-1} \) there exists \( U^{(n+1)}(0) \in \mathcal{V}_n(0) \) such that \( U^{(n)}(0) = E_n \cap U^{(n+1)}(0) \). Inductively we have
\[
\exists U^{(j+1)}(0) \in \mathcal{V}_j(0) \quad \text{such that } \quad U^{(j)}(0) = E_j \cap U^{(j+1)}(0) \quad \text{for all } j \geq n.
\]
By the relation
\[
U(0) \equiv \bigcap_{j \geq n} U^{(j)}(0) \quad \mathcal{V}_j(0) \equiv \bigcap_{j \geq n} U^{(j)}(0) \quad \mathcal{V}_j(0) \quad \text{def.} \quad (\forall j \geq n)
\]
and
\[
\{ U_m(0) \} \in \mathcal{F}(E) \quad (U_m(0) = \bigcap_{j \geq n} U^{(j)}(0) \quad \mathcal{V}_j(0) \quad \text{def.} \quad \{ U_m(0) \} \in \mathcal{F}(E_j) \quad (\forall j \geq n),
\]
we can show that \( E \) becomes a ranked space with indicator \( a_0 \):

(A) \( U(0) \equiv 0 \) is clear.

(B) For any \( U(0) \) and \( V(0) \) we have
\[
U(0) \cap V(0) = \bigcap_{j \geq n} U^{(j)}(0) \cap \bigcap_{j \geq n} V^{(j)}(0) \quad (n \leq m)
\]
\[
\geq \bigcap_{j \geq n} (U^{(j)}(0) \cap V^{(j)}(0))
\]
\[
\geq \bigcap_{j \geq n} W^{(j)}(0) \equiv \mathcal{F}W(0) \quad (U^{(j)}(0) \cap V^{(j)}(0) \equiv W^{(j)}(0), \quad \mathcal{F}W(0) \in \mathcal{V}(E_j)).
\]

(a) Since each \( E_n \) satisfies the axiom \( (a) \) there exists \( m \), \( l \leq m < a_0 \), such that
\[
U^{(j)}(0) \geq \bigcap_{j \geq l} \mathcal{F}V^{(j)}(0) \quad \mathcal{F}V^{(j)}(0) \in \mathcal{V}_m(0).
\]
Hence if we set \( V(0) = \bigcap_{j \geq n} V(j)(0) \) then:

\[
U(0) \supseteq V(0) \& \quad V(0) \in \mathcal{G}_m(0) \quad (l \leq m < a_0).
\]

Thus \( E \) satisfies the axiom (a).

Hence \( E = \bigcup_{n \in N} E_n \) is a ranked space with indicator \( a_0 \). Thus \( \lim \sup E_n \) is a ranked space with indicator \( a_0 \).

Since every \( E_n (n \in N) \) is a vector space, \( \bigcup_{n \in N} E_n \) is a vector space.

For any \( \{ U_n(0) \} \), \( \{ V_n(0) \} \in \mathcal{F}(E) \) we have, for every \( n \in N \),

\[
U_n(0) + V_n(0) = \bigcap_{j \geq k} U_n(j)(0) + \bigcap_{j > k} V_n(j)(0) \quad (k \geq k')
\]

\[
\subseteq \bigcap_{j \geq k} (U_n(j)(0) + V_n(j)(0))
\]

\[
\subseteq \bigcap_{j \geq k} W_n(j)(0) \equiv \mathcal{G} W_n(0) \quad (\mathcal{G} W_n(j)(0)) \in \mathcal{F}(E), \quad U_n(j)(0) + V_n(j)(0) \subseteq W_n(j)(0)
\]

and \( \{ W_n(0) \} \in \mathcal{F}(E) \).

Moreover for any \( \{ \lambda_n \} \in \Phi(\lambda \rightarrow 0) \) and any \( \{ U_n(0) \} \in \mathcal{F}(E) \) we have, for every \( n \in N \),

\[
\lambda_n \cdot U_n(0) = \lambda_n \cdot (\bigcap_{j \geq k} U_n(j)(0)) = \bigcap_{j \geq k} \lambda_n \cdot U_n(j)(0)
\]

\[
\subseteq \bigcap_{j \geq k} W_n(j)(0) \equiv W_n(0) \quad (\mathcal{G} W_n(j)(0)) \in \mathcal{F}(E), \quad \lambda_n \cdot U_n(j)(0) \subseteq W_n(j)(0)
\]

and \( \{ W_n(0) \} \in \mathcal{F}(E) \).

Hence \( \lim \sup E_n \) is a linear ranked space over \( \Phi \). (Q.E.D.)

**Definition 8.** Let \( A \) be an ordered set and \( \{ E_i, \mathcal{G}_{a_i}(\lambda) \} \) \( (\lambda \in \Lambda) \) a system of ranked spaces with same indicator. If an \( R \)-continuous mapping \( \psi_{\lambda, \mu} : E_{\mu} \rightarrow E_{\lambda} \) is defined for every \( \lambda, \mu \in \Lambda \) \( (\lambda \leq \mu) \) and if \( \lambda \leq \mu \leq \nu \Rightarrow \psi_{\lambda, \mu} \circ \psi_{\mu, \nu} = \psi_{\lambda, \nu} \) holds, then \( (E_i, \mathcal{G}_{a_i}(\lambda)) \) is called the *projective system* of ranked spaces \( \{ E_i, \mathcal{G}_{a_i}(\lambda) \} \) \( (\lambda \in \Lambda) \). Let \( E^* = \bigoplus_{\lambda \in \Lambda} E_{\lambda} \) be the direct product ranked space of \( E_{\lambda} (\lambda \in \Lambda) \) and set

\[
E = \{ x \in E^* \mid x = (x_\lambda) \text{ such that } x_\lambda = \psi_{\lambda, \mu}(x_\mu) \text{ if } \lambda \leq \mu \}
\]

Then \( E (\subseteq E^*) \) becomes a subspace of \( E^* \) by the induced rank. Such ranked space \( \{ E, \mathcal{G}_a \} \) is called the *projective limit* of \( \{ (E_i, \mathcal{G}_{a_i}(\lambda)) \} \) and it is denoted by \( (E, \mathcal{G}_a) = \lim_{\mathcal{R}} (E_i, \mathcal{G}_{a_i}(\lambda)) \) or \( E = \lim_{\mathcal{R}} E_i \).

Since \( \psi_{\lambda} : E^* \rightarrow E_{\lambda} \) is an \( R \)-continuous mapping, a mapping

\[
E \xrightarrow{\psi_{\lambda}} E_{\lambda} \quad \downarrow \psi_{\lambda, \mu} \quad \bigcirc \quad \psi_{\lambda, \mu} \quad E_{\mu} \xrightarrow{\psi_{\mu}} E_{\mu}
\]

is also \( R \)-continuous. Such \( \psi_{\lambda} : E \rightarrow E_{\lambda} \) is called the *canonical mapping*. From the definition we have \( \lambda \leq \mu \Rightarrow \psi_{\lambda, \mu} \cdot \psi_{\mu} = \psi_{\lambda} \).

**Definition 9.** (i) A sequence \( \{ x_n \}_{n \in \mathbb{N}} \) is called an \( r \)-Cauchy sequence in \( E \) if there exists a fundamental sequence \( \{ V_n(0) \} \in \mathcal{F}(0) \) and if, for each \( N \in \mathbb{N} \), we have \( m, n \geq N \Rightarrow x_m - x_n \in V_N(0) \).

(ii) \( E \) is said to be \( r \)-complete if every \( r \)-Cauchy sequence in \( E \) is \( r \)-convergent in \( E \).

**Remark.** In a linear ranked space, every \( r \)-convergent sequence is an \( r \)-Cauchy sequence, but the converse is not always true.

§ 3. Generalized Closed Graph Theorem.

Let \( E \) and \( F \) be two homogeneous linear ranked spaces over \( \Phi \). In this section we suppose that every \( V \in \mathcal{G}_m(0) \) \( \equiv \bigcup_{n \in N} \mathcal{G}_n(0) \) is \( r \)-closed and convex.

**** 1973年4月、日本数学会年会（於立教大学）にて一部講演。

7) We use only the fact \( \frac{1}{2} V + \frac{1}{2} V \subseteq V \) for \( V \in \mathcal{G}_0(0) \).
Condition \((\gamma)\). \(\forall V \in \mathfrak{Y}_+ r^{-1}(V)\) (0) holds for any \(x \in \Phi\) and any \(V \in \mathfrak{Y}_+(0)\) (0 \( \leq r \leq \omega_0\)).

Theorem 1. (Closed Graph Theorem). A linear mapping \(f\) of a linear ranked space \(E\) into an \(r\)-complete linear ranked space \(F\) satisfying the condition \((\gamma)\) is \(r\)-continuous and \(T\)-continuous provided \((i)\) the graph of \(f\) is \(r\)-closed in \(E \times F\),

\((ii)\) for each neighborhood \(V(0)\) of \(f\) in \(R\) the \(r\)-closure \(\text{cl}_r(f^{-1}(V(0)))\) is a neighborhood of \(0\) in \(E\), and

\((iii)\) \((a^*)\) for each neighborhood \(V(0)\) of \(0\) in \(E\) and each \(\{v_n(0)\} \in F(0; E)\) there exists an index \(m \in N\) such that \(v_m(0) \subseteq v(0)\).

Proof. (1). We begin by showing that for each \(V(0) \in \mathfrak{Y}(0)\) we have \(\text{cl}_r(f^{-1}(V(0))) \subseteq f^{-1}(V(0))\).

For any \(x \in \text{cl}_r(f^{-1}(V(0)))\) there exists an \(r\)-Cauchy sequence \(\{y_n\} \subseteq F\) such that

\[ y_n = 0 \text{ and } x \in \text{cl}_r(f^{-1}(y_n + U_n)) \text{ for all } n, 0 \leq n < \omega_0 \text{ (} U_0 = \frac{1}{2} V(0), U_n = \frac{1}{2} U_n \).

We will show this by mathematical induction. For \(n = 0\) this is clear. Suppose that \(y_n\) is defined for each \(n \geq 0\). Then we have

\[ x \in \text{cl}_r(f^{-1}(y_n + U_n)) \subseteq f^{-1}(y_n + U_n) + \text{cl}_r(f^{-1}(U_n)) \subseteq \text{cl}_r(f^{-1}(U_n)). \]

Hence there exists a point \(x' \in E\) such that

\[ f(x') \in y_n + U_n \text{ and } x \in x' + \text{cl}_r(f^{-1}(U_n)). \]

Now,

\[ x' + \text{cl}_r(f^{-1}(U_n)) = \text{cl}_r(x' + f^{-1}(U_n)) \subseteq \text{cl}_r(f^{-1}(f(x')) + f^{-1}(U_n)) \subseteq \text{cl}_r(f^{-1}(U_n)). \]

Set \(y_{n+1} = f(x')\). Then we have

\[ y_{n+1} \in y_n + U_n \text{ and } x \in \text{cl}_r(f^{-1}(y_n + U_n + U_{n+1})) \text{ for all } n, 0 \leq n < \omega_0. \]

Now set \(W_n = y_{n+1} + U_n\) for all \(n, 0 \leq n < \omega_0\). Then we have

\[ W_n = y_{n+1} + U_n \subseteq (y_{n+1} + U_n + U_{n+1}) = \text{cl}_r(f^{-1}(U_n)). \]

Thus for all \(n, 0 \leq n < \omega_0\), we have

\[ W_n \supseteq W_n + 1 \text{ and } W_n \nexists y_{n+1}, y_{n+2}, y_{n+3}, \ldots. \]

From \(W_n \supseteq W_n + 1 \text{ and } W_n \nexists y_{n+1}, y_{n+2}, y_{n+3}, \ldots\) we get

\[ y_j - y_n \in W_n - U_n \text{ for } \forall j > n \text{ and } y_{n+1} - y_j \in U_n - U_n \text{ for } \forall j \geq n. \]

Then there exists a \(U'_n \in F(0)\) such that

\[ y_j - y_n = (y_j - y_n) \in (U_n - U_n) \text{ for } \forall j \leq n. \]

Therefore \(\{y_n\}\) is an \(r\)-Cauchy sequence in \(F\). Thus we get \(\lim_{n \to \infty} y_n \exists y \in F\). (Because \(F\) is \(r\)-complete.)

Therefore, for all \(n \geq 0\), we get the following fact:

\[ y_n \in \text{cl}_r(W_n) \subseteq \text{cl}(W_n) = \text{cl}(y_1 + U_1) \subseteq \text{cl}(U_1 + U_0) \text{ (As } y_1 \in \mathfrak{Y}_+, U_0 \subseteq \mathfrak{Y}_0). \]

\[ \text{cl}_r(f^{-1}(V(0))) \subseteq \text{cl}_r(V(0)) \subseteq V(0) \text{ (Since } V(0) \text{ is } r\text{-closed),} \]

\(i. e.\), \(y \in V(0)\).

Now, as \(\lim_{n \to \infty} y_n \in F\) (as \(n \to \infty\)) there exist \(\{V_n(y)\}, \{W_n(y)\} \in F(y)\) such that for all \(n, 0 \leq n < \omega_0, 8\).

Such \(F\) has a structure as an extension of every locally convex, pseudo-metrizable and semi-normable TVS. In general \(F\) is not pseudo-metrizable.

9) If we define a topology as a TVS in \(E\) by a system of neighborhoods with ranks, if \(E\) is complete in the sense of ranked spaces (53) and if \(F\) is a pseudo-metrizable TVS, then \(E\) is 2nd Category (53).

Such \(E\) satisfies above condition (ii) (Kelley–Namioka [4]). Therefore Theorem 1 is an extension of the theorem of Kelley–Namioka [4].

10) First countability axiom \(\subseteq^\omega_{\omega_0}(a^*)\).
\[ W_n = y_{n+1} + U_n \subseteq \mathcal{F} V_{n}(y) + U_n \subseteq W_n(y) \quad (V_n(y) \not\ni y_{n+1} \text{ for each } n). \]

Hence, for all \( n, 0 \leq n < \omega_0 \), we have
\[ y_{n+1} + U_n \subseteq y_{n+1} + (\frac{1}{2} U_n + \frac{1}{2} U_n) \subseteq y_{n+1} + U_n = W_n \subseteq W_n(y). \]

Thus we get
\[ x \in \text{cl}_r (f^{-1}(y_{n+1} + U_n)) \subseteq \text{cl}_r (f^{-1}(W_n(y))) \text{ for all } n, 0 \leq n < \omega_0. \]

Therefore we have the followings:
\[ \mathcal{F} x_k(\cdot) \subseteq f^{-1}(W_n(y)), \text{ i.e., } f(\mathcal{F} x_k(\cdot)) \subseteq W_n(y) \text{ for } k = 0, 1, 2, \ldots \text{ and each } n, 0 \leq n < \omega_0, \]

such that
\[ x_k(\cdot) \to x \quad (\text{as } k \to \infty). \]

Thus there are \( \{ f_k(\cdot)(x) \} \in F(x) \quad (\{ u_k(\cdot)(x) \} \not\ni x_k(\cdot)) \text{ for each } n, 0 \leq n < \omega_0. \]

Therefore we get
\[ (u_k(\cdot)(x) \times W_n(y)) \cap (\text{the graph of } f) \not\ni \phi \quad (k = 0, 1, 2, \ldots; n = 0, 1, 2, \ldots). \]

Now, by condition (a*) and the homogeneity for \( E \), we can pick out a fundamental sequence of \((x, y)\) in \( E \times F \)
\[ \{ u_k(\cdot)(x) \times W_n(y) \} \quad (k(n) < k(n+1)) \]

from the family \( \{ u_k(\cdot)(x) \times W_n(y) \mid k = 0, 1, 2, \ldots; n = 0, 1, 2, \ldots \}. \)

Therefore we have
\[ (u_k(\cdot)(x) \times W_n(y)) \cap \text{(the graph of } f) \not\ni \phi \quad (\text{for all } n, 0 \leq n < \omega_0). \]

By assumption, the graph of \( f \) is \( r \)-closed set in \( E \times F \).

Thus we get
\[ f(x) = y \in V(0) \]
and
\[ x \in f^{-1}(y) \subseteq f^{-1}(V(0)). \]

Therefore
\[ \text{cl}_r (f^{-1}(\frac{1}{2} V(0))) \subseteq f^{-1}(V(0)) \]

is proved.

(II). Next, we will show that \( f \) is \( r \)-continuous on \( E \). Let \( \forall x \in E \) and let \( \{ \lim_n x_n \} \not\ni x \in E \).

From \( \{ \lim_n x_k \} \not\ni x \in E \), there exists \( \{ u_k(\cdot)(x) \} \in F(x) \) such that \( u_k(\cdot)(x) \not\ni x \) for all \( n, 0 \leq n < \omega_0 \). For any \( \{ V_n(0) \} \subseteq F(0) ; F \), by (I), we have \( \text{cl}_r (f^{-1}(\frac{1}{2} V_n(0))) \subseteq f^{-1}(V_n(0)). \)

Since \( V_n(0) = V_n(0) \subseteq V_n(0) \)

is a neighborhood of \( 0 \) in \( F \), using homogeneity for \( E \), we get that \( \text{cl}_r (f^{-1}(V_n(0))) + x \)

is a neighborhood of \( x \) in \( E \).

Now, by homogeneity for \( F \), since \( V_n(0) \ni V_n(0) + f(x) \)

is a neighborhood of \( f(x) \) in \( F \) we have \( \{ V_n(f(x)) \} \subseteq F(F) \)

and
\[ \text{cl}_r (f^{-1}(V_n(0))) + x \subseteq f^{-1}(V_n(0)) + x \subseteq f^{-1}(V_n(0) + f(x)) = f^{-1}(V_n(f(x))). \]

By (a*) there exists an \( m(N) \in N \) and we have
\[ x_{m(N)}(\cdot) \subseteq u_m(\cdot)(x) \subseteq \text{cl}_r (f^{-1}(V_{m(N)}(0))) + x \subseteq f^{-1}(V_{m(N)}(f(x))). \]

Hence
\[ f(x_{m(N)}) \subseteq V_{m(N)}(f(x)) \text{ for all } j \geq m(N). \]

Now, as \( f(x_{m(N)}) \subseteq V_{m(N)}(f(x)) \not\ni f(x_{m(N)-1}) \)

we set
\[ V_{m(N)}(f(x)) \equiv V_{m(N)+1}(f(x)) \equiv V_{m(N)+1}(f(x)) \equiv \ldots \equiv V_{m(N)+1}(f(x)) \equiv V_{m(N)+1}(f(x)) \equiv V_{m(N)+1}(f(x)) \equiv V_{m(N)+1}(f(x)) \]

and \( m(N+1) = m(N) + l(N). \)

Then we have
\[ f(x_{m(N)+p}) \subseteq V_{m(N)+p}(f(x)) \text{ for all } p, 0 \leq p \leq l(N). \]

Hence there exists \( \{ V_n(f(x)) \} \subseteq F(f(x)) \not\ni f(x_{m(N)}) \) for all \( n, 0 \leq n < \omega_0 \).

Thus we have
\[ \{ \lim_n x_n \} \not\ni x \text{ in } E \not\ni \{ \lim_n f(x_n) \} \not\ni f(x) \text{ in } F \]

and \( f \) is \( r \)-continuous on \( E \).

(III). Finally, we will show that \( f \) is \( T \)-continuous on \( E \). Let \( V(0) \)

be an arbitrary neighborhood of \( 0 \) in \( F \). By (I) and (a), for a \( V(0) \subseteq \mathcal{F}(0) \) in \( F \) we have
\[ \text{cl}_r (f^{-1}(\frac{1}{2} V(0))) \subseteq f^{-1}(V(0)) \subseteq f^{-1}(V(0)). \]
By assumption, \( cl_\epsilon (f^{-1}(\frac{1}{2}V(0))) \) is a neighborhood of 0 in \( E \). Hence, by axiom (a), there exists \( v(0) \in \mathcal{V}(0) \)

in \( E \) such that \( v(0) \subset cl_\epsilon (f^{-1}(\frac{1}{2}V(0))) \subset f^{-1}(V(0)) \).

Thus we have \( f(v(0)) \subset V(0) \).

Hence, \( f(v(0) + x) = f(v(0)) + f(x) \subset V(0) + f(x) \) for each \( x \in E \).

Therefore \( f \) is \( T \)-continuous on \( E \).

This completes the proof of the generalized closed-graph theorem. \( (Q.E.D.) \)

Let \( \{F_n, \mathcal{V}(n)\} \) be an increasing sequence of linear ranked spaces such that every \( F_n \) satisfies the condition \( (\star \star) \) and every \( U \in \mathcal{V}(n) \) is \( r \)-closed and convex in each \( F_n \). Then we have next theorem as a generalization of Theorem 1.

**Theorem 2.** A linear mapping \( f \) of a linear ranked space \( E \) into an \( r \)-complete \( F^* = \lim_{n \to \infty} F_n \) is \( r \)-continuous and \( T \)-continuous provided three conditions (i)—(iii) of Theorem 1 for \( E \) and \( F^* \).

**Proof.** For each \( U(0) \in \mathcal{V}(0) \) we have \( U(0) = \bigcap U_j(0) \) and \( U_j(0) \in \mathcal{V}(j)(0) \) \( (j \geq n) \). Moreover for each \( \{U_k(0)\}_{k=0,1,2,...} \) we have \( U_k(0) = \bigcap U_k(0) \) \( (k = 0,1,2,...) \) and \( \{U_k(0)\}_{k=0,1,2,...} \in F(0) \). Hence \( F^* \) satisfies the condition \( (\star \star) \). Thus we get this theorem by the method as is taken in the proof for Theorem 1. \( (Q.E.D.) \)

**§ 4. Examples of Linear Ranked Spaces.**

**Proposition 8.** The following spaces are linear ranked spaces:

(a) Pseudo-metrizable TVS (b) (Semi-) Normed space
(c) Countably normed space (d) Dual (or union) space of countably normed space
(e) \( D, D', \hat{D}, \hat{D}' \) (f) Nuclear space and its dual
(g) Space of \( (E.R.) \) integrable functions (h) Space with a system of semi-norms
(i) LCTVS and its conjugate (j) Inductive limit of LCTVS spaces
(k) Fréchet space, Banach space, LF-space, Bornological space, Barreled space, Montel space, \( L, Hörmander's \) space \( \mathfrak{F}(\mathcal{U}), \) etc

(1) Inductive limit for each space in (k).

In fact, we get this proposition from the result in \( [12] \). These spaces are included in the space \( F \). \( (13) \)

Moreover \( (s^*) \) and \( (\star \star) \) are held for these spaces.

**Definition 10.** \( (11) \). A netted space \( E \) is a LCTVS where exists a net of subsets of \( E \)

\[ \epsilon_{n_1} \cdots \epsilon_{n_k} ; \ n_1, \ldots, n_k = 1, 2, \ldots, \]

such that

(i) \( \epsilon_{n_1} \cdots \epsilon_{n_k} \) are absolutely convex,

(ii) \( E = \bigcup_{n_1} \epsilon_{n_1}, \epsilon_{n_1} = \bigcup_{n_2} \epsilon_{n_1} \epsilon_{n_2}, \ldots, \epsilon_{n_1} \cdots \epsilon_{n_k} = \bigcup_{n_{k+1}} \epsilon_{n_1} \cdots \epsilon_{n_k} \)

(iii) to every sequence \( n_k \) corresponds a sequence \( \lambda_k \geq 0 \) such that any series

\[ \sum_{k=1}^{\infty} \lambda_k f_k \epsilon_{n_1} \cdots \epsilon_{n_k} \]

converges in \( E \).

(iv) a series of the preceding form is such that

\[ \sum_{k=1}^{m} \lambda_k f_k \epsilon_{n_1} \cdots \epsilon_{n_k} \sum_{k=m+1}^{\infty} \lambda_k f_k \epsilon_{n_1} \cdots \epsilon_{n_p} \]

11) These are locally convex (cf. \([20]\), p. 28).

12) See On Generalized Continuous Groups II, pp. 63-64. And replace \( \leq \) by \( \leq \). Then, for \( v(0) \in \mathcal{V} \), we have \( cl_\epsilon (v(0)) = v(0) = v(0) \).

13) For (a) see Kelley-Namioka \([4]\), p. 97.
Proposition 9. Every netted space has a struture as a linear ranked space. Thus every netted space is included in the space $E$ of Theorem 1.

Proof. Let $E$ be a netted space defined by M. De Wilde.

We set $v(n, k ; 0) = e_{n \cdots n_k}(0)$.

\[ e_{n_0 \cdots n_k}(0) = \beta_{n_0} \cap e_{n_1 \cdots n_k}(0) \]  

(See H. G. Garnir [1], p. 368)

\[
\beta_{n_0} = \{ f \mid p(f) \leq \rho_{n_0} \} (\rho > 0), \quad e_{n_1 \cdots n_k}(0) = e_{n_1 \cdots n_k} \cup \{0\},
\]

\[
\text{max}(n_0, \ldots, n_k) = n \quad \text{for any} \quad n_0, \ldots, n_k, \quad k \in \mathbb{N},
\]

\[ \mathcal{S}_n(0) = \{ v(n, k ; 0) \mid k \in \mathbb{N} \}, \quad \mathcal{S}_n(0) = (E), \quad \mathcal{S}_n(f) \equiv \mathcal{S}_n(0) + \{ f \} (\forall f \in E)
\]

and

\[ \lambda \cdot v(n, k ; 0) \in \mathcal{S}_n(0) \quad (n=0,1,2,\ldots) \quad \text{for any} \quad \lambda \in \Phi.
\]

Then we have

\[ e_{n_0 \cdots n_k}(0) \in \mathcal{S}_0 \quad \text{and} \quad e_{n_0 \cdots n_k}(0) \in \mathcal{S}_n(0) \quad (n=0,1,2,\ldots) \quad \text{for any} \quad n \in \mathbb{N}.
\]

Axiom (a) is clear. Moreover we have

\[ e_{n_0 \cdots n_k}(0) = e_{n_0 \cdots n_k}(0) \cap e_{m_0 \cdots m_l}(0) = e_{n_0 \cdots n_k}(0) \cap e_{m_0 \cdots m_l}(0) \in \mathcal{S}_\max(n_0, \ldots, n_k, m_0, \ldots, m_l)(0).
\]

Axiom (b) is clear. Moreover we have

\[ e_{n_0 \cdots n_k}(0) + e_{m_0 \cdots m_l}(0) = e_{n_0 \cdots n_k}(0) + e_{m_0 \cdots m_l}(0) \in \mathcal{S}_0(0) \quad \text{and} \quad \lambda \cdot e_{n_0 \cdots n_k}(0) \in \mathcal{S}_0(0)
\]

\[ \forall \{ \lambda_m \} (\lambda_m \rightarrow 0) \quad \text{in} \quad \Phi.
\]

Moreover we have

\[ cl_1(e_{n_0 \cdots n_k}(0)) = e_{n_0 \cdots n_k}(0) = e_{n_0 \cdots n_k}(0).
\]

Thus $E$ is a linear ranked space. (Q.E.D.)

Definition 11. ([10]). (i) A filter $\Phi$ in a space $E$ is said to be an $S$-$\text{filter}$ if $\Phi$ has a countable basis $\{ S_n \}$ such that $\bigcap_{n=1}^{\infty} S_n = \emptyset$.

(ii) A filter $\Phi$ in a linear space $E$ is said to be an $LS$-$\text{filter}$ if $\Phi$ is generated by the complements of all the finite union of linear subspaces $E_n (n=1,2,\ldots)$ of $E$ such that $E = \bigcup_{n=1}^{\infty} E_n$.

(iii) A subset $A$ of a linear space $E$ is said to be $\text{linearly open}$ if for any straight line $L$ in $E$, $E \cap A$ is open in $L$ by its usual topology.

(iv) A filter $\Phi$ in a linear space $E$ is called a $P$-$\text{filter}$ if for every $x$ in $E$ there exists a linearly open set $A$ such that either $A$ is disjoint from $\Phi$ or $\Phi_A$, considered as a filter in $E$, is finer than an $LS$-$\text{filter}$.

Proposition 10. (1) If a sequence $E$ has a sequence of $S$-filters $\Phi_i (i=1,2,\ldots)$ then $E$ has a structure as a ranked space with indicator $\omega_0$.

(2) If a linear space $E$ has a sequence of $LS$-filters $\Phi_i (i=1,2,\ldots)$ then $E$ has a structure as a linear ranked space.

(3) If a linear space $E$ has a sequence of $P$-filters $\Phi_i (i=1,2,\ldots)$ then $E$ has a structure as a linear ranked space.

Proof. (1) Suppose that each $S$-filter $\Phi_i$ is generated by subsets $S_{k}^{(i)} (k=1,2,\ldots)$ such that $\bigcap_{k=1}^{\infty} S_{k}^{(i)} = \emptyset$.

We set, for each point $p \in E$,

\[ v(p; n) = \bigcap_{k=1}^{\infty} \bigcap_{i=1}^{n} S_{k}^{(i)} \cup \{ p \}, \quad \mathcal{S}_n(p) = \{ v(p; n) \} (n \in \mathbb{N}), \quad \mathcal{S}_n(p) = (E).
\]

Then $\{ E, \mathcal{S}_n \}$ fulfills the axioms (A), (B), (a) and (b). Hence $E$ is a ranked space with indicator $\omega_0$.

(2) Suppose that each $LS$-filter $\Phi_i$ is generated by the complements of all the finite union of linear subspaces $E_k^{(i)} (k=1,2,\ldots)$ of $E$ such that $E = \bigcup_{k=1}^{\infty} E_k^{(i)}$.

We set, for each $x \in E$,

\[ v(0; n) = \bigcap_{k=1}^{n} \bigcap_{i=1}^{n} (E_k^{(i)} \setminus 0) + v(0; n) + x, \quad \mathcal{S}_n(x) = \{ v(0; n) \} (n \in \mathbb{N}), \quad \mathcal{S}_n(x) = (E).
\]

Then $\{ E, \mathcal{S}_n \}$ fulfills the axiom (A), (B), (a) and (b). Now, for any \{$v_n(0)$\}, \{$v_n(0)$\} $\in F(0)$

14 Let $A$ be a subset of a set $E$ and $\Phi$ a filter in $E$. We say that $A$ is disjoint from $\Phi$ if there is $B$ in $\Phi$ such that $A \cap B = \emptyset$. 

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\( u_n(0) \in \mathcal{V}(n)(0), \ v_n(0) \in \mathcal{V}(n)(0) \), we have
\[
\begin{align*}
\lim_{n \to \infty} u_n(0) + v_n(0) &= \bigcap_{k=1}^{r(n)} \bigcap_{i=1}^{s(n)} E_k^{(i)} \subseteq \bigcap_{k=1}^{r(n)} \{ w_n(0) \mid f(n) = \text{Min} \{ r(n), s(n) \} \}
\end{align*}
\]
and \( \{ w_n(0) \} \in \mathcal{F}(0) \). Moreover, for any \( \{ \lambda_n \} \subseteq \varphi(\lambda_n \rightarrow 0) \) and \( \{ u_n(0) \} \in \mathcal{F}(0) \), we have
\[
\lambda_n \cdot u_n(0) = \lambda_n \cdot \bigcap_{k=1}^{r(n)} \bigcap_{i=1}^{s(n)} E_k^{(i)} = \bigcap_{k=1}^{r(n)} \bigcap_{i=1}^{s(n)} \lambda_n \cdot E_k^{(i)} = v_n(0) \in \mathcal{F}(0).
\]
Therefore \( E \) is a linear ranked space.

(3). For any \( x \in E \) there exists a linearly open set \( A \ni x \) such that \( \phi_i \cap A = \varphi \) or \( \phi_i \nsubseteq \mathcal{F} \phi_i \)\( (i = 1, 2, 3, \ldots) \). We assume \( \phi_i \cap A = \varphi \) without losing generality. Let each \( \phi_i \) be generated by the complements of all the finite union of linear subspaces \( E_k^{(i)} \) such that \( E = \bigcup_{k=1}^{r(n)} E_k^{(i)} \ (i = 1, 2, 3, \ldots) \).

We will give a rank for \( E \) by the method as is taken in (2).

Then \( E \) becomes a linear ranked space. By using (1), (2) and (3) we can show this Proposition. \( Q.E.D. \)

M. Nakamura proved the following facts ([10]):

1. Netted spaces by M. De Wilde-H. G. Garnir \( \subseteq \mathcal{G}N \)-spaces,
2. \( a \beta \gamma \)-representable spaces by W. Słowiński \( \subseteq \mathcal{G}N \)-spaces,
3. Souslin spaces by A. Martineau-L. Schwartz \( \subseteq \mathcal{K} \)-Souslin spaces by A. Martineau \( \subseteq \mathcal{Q} \)-Souslin spaces by M. Nakamura.

4. Spaces with nets of type \( P \) by M. De Wilde \( \subseteq \mathcal{G} \)-spaces.

Since \( \mathcal{G} \), \( \mathcal{M} \)-space, \( \mathcal{G}N \)-space, quasi-Souslin space and \( \mathcal{G} \)-space are defined by using \( L \)-covering or filters of Definition 11, these spaces have the structures as the linear ranked spaces. Moreover these spaces hold the condition (\( \alpha \ast \)) and (\( \alpha \star \)).

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References


