On "Fundamental and Applied Aspects of Mathematics" Especially an Account of "The Complex Numbers Plane"

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Synopsis

This thesis is to find the conditions that are sufficient to enable three points on the complex number plane to form an equilateral triangle, and further to apply this condition for the solution of several problems and still to expand the theories of this condition for the solution of Morley's equilateral triangle.

Preface:

If we take the analytical methods in modern mathematics to solve arithmetic problems, geometric ones in elementary course and further, many fundamental problems in mathematics or in applied mathematics, probalility, mathematical statistics and so on, we should know how we are helped by the intuitive cognitions in attacking methods of mathematical problems in the elemental course.

However, even though from the view point of the modern science, if we want to found some theories, we can not help cognizing some intuitions.

For example, in the theory of group, we must even cognize intuitively the important fact in the system of its axioms. They have no preliminary criterion to discriminate the difference between a + b and b + a (or between a × b and b × a), while they need to cognize their diffrence.

Modern Mathematics often treats the vector spaces properly elected, on the study of some Geometrical Relations or Characters.

§ 1 As, the plane figures in Euclidian Space are usually considered in the complex (numbers) plane, which is the two dimentions normed vector space and in "complete", that is further, Banach space.

§ 2 A complex plane so called is a corresponding geometrical counterpart to the set of all the complex numbers C, which we define as "Norm" and the caluclus laws concering "addition" and "multiplication":

$$\forall \alpha,\ \beta \in \mathbf{C}$$

$$\alpha = (a_1,\ a_2),\ \beta = (b_1,\ b_2)$$

a_i, b_i *and* λ *are the real numbers.*

$$\|\alpha\| = |\alpha| = \sqrt{a_1^2 + a_2^2}$$

$$\alpha + \beta = (a_1,\ a_2) + (b_1,\ b_2) = (a_1+b_1,\ a_2+b_2)$$

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$$\lambda \cdot \alpha = \lambda \cdot (a_1, a_2) = (\lambda a_1, \lambda a_2)$$
$$\alpha \times \beta = (a_1 b_1 - a_2 b_2, a_1 b_2 + a_2 b_1)$$

where the complex plane is also considered as a metric space (Euclidian Space) which further defines the distance of the two points, as

$$\rho(\alpha, \beta) = |\alpha - \beta| = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}$$

§ 3 Examples:

For three of these points of the complex plane, $\alpha, \beta, \gamma \in \mathbf{C}$ we have under certain some conditions

$$\alpha + \omega\beta + \omega^2\gamma = 0 \qquad |\beta - \gamma| = |\gamma - \alpha| = |\alpha - \beta| \neq 0$$

where $\omega^2 + \omega + 1 = 0$

From the above proposition are derived some trivial but rather interesting facts; that is, we may verify a series of interesting problems concerning the characteristics of "equilateral triangles" and further Frank Morley's problem.

(1) If $\alpha_i \in \mathbf{C}$ $\quad i = 1, 2, 3, \quad -\infty < \lambda < +\infty$

$$Z_1 = (1 - \lambda)\alpha_2 + \lambda\alpha_3$$
$$Z_2 = (1 - \lambda)\alpha_3 + \lambda\alpha_1$$
$$Z_3 = (1 - \lambda)\alpha_1 + \lambda\alpha_2$$

and $\quad Z_1 + \omega Z_2 + \omega^2 Z_3 = 0,$

then we have

$$\alpha_1 + \omega\alpha_2 + \omega^2\alpha_3 = 0$$

(2) If $\alpha_i \in \mathbf{C}, \quad \lambda_i > 0, \quad i = 1, 2, 3$

$$Z_1 = (1 - \lambda_1)\alpha_2 + \lambda_1\alpha_3$$
$$Z_2 = (1 - \lambda_2)\alpha_3 + \lambda_2\alpha_1$$
$$Z_3 = (1 - \lambda_3)\alpha_1 + \lambda_3\alpha_2$$
$$Z_1 - \alpha_1 = e^{i\theta_3}(Z_3 - \alpha_3), \qquad Z_2 - \alpha_2 = e^{i\theta_3}(Z_1 - \alpha_1),$$
$$Z_3 - \alpha_3 = e^{i\theta_3}(Z_1 - \alpha_1)$$

where $\quad \sum_{i=1}^{3} \theta_i = 2\pi, \quad \theta_i > 0$

and $\quad Z_1 + \omega Z_2 + \omega^2 Z_3 = 0$

then we have

$$\alpha_1 + \omega\alpha_2 + \omega^2\alpha_3 = 0.$$

(3) If $\alpha_i, \beta_i \in \mathbf{C}, \quad i = 1, 2, 3 \qquad \lambda \in \mathbf{R}$

$$\alpha_1 + \omega\alpha_2 + \omega^2\alpha_3 = 0$$
$$\beta_1 + \omega\beta_2 + \omega^2\beta_3 = 0$$

and $\quad Z_i = (1 - \lambda)\alpha_i + \lambda\beta_i \qquad i = 1, 2, 3$

then we have

$$Z_1+\omega Z_2+\omega^2 Z_3 = 0.$$

(4) If α_i, β_i, $\gamma_i \in \mathbf{C}$, $i = 1, 2, 3$ $-\infty < m, n < +\infty$

$$\alpha_1+\omega\alpha_2+\omega^2\alpha_3 = 0$$

$$\beta_1+\omega\beta_2+\omega^2\beta_3 = 0$$

$$\gamma_1+\omega\gamma_2+\omega^2\gamma_2 = 0$$

and

$$Z_i = (1-\lambda)\alpha_i+\lambda\frac{m\beta_i+n\gamma_i}{m+n}, \quad i = 1, 2, 3 \quad \lambda \in \mathbf{R}$$

then we have

$$Z_1+\omega Z_2+\omega^2 Z_3 = 0.$$

(5) If α_i, $\beta_i \in \mathbf{C}$, $i = 1, 2, 3$

$$\beta_1+\omega\alpha_3+\omega^2\alpha_2 = 0$$

$$\beta_2+\omega\alpha_1+\omega^2\alpha_3 = 0$$

$$\beta_3+\omega\alpha_2+\omega^2\alpha_1 = 0$$

and

$$Z_i = (1-\lambda_i)\alpha_i+\lambda_i\beta_i, \quad i = 1, 2, 3,$$

$$\sum_{i=1}^{3}\lambda_i = 1, \qquad \lambda_i > 0$$

then we have

$$Z_1+\omega Z_2+\omega^2 Z_3 = 0.$$

(6) If α_i, β_i, $\gamma_i \in \mathbf{C}$, $i = 1, 2, 3$

and

$$\alpha_1+\omega\alpha_2+\omega^2\alpha_3 = 0$$

$$\alpha_i+\omega\beta_i+\omega^2\gamma_i = 0, \quad i = 1, 2, 3$$

$$Z_1 = (1-\lambda)\gamma_1+\lambda\beta_2, \qquad Z_2 = (1-\lambda)\gamma_2+\lambda\beta_3,$$

$$Z_3 = (1-\lambda)\gamma_3+\lambda\beta_1, \qquad \lambda = \frac{1}{2}$$

then we have

$$Z_1+\omega Z_2+\omega^2 Z_3 = 0.$$

(7) If $\forall\alpha,\beta,\gamma \in \mathbf{C}$, $\alpha \neq \beta$, $\beta \neq \gamma$, $\gamma \neq \alpha$

and $|\alpha| = |\beta| = |\gamma| = 1$

and putting

$$Z_1 = \beta+\lambda_{1\beta}(\alpha^{\frac{1}{3}}\lambda^{\frac{2}{3}}-\beta)$$

$$= \gamma+\lambda_{1\gamma}(\alpha^{\frac{1}{3}}\beta^{\frac{2}{3}}-\gamma)$$

$$Z_2 = \gamma+\gamma_{2\gamma}(\beta^{\frac{1}{3}}\alpha^{\frac{2}{3}}-\gamma)$$

$$= \alpha+\lambda_{2\alpha}(\beta^{\frac{1}{3}}\gamma^{\frac{2}{3}}-\alpha)$$

$$Z_3 = \alpha + \lambda_{3\alpha}\ (\gamma^{\frac{1}{3}}\ \beta^{\frac{2}{3}} - \alpha)$$

$$= \beta + \lambda_{3\beta}\ (\gamma^{\frac{1}{3}}\ \alpha^{\frac{2}{3}} - \beta)$$

where $\lambda's$ are real numbers, then we have

$$Z_1 + \omega Z_2 + \omega^2 Z_3 = 0 \qquad or \qquad |Z_2 - Z_3| = |Z_3 - Z_1| = |Z_1 - Z_2|$$

This is the proposition so called Frank Morley's.
Professor Frank Morley has once written to Professor T. Hayashi concerning the classical and elegant solution of this proposition. (cf. J. M. E vol. 6 Japan).

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(To be continued.)

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