Views on Extensors in Higher Order Spaces.

TAKASHI ONODERA *

Synopsis.

The purpose of the present paper is to explain systematically the outline of various extensors according to the steps of their development. We do not refer to how these extensors are useful in higher order spaces. Therefore, we must consult the reference papers, if we would like to know about their uses.

§ 0. **Introduction.** The notion of an extensor which has played an important role in geometry of higher order spaces of connections was first used by H. V. Craig [1]. Prof. A. Kawaguchi [6] has immediately extended it to multiple parameter extensor on n—dimensional space and formed the complete system of extensor theory [6], [7], [8], [9]. Afterwards, H. V. Craig and W. T. Guy, Jr. [4] have introduced the *Jacobian extensor* which is one of the generalizations of the extensor, and studied its properties, and H. V. Craig [5] has also extended it to multiple parameter case. On the other hand, M. Kawagnchi [10] has defined the *generalized extensor*, and arranged its theory [11], [12], [13]. In this case, it has several properties which are different from those of the ordinary extensor.

Notation. 1) Following H. V. Craig [1], [2], [3], we shall use the one root letter x for all coordinate systems.

2) We shall distinguish between different coordinate systems by means of the letters employed indices, i, j ; α, β.

3) Furthermore, we shall introduce the following notations:

$$x^{(0)i}=x^i, \qquad x'^i=x^{(1)i}=\frac{dx^i}{dt}, \qquad x''^i=x^{(2)i}=\frac{d^2x^i}{dt^2},$$

$$x^{(\alpha)i}=\frac{d^\alpha x^i}{dt^\alpha}, \qquad F_{(\alpha)i}=\frac{\partial F}{\partial x^{(\alpha)i}}, \qquad X^a_i=X^{(0)a}_{(0)i}=\frac{\partial x^a}{\partial x^i},$$

$$X^a_{ij}=\frac{\partial^2 x^a}{\partial x^i \partial x^j}, \qquad X^{(\alpha)a}_{(\beta)i}=\frac{\partial x^{(\alpha)a}}{\partial x^{(\beta)i}}, \qquad X^{a(\alpha)}_i=\frac{d^\alpha}{dt^\alpha}(X^a_i).$$

4) We shall denote the number of A things, P being taken, by the symbol $\binom{A}{P}$, that is, a binominal coefficient. In addition, we shall give $\binom{A}{P}$ the value zero, if $A<P$.

5) Finally, we shall use the summation conventions consisting of three parts as follows:

---

* 助教授, 一般教科

(a) repeated Latin indices indicate summation from I to N,

(b) repeated Greek indices indicate summation from O to M,

(c) but capital indices do not indicate sums.

In the case of (c) we shall frequently replace α with A, β with B, γ with Γ, etc.

§ 1. **Ordinary extensors.** In the manifold X_n, let us consider a coordinate transformation

$$x^a = x^a(x^i) \qquad (a,\ i = 1,\ 2,\ \cdots,\ N), \tag{1.1}$$

where the function $x^a(x^i)$ is differentiable as often as necessary, and the Jacobian $\left|X^a_i\right|$ is different from zero in our domain.

When the N components V^a and N^2 components T_{ab} vary by (1.1), satisfying the rule

$$\begin{cases} V^a = X^a_i V^i, \\ T_{ab} = X^i_a X^j_b T_{ij}, \end{cases} \tag{1.2}$$

these components are called the components of a *vector* and a *tensor*, respectively.

Now, by differentiating (1.1) repeatedly with respect to a curve parameter t, we can find the following series of extended transformation:

$$\begin{cases} x^a = x^a(x^i), \\ x^{(1)a} = X^a_i x^{(1)i}, \\ x^{(2)a} = X^a_i x^{(2)i} + X^a_{ij} x^{(1)i} x^{(1)j}, \\ x^{(3)a} = X^a_i x^{(3)i} + 3X^a_{ij} x^{(1)i} x^{(2)j} + X^a_{ijk} x^{(1)i} x^{(1)j} x^{(1)k}, \ \cdots\cdots, \\ x^{(M)a} = X^a_i x^{(M)i} + MX^a_{ij} x^{(M-1)i} x^{(1)j} + \cdots \ (M>2). \end{cases} \tag{1.3}$$

According to A. Kawaguchi [9], this transformation (1.3) can be rewritten in the following condensed form:

$$x^{(\alpha)a} = \sum_{(\beta)} \frac{\alpha!}{\beta_1!\,\beta_2!\cdots\beta_\lambda!\,\gamma_1!\,\gamma_2!\cdots\gamma_{\alpha-\lambda+1}} X^a_{i_1 i_2 \cdots i_\lambda} x^{(\beta_1)i_1} x^{(\beta_2)i_2} \cdots x^{(\beta_\lambda)i_\lambda} \tag{1.4}$$

$$(\beta_1 + \beta_2 + \cdots + \beta_\lambda = \alpha),$$

where the symbol $\sum_{(\beta)}$ signifies the summation with respect to all combinations $(\beta_1,\ \beta_2,\ \cdots,\ \beta_\lambda)$ and with γ_ν we denote the number of β_μ's which have the same value ν. Under (1.4) the coordinate of a line element of order M $(x^i, x^{(1)i}, \cdots, x^{(M)i})$ is transformed to $(x^a, x^{(1)a}, \cdots, x^{(M)a})$, but the right side of (1.4) is not linear with respect to $x^{(\alpha)i}$. Accordingly, the line element can not keep the properties of an ordinary vector. Now, we employ the following generalized form in place of (1.4),

$$(1.5) \qquad x^{(\alpha)a} = x^{(\alpha)a}(x^i, x^{(1)i}, \cdots, x^{(M)i}).$$

When the $N(M+1)$ components $V^{\alpha a}$ vary by (1.5), satisfying the rule

$$(1.6) \qquad V^{\alpha a} = X^{(\alpha)a}_{(\beta)i} V^{\beta i} \quad (\alpha, \beta = 0, 1, \cdots, M; a, i = 1, 2, \cdots, N),$$

these components are called the components of an *excontravariant exvector* of grade M. Because of Jacobian $\left|X^a_i\right| \neq 0$, we shall have the only inverse transformation of (1,6) as follows:

$$(1.7) \qquad W_{\alpha a} = X^{(\beta)i}_{(\alpha)a} W_{\beta i},$$

and this coefficient $X^{(\beta)i}_{(\alpha)a}$ is satisfied by

$$(1.8) \qquad X^{(\beta)i}_{(\alpha)a} X^{(\alpha)a}_{(\gamma)j} = \delta^\beta_\gamma \delta^i_j \quad \text{or} \quad X^{(\beta)i}_{(\alpha)a} X^{(\gamma)b}_{(\beta)i} = \delta^\gamma_\alpha \delta^b_a.$$

Then these labelled numbers will be called the components of an *excovariant exvector* of grade M. Moreover, when the $N^3(M+1)^2$ components $T^{\alpha a}_{c\beta b}$ vary by (1.5), satisfying the rule

$$(1.9) \qquad T^{\alpha a}_{c\beta b} = T^{\gamma i}_{k\delta j} X^{(\alpha)a}_{(\gamma)i} X^{(\delta)j}_{(\beta)b} X^k_c,$$

these components of a *mixed third degree extensor* which is *excontravariant, covariant,* and *excovariant* — each of order one.

Hence, for the coefficient $X^{(\alpha)a}_{(\beta)i}$ of (1.6) we have the following equation,

$$(1.10) \qquad X^{(\alpha)a}_{(\beta)i} = \binom{A}{B}\left(X^a_i\right)^{(A-B)}$$

which is so-called the Craig's formula, and making use of (1.10), we obtain

$$(1.11) \qquad X^{(\alpha+1)a} = \frac{dx^{(\alpha)a}}{dt} = X^{(\alpha)a}_{(\beta)i} \frac{dx^{(\beta)i}}{dt} = X^{(\alpha)a}_{(\beta)i} x^{(\beta+1)i}$$

$$= \binom{\alpha}{\beta}\left(X^a_i\right)^{(\alpha-\beta)} x^{(\beta+1)i},$$

Therefore, in the place of (1.4) we can consider the following transformation:

$$
(1.12)\qquad
\begin{aligned}
x^{a} &= x^{a}\,(x^{i}),\\
x^{(\alpha)a} &= X^{(\alpha-1)a}_{(\beta-1)i}\, x^{(\beta)i} \qquad (\alpha,\beta = 1,\ 2,\cdots,\ M),\\
&= \sum_{\beta=1}^{\alpha} \binom{\alpha-1}{\beta-1} \left(X^{a}_{i}\right)^{(\alpha-\beta)} x^{(\beta)i},
\end{aligned}
$$

which is called the *expoint transformation* of Craig. Equation (1.12) is linear with respect to $x^{(\alpha)i}$, but equation (1.4) is not.

The following properties of exvector which we find in the proofs by A. Kawaguchi [7] are very useful.

1) When $V^{\alpha i}$ and $W_{\alpha i}$ are the components of an excontravariant exvector and of an exco-variant exvector, and those of grade M, respectively, the following contraction

$$
(1.13)\qquad V^{\alpha i}\, W_{\alpha i}
$$

is invariable by (1.5), that is, a scalar. But unlike ordinary vector analysis there are $M+1$ kinds of contractions in our exvector analysis as follows:

$$
(1.14)\qquad
\begin{aligned}
\overset{A}{V} &= \sum_{\alpha=A}^{M} \binom{\alpha}{A} V^{\alpha-A,\ i}\, W_{\alpha i},\\
&= \sum_{\alpha=0}^{A} \binom{A+\alpha}{\alpha} V^{\alpha i}\, W_{A+\alpha,\ i} \qquad (A = 0,\ 1,\cdots M),
\end{aligned}
$$

and $\overset{A}{V}$ is another scalar.

2) Furthrmore, we can obtain various exvectors out of one exvector and its derivativds. Let $V^{\alpha i}$ be an excontravarant exvector of grade M, and differentiate it with respect to the parameter t, then we obtain an excontravariant exvector by the following operation which is known as $\mathfrak{S}$-operation ([7] p. 28).

$$
(1.15)\qquad \overset{H}{\mathfrak{S}} V^{\alpha i} = \sum_{\lambda=0}^{H} (-1)^{H-\lambda} \binom{H}{\lambda} V^{\alpha+\lambda,\, i\,(H-\lambda)} \quad (H=0,\ 1,\cdots,M),
$$

where $V^{\alpha+\lambda,\, i\,(H-\lambda)} = d^{H-\lambda}/dt^{H-\lambda}\, V^{\alpha+\lambda,\, i}$, and the grade of this exvector is M-H, lower than that of $V^{\alpha i}$.

In the same manner from $\mathfrak{Z}$-operation ([7], p. 37), we have

$$
(1.16)\qquad \overset{H}{\mathfrak{Z}} V^{\alpha i} = \sum_{\lambda=0}^{\alpha} \binom{\alpha}{\lambda} 2^{H\lambda} \left(1-2^{H}\right)^{\alpha-\lambda} V^{\lambda i\,(\alpha-\lambda)} \quad (1 \leqq H),
$$

where the grade of this exvector is the same as that of $V^{\alpha i}$.

Furthermore, from $\mathfrak{Y}$-operation ([7], p. 44), we have

$$(1.17)\quad \begin{aligned} \mathfrak{Y}^H V^{\alpha i} &= H! \sum_{\nu=0}^{\alpha} \binom{\alpha}{\nu} \binom{M+H-\alpha}{H-\nu} V^{\alpha-\nu, i(\nu)} \quad (\alpha=0, 1, \cdots, H), \\ &= H! \sum_{\nu=0}^{H} \binom{\alpha}{\nu} \binom{M+H-\alpha}{H-\nu} V^{\alpha-\nu, i(\nu)} \quad (\alpha=H, H+1, \cdots, M), \\ &= H! \sum_{\nu=\alpha-M}^{H} \binom{\alpha}{\nu} \binom{M+H-\alpha}{H-\nu} V^{\alpha-\nu, i(\nu)} \quad (\alpha=M, M+1, \cdots, M+H), \end{aligned}$$

where the grade of this exvector is $M+H$, higher than that of $V^{\alpha i}$.

We can find the same operations about an excovariant exvector $W_{\alpha i}$.

3) Let $V^{\alpha i}$ and $W_{\alpha i}$ be an excontravariant exvector and an excovariant exvector of grade M, respectively, then the sets of the components

$$V^{\alpha i} \text{ and } \binom{M-K+\alpha}{\alpha} W_{M-K+\alpha, i} \quad (\alpha = 0, 1, \cdots, K \leqq M)$$

are also the components of an excontravariant and an excovariant of grade K, respectively. From the contraction of these exvectors $V^{\alpha i}$ and $\binom{M-K+\alpha}{\alpha} W_{M-K+\alpha, i}$ we obtain the scalar which is the very scalar $V^{(A)}$ of (1.14).

We can see, of course, the same properties regarding extensors as the exvectors given above ([7], p. 69–139).

§ 2. **K-dimentional extensors.** The first extention of the extensor has been studied by A. Kawaguchi [6]. Now we shall consider a surface element of order M $(x^i, P^i_\lambda, P^i_{\lambda_{(1)}}, \cdots, P^i_{\lambda_{(M)}})$ instead of a line element of order M $(x^i, x^{(1)i}, \cdots, x^{(M)i})$ in the above section,

where $\quad x^i = x^i(u^\lambda) \qquad (i = 1, 2, \cdots, N;\ \lambda = 1, 2, \cdots, K),$

$$P^i_{\lambda_1} = \frac{\partial x^i}{\partial u^{\lambda_1}}, \quad \cdots\cdots,$$

and $$P^i_{\lambda_{(K)}} = P^i_{\lambda_1 \lambda_2 \cdots \lambda_K} = \frac{\partial^K x}{\partial u^{\lambda_1} \partial u^{\lambda_2} \cdots \partial u^{\lambda_K}} \quad (K = 1, 2, \cdots, M).$$

Since $P^i_{\lambda_{(K)}}$ are symmetry with respect to the indices $\lambda_1, \lambda_2, \cdots, \lambda_K$, we can find the following transformation from (1.1) by differentiation with respect to the K parameters u^λ:

$$(2.1)\quad \begin{aligned} x^a &= x^a(x^i), \\ P^a_{\lambda_1} &= X^a_i P^i_{\lambda_1}, \\ P^a_{\lambda_{(2)}} &= X^a_i P^i_{\lambda_{(2)}} + X^a_{ij} P^i_{\lambda_1} P^j_{\lambda_2}, \quad \cdots\cdots, \end{aligned}$$

$$P^{a}_{\lambda(M)} = X^{a}_{i} P^{i}_{\lambda(M)} + MX^{a}_{ij} P^{i}_{(\lambda(M-1)} P^{j}_{\lambda(M))} + \cdots.$$

In this case, we have the following formula corresponding with Craig's (1.10),

$$(2.2) \qquad \frac{\bar{\partial} P^{a}_{\lambda(r)}}{\partial P^{i}_{\mu(t)}} = \binom{r}{t} \delta^{\mu(t)}_{(\lambda(r)} \left(\frac{\partial x^a}{\partial x^i}\right)_{/\lambda(r-t))},$$

where $\delta^{\mu(r)}_{\lambda(r)} = \delta^{\mu_1}_{(\lambda_1} \delta^{\mu_2}_{\lambda_2} \cdots \delta^{\mu_r}_{\lambda_r)}$, $\Phi/\lambda_{(K)} = \dfrac{\partial^K \Phi}{\partial u^{\lambda_1} \partial u^{\lambda_2} \cdots \partial u^{\lambda_K}}$,

and $$\frac{\bar{\partial}}{\partial P^{b}_{\mu(t)}} = \frac{\nu_1 !\ \nu_2 ! \cdots \nu_u !}{t!} \cdot \frac{\partial}{\partial P^{b}_{\mu(t)}}.$$

By using ν_s we denote the number of μ_r's which has the same value s, and besides, with restriction $\nu_1 + \nu_2 + \cdots + \nu_u = t$.

In like manner of the definition of extensor we have a following definition; when the quantities $V^{a}_{\lambda(t)}$ and $W^{\lambda(t)}_{a}$ vary by (1.5), satisfying the rule

$$(2.3) \qquad V^{a}_{\lambda(t)} = \frac{\bar{\partial} P^{a}_{\lambda(t)}}{\partial P^{i}_{\mu(r)}} V^{i}_{\mu(r)} \quad \text{and} \quad W^{\lambda(t)}_{a} = \frac{\bar{\partial} P^{i}_{\mu(r)}}{\partial P^{a}_{\lambda(t)}} W^{\mu(r)}_{i}$$

$$(t, r = 0, 1, \cdots, M;\ a,\ i = 1,\ 2, \cdots N;\ \mu, \lambda = 1,\ 2, \cdots, K),$$

these quantities are called the components of an excontravariant and an *exconvariant K-dimentional extensor* of grade M, respectively.

These coefficients are satisfied by

$$(2.4) \qquad \frac{\bar{\partial} P^{i}_{\mu(r)}}{\partial P^{a}_{\lambda(t)}} \frac{\bar{\partial} P^{a}_{\lambda(t)}}{\partial P^{j}_{\nu(s)}} = \delta^{i}_{j}\ \delta^{\nu(s)}_{\mu(r)}.$$

In this case the foregoing properties of ordinary extensors are extended as follows: In the first place, we have $M+1$ kinds of contractions

$$(2.5) \qquad V^{\lambda(A)} = \sum_{S=A}^{M} \binom{S}{A} V^{a}_{\lambda(S-A)} W^{\lambda(S-A)\ \lambda(A)}_{a} \quad (A=0,\ 1, \cdots,\ M).$$

Secondly, when $V^{a}_{\lambda(t)}$ $(t = 0,\ 1, \cdots,\ M)$ is given, $V^{a}_{\lambda(t)}$ $(t = 0,\ 1, \cdots, H \leqq M)$ is also the components of an excontravariant K-dimentional extensor of grade H, and as for $W^{\lambda(t)}_{a}$ $(t=0,\ 1\cdots,\ M)$, $\binom{M-H+t}{t} W^{\lambda(M-H+t)}_{a}$ $(t = 0,\ 1, \cdots, H)$ is the components of an excovariant K-

dimentional extensor of grode H. Therefore, we can see that each of $V^{\lambda(A)}$ (2.5) is a scalar.

Finally, from $\mathfrak{S}$-operation we have

$$(2.6)\qquad \underset{\lambda(H)}{\mathfrak{S}^{H}} V^{a}_{\lambda(t)} = \sum_{r=0}^{H} (-1)^{H-r} \binom{H}{r} V^{a}_{(\lambda(r+t)/\lambda(H-r))},$$

but when we consider the transformation equation of (2.6), we have

$$(2.7)\qquad \underset{\lambda(H)}{\mathfrak{S}^{H}} V^{a}_{\lambda(t)} = \frac{\bar{\partial} P^{a}_{(\lambda(t)}}{\partial P^{i}_{\mu(r)}} \underset{\lambda(H))}{\mathfrak{S}^{H}} V^{i}_{\mu(r)} \qquad (0 \leqq H \leqq M).$$

That is, we can not obtain the same exvector as the given one.

On the contrary, as for $W^{\lambda(t)}_{a}$, we have the following exvector

$$(2.8)\qquad \mathfrak{S}^{H} W^{\lambda(t)}_{a} = \sum_{r=0}^{H} (-1) \binom{t+r}{r} \binom{M-t-r}{M-t-H} W^{\lambda(t)\mu(r)}_{a} / \mu_{(r)},$$

where the grade of this exvector is M-H.

As for $\mathfrak{Y}$-and $\mathfrak{Z}$-operation, $\mathfrak{Y} V^{a}_{\lambda(t)}$ and $\mathfrak{Z} V^{a}_{\lambda(t)}$ are excontravariant K-dimentional exvectors of which grades are $M+H$ and M, respectively. And as for $W^{\lambda(t)}_{a}$, in this case, we can not obtain an exvector.

§ 3, **Jacobian extensors.** By H. V. Craig [4] the notion of a *Jacobian extensor,* a generalization of the extensor, has been introduced and extended to the case of multiple parameters functions [5].

We shall find that Jacobian extensors include weighted and absolute tensors, and scalars as special cases, and that their properties are similar to those of ordinary extensors.

The point transformation is denoted by

$$(1.1)\qquad x^{a} = x^{a}(x^{r}) \quad (a,\ r = 1,\ 2,\ \cdots,\ N),$$

and the expoint transformation by

$$(1.11)\qquad x^{\alpha+1,a} = X^{(\alpha)a}_{(\rho)r}\, x^{\rho+1,r} \qquad (\alpha,\ \rho = 0,\ 1,\cdots,M-1).$$

In addition to coefficients X^{a}_{r} and $X^{(\alpha)a}_{(\rho)r}$ of tensor and extensor analysis, the new transformation equations contain the symbols X^{α}_{ρ} and X^{ρ}_{α} which are based on X and $\bar{X}$, namely, on the weighted Jacobian $\left| X^{a}_{r} \right|^{w}$ and $\left| X^{r}_{a} \right|^{w}$, and are defined as follows;

(3.1) $X^{\alpha}_{\rho} = \binom{A}{P} \underset{-}{x}^{(A-P)}, \quad X^{\rho}_{\alpha} = \binom{P}{A} \bar{x}^{(P-A)} \quad (A=\alpha,\ P=\rho).$

If x^{α}, x^{i} and x^{r} are any three coordinate systems and we correlate the indices α, λ, and ρ to these coordinate systems, then

(3.2) $X^{\lambda}_{\alpha} X^{\alpha}_{\rho} = X^{\lambda}_{\rho}.$

Therefore, when we correlate indices ρ and σ to the same coordinate system, we have

$$X^{\rho}_{\alpha} X^{\alpha}_{\sigma} = \delta^{\rho}_{\sigma}$$

Now, when the quantities V^{α} and W_{α} vary by (1.1) satisfying the rule

(3.3) $V^{\alpha} = V^{\rho} X^{\alpha}_{\rho}$ and $W_{\alpha} = W_{\rho} X^{\rho}_{\alpha},$

these quantities are called the components of a *Jacobian contra-* and *covariant exvector*, or briefly a *J-contra* and a *J-covariant exvector*, respectively.

Moreover, when the $N^4\ (M+1)^4$ labelled numbers $E^{\alpha\cdot\gamma\, c\cdot e}_{\beta\cdot\delta\, d\cdot f}$ vary, satisfying the rule,

(3.4) $E^{\alpha\cdot\gamma\, c\cdot e}_{\beta\cdot\delta\, d\cdot f} = E^{\rho\cdot\tau\, t\cdot u}_{\sigma\cdot\omega w\cdot v} X^{\alpha}_{\rho} X^{\sigma}_{\beta} X^{(\gamma)c}_{(\tau)t} X^{(\omega)w}_{(\delta)d} X^{e}_{u} X^{v}_{f},$

these labelled numders will be called the components of an *extensor* of grade M and weight w which is *J-contravariant, J-covariant, excontravariant, excovariant, contravariant*, and *covariant*— each of order one.

As in extensor analysis, there are $M+1$ kinds contractions as follows;

(3.5) $V^{\theta} = \sum_{\alpha=\theta}^{M} \binom{\alpha}{\theta} E^{\alpha-\theta}_{\alpha} = \sum_{\alpha,\rho=\theta}^{M} \binom{\alpha}{\theta} E^{\rho-\theta}_{\sigma} X^{\alpha-\theta}_{\rho-\theta} X^{\sigma}_{\alpha} \quad (\theta = 0,\ 1, \cdots, M).$

Namely, V^{θ} is an absolute scalar.

For the examples *J*-extensors, let us consider the quantities $V^{o(\alpha)}$ and $V_{\alpha} = \binom{M}{A} V^{(M-A)}_{M}$ $(0 \leqq \alpha \leqq M)$, where V^{o} and V_{M} are a scalar of weight$-w$ and w, respectively. Furthermore, we shall show that similar examples exist for higher order weighted tensors, that is, if T^{abc}_{M} is a contravariant tensor of order 3, of weight w, then

(3.6) $E^{\alpha a\cdot\beta b\cdot\gamma c}_{\delta} = \begin{Bmatrix} A,\ B,\ \Gamma \\ M,\ \Delta \end{Bmatrix} T^{abc\ (A+B-2M-\Delta)}_{M}$

is an extensor of excontravariant of order 3 and *J*-covariant of order one, where

$$\left\{\begin{matrix} A, B, \Gamma \\ M, \varDelta \end{matrix}\right\} = \frac{A!\ B!\ \Gamma!}{M!^2 \varDelta!\ (A+B+\Gamma-2M-\varDelta)!} \ or\ 0$$

$$for\ (A+B+\Gamma-2M-\varDelta) \geqq 0\ or < 0$$

On the other hand, if T_{Mbc} is a covariant tensor of order 2, of weight w, then

$$(3.7) \qquad E_{\alpha\cdot\beta b\cdot\gamma c} = \binom{M}{A, B, \Gamma} T^{(M-A-B-\Gamma)}_{M\cdot bc}$$

is an extensor of J-covariant order 1 and excovariant order 2.

Finally, we shall obtain the following effective formula by means of the Leibnitz's rule and by the same method as in the proofs of the foregoing equations:

$$(T^{obc} B_b C_c)^{\alpha} = E^{\alpha\cdot\beta b\cdot\gamma c} B_{\beta b} C_{\gamma c} = E^{\alpha\cdot\beta b\cdot\gamma c} L^f_{\beta b} B_f L^g_{\gamma c} C_g,$$

$$(3.8) \quad \begin{aligned} \binom{M}{\varDelta}\left(T^{abc}_{M} A_a B_b C_c\right)^{(M-\varDelta)} &= E^{\alpha a\cdot\beta b\cdot\gamma c}_{\delta} A_{\alpha a} B_{\beta b} C_{\gamma c} \\ &= E^{\alpha a\cdot\beta b\cdot\gamma c}_{\delta} L^e_{\alpha a} A_e L^f_{\beta b} B_f L^h_{\gamma c} C_h, \end{aligned}$$

$$\binom{M}{A}\left(T_{Mbc} B^b C^c\right)^{(M-A)} = E_{\alpha\cdot\beta b\cdot\gamma c} B^{\beta b} C^{\gamma c} = E_{\alpha\cdot\beta b\cdot\gamma c} L^{\beta b}_d B^d L^{\gamma c}_e C^e,$$

where this quantities L can be taken to be the extended components of connection of our space.

§ 4. **Generalized extensors.** M. Kawaguchi has introduced the notion of a generalized extensor, another generalization of the extensor, calculated and studied powerfully ([11], [12], [13]).

The point transmation is denoted by

$$(1.1) \qquad x'^i = x'^i(x^j) \quad (i, j = 1, 2, \cdots, N),$$

and the expoint transformation by

$$(1.11) \qquad x'^{\alpha\,:\,1,\,i} = X^{\alpha i}_{\beta j}\, x^{\beta\,:\,1,\,j} \quad (\alpha, \beta = 0, 1, \cdots, M-1).$$

Here, in the place of (1.11) we employ the following condensed form:

$$(4.1) \qquad x'^I = X^I_J\, x'^J \quad (I, J = 1, 2, \cdots, MN),$$

where we use the capital indices I, J, K which denote the expoint indicies αi, βj, γk, that is, $I = \alpha N + i$, $J = \beta N + j$.

When F is a function of the expoint x^I, the derivatives $F_{I_1 I_2 \cdots I_r}$ $(r = 1, 2, \cdots, k)$ are transformed by the expoint transformation (4.1) as follows:

$$F_{I_1} = X^{J_1}_{I_1} F_{J_1},$$

(4.2) $$F_{I_1 I_2} = X^{J_1}_{I_1} X^{J_2}_{I_2} F_{J_1 J_2} + X^{J_1}_{I_1 I_2} F_{J_1}, \cdots \cdots,$$

$$F_{I_1 I_2 \ldots I_k} = X^{J_1}_{I_1} X^{J_2}_{I_2} \cdots X^{J_k}_{I_k} F_{J_1 J_2 \ldots J_k} + \cdots + X^{J_1}_{I_1 I_2 \ldots I_k} F_{J_1}.$$

$F_{I_1 I_2 \ldots I_r}$ $(r = 1, 2, \cdots, k)$ is symmetry with respect to $I_1, I_2, \cdots, I_r$. When we choose the dictionary permutation which will be denoted by $I_{(r)}$ among the sets of indices of which combination is the same, we have the independent equations of (4.2) in a brief form:

(4.3) $$F_{I(r)} = U^{J(s)}_{I(r)} F_{J(s)} \quad (s = 1, 2, \cdots, r),$$

where, according to the same symbols used in (1.4), we have

(4.4) $$U^{J(s)}_{I(r)} = \frac{r!}{a_1!\, a_2! \cdots a_s!\, b_1\, b_2 \cdots b_{r-s+1}!} X^{(J_1}_{(I(a_1)} X^{J_2}_{I(a_2)} \cdots X^{J_s)}_{I(a_s))}.$$

Since we have $$\left| U^{J(s)}_{I(r)} \right| = \left| X^{i}_{j} \right|^{\sum_{r=1}^{k} M^r \binom{M+r-1}{r-1}}$$

there exists the inverse transformation of (4.3), and we denote the coefficients with $V^{I(r)}_{J(s)}$, then $U^{J(s)}_{I(r)} V^{I(r)}_{J(s)} = \delta^{J(s)}_{J(s)}$.

In a similar way of the foregoing definitions, when the quantities $\Phi^{I(r)}$ and $\Psi_{I(r)}$ vary by (4.1), satisfying the rule

(4.5) $$\Phi^{I(r)} = V^{I(r)}_{J(s)} \Phi^{J(s)} \text{ and } \Psi_{I(r)} = U^{J(s)}_{I(r)} \Psi_{J(s)},$$

these quantities are called the components of a *generalized contra*-and *covariant exvector*, or briefly a *g-contra* and *g-covariant exvector*, respectively. If $k = 1$, our g-exvector is reduced to the ordinary exvector.

Now we repeat the partial differential operation $l-1$ times for the scalar product of two ordinary extensors $U^{\alpha i}$, $V_{\alpha i}$ with respect to the expoint $x^{\beta j}$ as follows:

$$\rho_1 = U^{\alpha i} V_{\alpha i},$$

(4.6) $$\rho_2 = U^{\alpha_2 i_2} (U^{\alpha_1 i_1} V_{\alpha_1 i_1})_{;\alpha_2 i_2} = U^{\alpha_2 i_2} U^{\alpha_1 i_1} V_{(\alpha_1 i_1 | \alpha_2 i_2)}$$

$$+ U^{\alpha_2 i_2} U^{\alpha_1 i_1}_{;\alpha_2 i_2} V_{\alpha_1 i_1}, \quad \cdots \cdots, \quad \text{where } U_{;\alpha_r i_r} = \frac{\partial U}{\partial x^{\alpha_r i_r}}.$$

Using (1.14), we have following scalars

$$(4.7)\quad \begin{cases} \rho^{A_1 j} = \sum_{\alpha_1=0}^{M-A_1} \binom{A_1+\alpha_1}{A_1} U^{\alpha_1 i_1} V_{(A_1+\alpha_1) i_1}, \quad (A_1 = 0, 1, \cdots, M), \\ \rho^{A_1 A_2 j} = \sum_{\alpha_1=0}^{M-A_1} \sum_{\alpha_2=0}^{M-A_2} \binom{A_1+\alpha_1}{A_1} \binom{A_2+\alpha_2}{A_2} U^{\alpha_2 i_2} \left(U^{\alpha_1 i_1} V_{(A_1+\alpha_1) i_1} \right)_{/(A_2+\alpha_2) i_2}, \\ \cdots\cdots. \end{cases}$$

Then,

$$(4.8)\quad \Psi_{l\, I(s)} = \binom{A_1+\alpha_1}{A_1} \binom{A_2+\alpha_2}{A_2} \cdots \binom{A_S+\alpha_S}{A_S} V_{((A_1+\alpha_1) i_1 / (A_2+\alpha_2) i_2 \cdots (A_s+\alpha_s) i_s)}$$

is a reduced covariant g-extensor, and

$$(4.9)\quad \Phi_l^{I(S)} = \binom{A_{s+1}+\alpha_{s+1}}{A_{s+1}} \cdots \binom{A_l+\alpha_l}{A_l} \sum_{(\alpha)} \frac{(l-s)!}{a_1!\, a_2! \cdots a_s!\, b_1!\, b_2! \cdots b_{l-2s+1}!}$$

$$\times\, U^{(\alpha_1 i_1}_{//(\alpha+A)(\alpha_1)} U^{\alpha_2 i_2}_{//(\alpha+A)(\alpha_2)} \cdots U^{\alpha_s i_s)}_{//(\alpha+A)(\alpha_s)}$$

$$(\alpha_1 = 0, 1, \cdots, M-A_1; \cdots; \alpha_l = 0, 1, \cdots, M-A_l)$$

is a reduced contravariant g-extensor,

where

$$U^{\alpha_1 i_1}_{//(\alpha+A)(\alpha_1)} = (\cdots((U^{\alpha_1 i_1}_{/(\alpha_{s+1}+A_{s+1})\, i_{s+1}} U^{\alpha_{s+1} i_{s+1}})_{/(\alpha_{s+2}+A_{s+2})\, i_{s+2}}$$

$$\times\, U^{\alpha_{s+2} i_{s+2}}) \cdots)_{/(\alpha_{s+\alpha_1}+A_{s+\alpha_1})\, i_{s+\alpha_1}} U^{\alpha_{s+\alpha_1} i_{s+\alpha_1}}, \quad \text{etc.}$$

We obtain the following contraction for $U^{\alpha i}$ and $\mathfrak{S}^H V_{\alpha i}$, the latter has been introduced in (1.18):

$$\mathfrak{S}^H V_{\alpha i} = H! \sum_{\nu=0}^{H} (-1)^\nu \binom{\alpha+\nu}{\alpha} \binom{M-\alpha-\nu}{M-H-\alpha} V^{(\nu)}_{(\alpha+\nu) i},$$

$$(4.10)\quad \rho^H_A = H! \sum_{\alpha=0}^{M-A-H} \binom{\alpha+A}{A} U^{\alpha i} \sum_{\nu=0}^{H} (-1)^\nu \binom{\alpha+A+\nu}{\alpha+A} \binom{M+\alpha-A-\nu}{M-H-\alpha-A} V^{(\nu)}_{(\alpha+A+\nu) i}$$

$$(H = 0, 1, \cdots, M;\ A = 1, 2, \cdots, M-H),$$

and applying this operation $l-1$ times, we obtain

$$(4.11)\quad \rho^{H_1 H_2}_{K_1 K_2} = \prod_{a=1}^{2} H_a! \sum_{\alpha_a=0}^{M-A_a+H_a} \binom{\alpha_a+A_a}{A_a} \sum_{\nu_a=0}^{H_a} (-1)^{\nu_a} \binom{\alpha_a+A_a+\nu_a}{\alpha_a+A_a}$$

$$\times \binom{M-\alpha_a-A_a-\nu_a}{M-H_a-\alpha_a-A_a} \left(V^{(\nu_1)}_{(\alpha_1+A_1+\nu_1) i_1} U^{\alpha_1 i_1} \right)^{(\nu_2)}_{,(\alpha_2+A_2+\nu_2) i_2} U^{\alpha_2 i_2},$$

$\cdots\cdots$, anb so on.

For $\mathfrak{Z} V_{\alpha i}$ and $\mathfrak{Y} V_{\alpha i}$, by means of the same manner as in the previous cases, we can find many reduced extensors out of more generalized g-extensors.

Bibliography

〔1〕 H. V. Craig, On tensor relative to the expoint transformation, *Amer. J. M.*, 59 (1937), 764-774.

〔2〕 H. V. Craig, On extensors and a Euclidian basis for higher order speces, *Amer. J. M.*, 61 (1939), 791-808.

〔3〕 H. V. Craig, Vector and tensor analysis, *New York* (1943).

〔4〕 H. V. Craig and W. T. Guy, Jr., Jacobian extensors, *Amer. J. M.*, 62 (1950), 229-246.

〔5〕 H. V. Craig, On the multiple parameter Jacodian extensor, *Tensor, N. S.*, 2 (1952), 27-35.

〔6〕 A. Kawaguchi, Eine Verallgemeinerung von Extensoren, *Monatsh. für Math. u. Phys.*, 48 (1939), 329-339.

〔7〕 A. Kawaguchi, Die Differentialgeometrie höherer Ordnung I, Erweiterte Koordinaten-Transformationen und Extensoren, *Jour. Fac. Sci. Hokkaido Imp. Univ.*, 9 (1940), 1-152.

〔8〕 A. Kawaguchi, Die Differentialgemetrie höherer Ordnung III, Erweiterte Parameter-Transformationen und P-Tensoren, *Jour. Fac. Sci. Hokkaido Imp. Univ.*, 10 (1941), 77-156.

〔9〕 A. Kawaguchi, Die Differentialgeometrie höherer Ordnung IV. Erweiterung der verallgemeinerten Rheonomtransformation von Flächenelementen höherer Ordnung und R_K-Extensoren, *Public. Math, Debrecen*, 7 (1960), 256-276.

〔10〕 M. Kawaguchi, A Generalization of the extensor, *Tensor, N. S.* 2 (1952), 59-66.

〔11〕 M. Kawaguchi, On a generalization of the mulitple parameter extensor, *Tensor, N. S.*, 2 (1952), 99-101.

〔12〕 M, Kawaguchi, On some theorems in the g-extensor analysis, *Tensor, N. S.*, 3 (1953), 46-52.

〔13〕 M. Kawaguchi, Sur les derivées covariantes des g-extensors, *Tensor, N. S.*, 11 (1961), 74-98.

Received on Dec, 4, 1965