On the Convergence of Stochastic Integrals. II*

By

T. SHINTANI**

Abstract

Stochastic integral converges always and it is the Riemann-Stieltjes integral defined on a probability space.

Let (Ω, a, P) be a probability space and $\{a_t\}_{t\geq 0}$ a family of fixed increasing sub-σ-fields of a. Throughout the pape consider a probability space $(\Omega, a, P; \{a_t\})$.

§ 1. Integral representations of a random variable, expectation and conditional expectation.

THEOREM 1 ([8]). Let X be a random variable. If X is integrable then

$X=\int_{-\infty}^{\infty} x\, dI_{\{X\leqq x\}}$. Here $I_{\{X\leqq x\}}$ denotes the indicator function of $\{X\leqq x\}$.

In fact, for $-\infty<\alpha<\beta<\infty$ the integral $\int_{\alpha}^{\beta} x\, dI_{\{X\leqq x\}}$ is defined as the strong limit of the Riemann-Stieltjes sum $\sum_{j=1}^{n} \xi_j(I_{\{X\leqq x_j\}}-I_{\{X\leqq x_{j-1}\}})$ as $\max_j(x_j-x_{j-1})$ tends to 0, where

$\alpha=x_0<x_1<\cdots<x_n=\beta$ and $x_{j-1}\leqq \xi_j\leqq x_j$.

Because,

$$\left|\int_{\alpha}^{\beta} x\, dI_{\{X\leqq x\}}-\sum_{j=1}^{n} \xi_j(I_{\{X\leqq x_j\}}-I_{\{X\leqq x_{j-1}\}})\right| \leqq \max_{1\leqq j\leqq n}(x_j-x_{j-1})。$$

When $\int_{\alpha}^{\beta} x\, dI_{\{X\leqq x\}}$ is strongly convergent as $\alpha\to-\infty$ and $\beta\to+\infty$, the limit will be denoted by $\int_{-\infty}^{\infty} x\, dI_{\{X\leqq x\}}$. If X is integrable then $X=\int_{-\infty}^{\infty} x\, dI_{\{X\leqq x\}}$.

Let $E(X)$ be the expectation of X. Then

THEOREM 2. $E(X) \underset{=\!=}{\text{def.}} \int_{\Omega} X\, dP$

$$= \int_{-\infty}^{\infty} x\, dP(X\leqq x).$$

Here $F(x)=P(X\leqq x)$ is the distribution function of a random variable X.

Let $E(X/B)$ be the conditional expectation of X relative to $B\subseteq a$. Then

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** 助教授，一般教科，数学

THEOREM 3.

(i) $\int_C X dP = \int_C E(X/B)\, dP$ for each $C \in B$.

(ii) $E(X/B) = \int_{-\infty}^{\infty} x\, dP(X \leqslant x/B)$.

Here $P(X \leqslant x/B) = P(X \in dx/B)$ is the conditional distribution of X relative to B. (See [3], p. 27, Theorem 9. 1 and [7], p. 174.)

COROLLARY. (i) If X is measurable B, then $E(X/B) = X$ a. e. and $E(XY/B) = X \cdot E(Y/B)$ a. e.

(ii) $P(X \leqslant x/\{\phi, \Omega\}) = P(X \leqslant x)$.

(iii) $E(X/\{\phi, \Omega) = E(X)$.

§ 2. Convergence Theorem of Stochastic Integrals.

Let $f = \{f(t),\ a_t,\ t \geqslant 0\}$ be a martingale with continuous sample paths and $v = \{v(t), a_t, t \geqslant 0\}$ a stochastic process where $v(t)$ is measurable a_t for each t.

Let $\Delta = \{\Delta_m\}$, where $\Delta_m = \{t_{m,k} : 0 = t_{m,0} < \cdots < t_{m,r} < t\}$ be a sequence of partitions of $[0, t]$ with $|\Delta| = \operatorname*{Max}_k\ (t_{m,k+1} - t_{m,k}) \to 0\ (m \to \infty)$ and $d(t_{m,k}) = f(t_{m,k+1}) - f(t_{m,k})\ \ (r > k \geqslant 0) =$

$$f(t) \quad - f(t_{m,r})\ \ (t_{m,k} \leqslant t,\ k = r)$$

denotes the increment for f. Let $\xi_{m,k} \in [t_{m,k},\ t_{m,k+1}]$, $k \geqslant 0$. $\|f\|_1$ is the L^1-norm of f.

LEMMA 1 ([11], [13]). For $f = \{f(s),\ 0 \leqslant s \leqslant t\}$, $t \geqslant 0$ and $\varepsilon > 0$ there exists an L^∞ martingale $f^{(\varepsilon)} = \{f^{(\varepsilon)}(s),\ 0 \leqslant s \leqslant t\}$ such that $\|f(s) - f^{(\varepsilon)}(s)\|_1 < \varepsilon$ for all s, $0 \leqslant s \leqslant t$.

LEMMA 2 ([12], [13]). Let $\{A_t,\ a_t,\ t \geqslant 0\}$ be an increasing continuous process. For $v = \{v(t), a_t,\ t \geqslant 0\}$ and $\varepsilon > 0$ there exists a continuous (and of locally bounded variation) process $v^{(\varepsilon)} = \{v^{(\varepsilon)}(t), a_t,\ t \geqslant 0\}$ such that

$$E\left[\int_\alpha^\beta |v(t) - v^{(\varepsilon)}(t)|^p\, dA_t\right] < \varepsilon\ (p \geqslant 1) \text{ where } 0 \leqslant \alpha \leqslant \beta.$$

The existence of Stochastic integrals is shown by the following theorem.

Convergence Theorem ([11], [13]). For v and f, when $m \to \infty$,

$$\theta_m = \sum_K v(\xi_{m,K}) \cdot d(t_{m,K})$$

converges in L^p and a. e. if $p > 1$ and it converges a. e. if $p = 1$. The limit $\lim_{m \to \infty} \theta_m$ defines a **stochastic integral** which will be denoted by $\int_0^t v(s)\ df(s)$.

PROOF. (I). Case p > 1. By Lemma 2 there is a continuous (so that uniformly continuous) process $v^{(\varepsilon)}(s)$ such that $E\left[\int_0^t |v(s) - v^{(\varepsilon)}(s)|^p\, d\langle f\rangle_s\right] < \varepsilon^p$.

Let $\theta_m^{(\varepsilon)} = \sum_K v^{(\varepsilon)}(\xi_{m,K}) \cdot d(t_{m,K})$. It may be supposed that $p = 2$.

$$\begin{aligned}
\text{(a)} \qquad E[E((f(t) - f(s))^2/a_t)] &= E[(f(t) - f(s))^2] \\
&= E[E((f(t) - f(s))^2/a_s)] \\
&= E[E(\langle f\rangle_t - \langle f\rangle_s/a_s)] \\
&= E[E(\langle f\rangle_t - \langle f\rangle_s/a_t)] \quad (t > s).
\end{aligned}$$

Similarly, from $E(d(t_{\cdot,K}) \cdot d(t_{\cdot,l})/a_{t\cdot,l}) = 0\ \ (k > l)$

(b) $\quad E[E(d(t_{\cdot,K}) \cdot d(t_{\cdot,l})/a_{t\cdot,K+1})] = 0 \quad (k > l)$.

$$\lim_{m\to\infty} E[|\theta_m - \theta_m^{(\varepsilon)}|^2] = \lim_{m\to\infty} E[\sum_K \{v(\xi_{m,K}) - v^{(\varepsilon)}(\xi_{m,K})\}^2 \cdot d(t_{m,K})^2] \quad \text{(by (a), (b))}$$
$$= E\left[\int_0^t |v(s) - v^{(\varepsilon)}(s)|^2\, d\langle f\rangle_s\right] < \varepsilon^2.$$
$$\lim_{m,n\to\infty} E[|\theta_m^{(\varepsilon)} - \theta_n^{(\varepsilon)}|^2] = \lim_{m,n\to\infty} E[|\sum_K \{v^{(\varepsilon)}(\xi_{m;K}) - v^{(\varepsilon)}(\xi_{n;K})\}\cdot d(t_{.,K})|^2]$$
$$\leqslant K.\lim_{|\Delta|\to 0} E^{1/2}[\sup |v^{(\varepsilon)}(\xi_{m;K}) - v^{(\varepsilon)}(\xi_{n;K})|^4] \quad (K > 0 \text{ is constant})$$
$$= 0 \qquad (v^{(\varepsilon)}(s) \text{ is uniformly continuous}).$$

So

$$\lim_{m,n\to\infty} E^{1/2}[|\theta_m - \theta_n|^2] \leqslant \lim_{m\to\infty} 2 \cdot E^{1/2}[|\theta_m - \theta_m^{(\varepsilon)}|^2] + \lim_{m,n\to\infty} E^{1/2}[|\theta_m^{(\varepsilon)} - \theta_n^{(\varepsilon)}|^2]$$
$$< 2\varepsilon + 0 \qquad \text{for all} \quad \varepsilon > 0.$$

Since θ_m converges in L^2, θ_m converges in probability. Then, by using *K.* Itô' s method, it is shown that θ_m converges a. e. as follows.

Since θ_m converges in probability, for sufficiently large m and n, $P(|\theta_m - \theta_n| > \varepsilon) < 1/2$ uniformly in m and n. For $\varepsilon > 0$ take a large m. For all n, by Ottaviani's inequality,

$$P(\max_{1\leq p\leq n} |\theta_{m+p} - \theta_m| > 2\,\varepsilon) \leqslant 2\cdot P(|\theta_{m+n} - \theta_m| > \varepsilon)$$
$$\leqslant 2\cdot \sup_n P(|\theta_{m+n} - \theta_m| > \varepsilon).$$

So

$$P(\sup_{p,q} |\theta_{m+p} - \theta_{m+q}| > 4\,\varepsilon) \leqslant 4\cdot \sup_n P(|\theta_{m+n} - \theta_m| > \varepsilon).$$

Since θ_m converges in probability, when $m \to \infty$

$$P(\lim_{m\to\infty} \sup_{p,q} |\theta_{m+p} - \theta_{m+q}| > 4\,\varepsilon) \leqslant 4\cdot \sup_n P(|\theta_{m+n} - \theta_m| > \varepsilon) \to 0$$

as the desired result.

(II). Case $p=1$. By Lemma 1, for f, $t \geqslant 0$ and $\varepsilon > 0$ there is an L^∞ martingale $f^{(\varepsilon)}$ such that $\|f(s) - f^{(\varepsilon)}(s)\|_1 < \varepsilon^2, 0 \leqslant s \leqslant t$.

Let $d^{(\varepsilon)}$ be the increment for $f^{(\varepsilon)}$ and let $\theta_m^{(\varepsilon)} = \sum_K v(\xi_{m,K})\cdot d^{(\varepsilon)}(t_{m,K})$

Then, from the weak L^1-inequality of Burkholder (See [1], [2], [10], [11]) it follows that

$$P(\lim_{m,n\to\infty} \sup |\theta_m - \theta_n| > 3\,\varepsilon)$$
$$\leqslant 2\cdot P(\sup_S |\sum_K^S v(\xi_{m,K})\cdot[d(t_{m,K}) - d^{(\varepsilon)}(t_{m,K})]| > \varepsilon) + P(\lim_{m,n\to\infty} \sup |\theta_m^{(\varepsilon)} - \theta_n^{(\varepsilon)}| > \varepsilon)$$
$$\leqslant 2\,C\cdot\varepsilon^{-1}\cdot\|f - f^{(\varepsilon)}\|_1 + 0 \qquad (C \text{ is constant})$$
$$< 2\,C\cdot\varepsilon \qquad \text{for all } \varepsilon > 0.$$

This completes the proof of Theorem.

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