Complements for On the convergence of Stochastic Integrals

T. SHINTANI*

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1) In the proof of theorem (p. 172), of course, it may be supposed that an L^2 martingale f is an L^∞ martingale. In fact, by the similar method to lemma 1 (p. 171), for L^p martingale f and $\varepsilon > 0$ there is a uniformly bounded L^∞ martingale $f^{(\varepsilon)}$ such that $\| f(s) - f^{(\varepsilon)}(s) \|_p < \varepsilon \,(0 \leqslant s \leqslant t)$ when p $\geqslant$ 1.

Let $\theta_m^{(\varepsilon)} := \sum_K v(t_{m,k}) \cdot d^{(\varepsilon)}(t_{m,k})$, $d^{(\varepsilon)}(t_{m,k}) := f^{(\varepsilon)}(t_{m,k+1}) - f^{(\varepsilon)}(t_{m,k})$.

The proof of theorem shows that when $m \to \infty$ $\theta_m^{(\varepsilon)}$ converges in L^2.

Here recall that $\theta_m := \sum_K v(t_{m,k}) \cdot d(t_{m,k})$, $d(t_{m,k}) := f(t_{m,k+1}) - f(t_{m,k})$.

Let $p = 2$ then

$$\begin{aligned}
\lim_{m,n\to\infty} \| \theta_m - \theta_n \|_2 &\leqslant \lim_{m\to\infty} 2 \cdot \| \theta_m - \theta_m^{(\varepsilon)} \|_2 + \lim_{m,n\to\infty} \| \theta_m^{(\varepsilon)} - \theta_n^{(\varepsilon)} \|_2 \\
&= 2 \cdot \lim_{m\to\infty} \| \theta_m - \theta_m^{(\varepsilon)} \|_2 + 0 \\
&\leqq 2 \cdot \lim_{m\to\infty} c \cdot \| \sum_K [d(t_{m,k}) - d^{(\varepsilon)}(t_{m,k})] \|_2 \qquad (c > 0 \text{ is constant}) \\
&= 2c \cdot \| f(t) - f^{(\varepsilon)}(t) \|_2 \\
&< 2c \cdot \varepsilon \quad \text{for } \underline{\text{all}} \ \varepsilon > 0.
\end{aligned}$$

2) Let $1 < p \leqslant 2$ then by the same method to above 1)

$$\lim_{m,n\to\infty} \| \theta_m - \theta_n \|_p < 2c \cdot \varepsilon \qquad \text{for } \underline{\text{all}} \ \ \varepsilon > 0 \quad (c > 0 \text{ is constant}).$$

3) Let $p > 2$.

$$E[| \theta_m |^p] \leqslant c_p \cdot E[| f |^p] < \infty, \qquad E[| \theta_m^{(\varepsilon)} |^p] \leqslant c'_p \cdot E[| f^{(\varepsilon)} |^p] \leqslant K$$

($c_p > 0$, $c'_p > 0$ are constant. $K > 0$ is also constant since $f^{(\varepsilon)}$ is uniformly bounded.) As $\theta_m^{(\varepsilon)}$ converges in L^2, $\theta_m^{(\varepsilon)}$ converges in L^1.

Since

$$\begin{aligned}
E^{1/2}[| \theta_m^{(\varepsilon)} - \theta_n^{(\varepsilon)} |^{2p-1}] &= (\| \theta_m^{(\varepsilon)} - \theta_n^{(\varepsilon)} \|_{2p-1})^{2p-1}/_2 \\
&\leqslant (\| \theta_m^{(\varepsilon)} \|_{2p-1} + \| \theta_n^{(\varepsilon)} \|_{2p-1})^{2p-1}/_2 \qquad (2p-1 > 1) \\
&\leqslant (2 \cdot \sqrt[2p-1]{C'_{2p-1}} \cdot \| f^{(\varepsilon)} \|_{2p-1})^{2p-1}/_2 \\
&\leqslant K' \ \ (K' > 0 \text{ is constant}),
\end{aligned}$$

$$\begin{aligned}
E[| \theta_m^{(\varepsilon)} - \theta_n^{(\varepsilon)} |^p] &= E[| \theta_m^{(\varepsilon)} - \theta_n^{(\varepsilon)} |^{1/2} \cdot | \theta_m^{(\varepsilon)} - \theta_n^{(\varepsilon)} |^{p-\frac{1}{2}}] \\
&\leqslant E^{1/2}[| \theta_m^{(\varepsilon)} - \theta_n^{(\varepsilon)} |] \cdot E^{1/2}[| \theta_m^{(\varepsilon)} - \theta_n^{(\varepsilon)} |^{2p-1}]
\end{aligned}$$

* 助教授　一般教科数学

$$\leqslant K' \cdot E^{1/2}[| \theta_m^{(\varepsilon)} - \theta_n^{(\varepsilon)} |] \longrightarrow 0 \quad \text{as } m, n \to \infty.$$

So, by the same method to 1), $\lim_{mn\to\infty} \| \theta_m - \theta_n \|_p = 0.$

4) Let $p = 1$. From the proof for $p = 2$ (p. 172), $\theta_m^{(\varepsilon)}$ converges a. e. Let $\| f - f^{(\varepsilon)} \|_1 < \varepsilon^2$.

$$P(\limsup_{m,n\to\infty} | \theta_m - \theta_n | > 3\varepsilon)$$

$$\leqslant 2 \cdot P(\sup_S | \sum_{K=0}^{S} v(t_{m,k}) [d(t_{m,k}) - d^{(\varepsilon)}(t_{m,k})] | > \varepsilon)$$

$$+ P(\limsup_{m,n\to\infty} | \theta_m^{(\varepsilon)} - \theta_n^{(\varepsilon)} | > \varepsilon)$$

$$\leqslant 2_c \cdot \varepsilon^{-1} \cdot \| f - f^{(\varepsilon)} \|_1 + 0 \quad (c > 0 \text{ is constant})$$

$$< 2_c \cdot \varepsilon \qquad \text{for } \underline{\text{all}}\ \varepsilon > 0.$$

REFERENCE

T. Shintani, On the convergence of Stochastic Integrals, Memoires of Tomakomai Technical College 16 (1981), 171-172.