A complement for On the convergence of Stochastic Integrals. II

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In the proof of the convergence theorem ([4], p. 129), of course, it may be supposed, as [3], that L^2 martingale f is an L^∞ martingale. In fact, by the similar method to Lemma 1, for L^2 martingale f and $\varepsilon > 0$ there is an L^∞ martingale g such that $\| f(s) - g(s) \|_2 < \varepsilon$ $(0 \leqslant s \leqslant t)$. The proof of the convergence theorem shows that when $m \to \infty$

$$\tilde{\theta}_m := \sum_K v(\xi_{m,k}) \cdot [g(t_{m,k+1}) - g(t_{m,k})]$$

converges in L^2. Here, recall $\theta_m := \sum_K v(\xi_{m,k}) \cdot [f(t_{m,k+1}) - f(t_{m,k})]$.

$$\lim_{m,n\to\infty} \| \theta_m - \theta_n \|_2 \leqslant 2 \cdot \lim_{m\to\infty} \| \theta_m - \tilde{\theta}_m \|_2 + \lim_{m,n\to\infty} \| \tilde{\theta}_m - \tilde{\theta}_n \|_2$$

$$= 2 \cdot \lim_{m\to\infty} \| \theta_m - \tilde{\theta}_m \|_2 + 0$$

(from the proof of convergence theorem [4], p. 129)

$$\leqslant 2c \cdot \lim_{m\to\infty} \| \sum_K [\{f(t_{m,k+1}) - f(t_{m,k})\} - \{g(t_{m,k+1}) - g(t_{m,k})\}] \|_2$$

(by an inequality of Burkholder [1], p. 858 and [2], p. 592)

$$= 2c \cdot \lim_{m\to\infty} \| f(t) - g(t) \|_2 \qquad (c > 0 \text{ is constant})$$

$$< 2c \cdot \varepsilon \qquad \text{for all } \varepsilon > 0.$$

The convergence theorem was established.

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Remark on the 1 st line of ([1], p. 129)

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Remark. From the proof of lemma 2 ([1], p. 128 and [2], p. 189), for a stochastic process $\{v(t)\}$, there is a sequence of continuous process $\{v_n(t)\}$ such that $\lim_{n\to\infty}\int_o^t |v_n(t)-v(t)|^p dt=0$ $(p\geqslant 1)$ for almost all $\omega\in\Omega$. This implies that if $v(t)$ is measurable then, for every $s\in[0,t]$, $v_n(t)$ is measurable a_s (define $v_n(t)$ by $v_n(t,\omega)\cdot I_{[0,s_n]}(t)(s_n=s,\ s_{n+1}\uparrow\infty)$ and so that there is a sequence of such measurable functions $\{v_n(t)\}$ satisfying $v_n(t)\to v(t)$ for almost all $\omega\in\Omega$ as $n\to\infty$. So "if $E(X/a_s)=E(Y/a_s)$ then $E(v(t)\cdot X/a_s)=E(v(t)\cdot Y/a_s)$"$\cdot$In fact, since $v(t)$ is measurable a_s $(s\in[0,t])$,

$$E(v_n(t)\cdot X/a_s)=v_n(t)\cdot E(X/a_s)=v_n(t)\cdot E(Y/a_s)=E(v_n(t)\cdot Y/a_s)$$

and $v_n(t)\cdot X\to v(t)\cdot X$ a.e., $v_n(t)\cdot Y\to v(t)\cdot Y$ a.e. as $n\to\infty$ so, by Lebesgue's convergence theorem,

$$E(v(t)\cdot X/a_s)=\lim_{n\to\infty}E(v_n(t)\cdot X/a_s)=\lim_{n\to\infty}E(v_n(t)\cdot Y/a_s)=E(v(t)\cdot Y/a_s)$$

as the desired result.

So 1): from $E(d(t_{m,k})^2/a_{t_{m,k}}]=E[\langle f\rangle_{t_{m,k+1}}-\langle f\rangle_{t_{m,k}}/a_{t_{m,k}}]$ it follows that

$$E[\{v(\xi_{m,k})-v^{(\varepsilon)}(\xi_{m,k})\}^2\cdot d(t_{m,k})^2/a_{t_{m,k}}]=E[\{v(\xi_{m,k})-v^{(\varepsilon)}(\xi_{m,k})\}^2\cdot(\langle f\rangle_{t_{m,k+1}}-\langle f\rangle_{t_{m,k}})/a_{t_{m,k}}]$$

and 2): from $E[d(t_{m,k})\cdot d(t_{m,l})/a_{t_{m,l}}]=0$ $(k>l)$ it follows that

$$E[v(\xi_{m,k})\cdot v(\xi_{m,l})\cdot d(t_{m,k})\cdot d(t_{m,l})/a_{t_{m,l}}]=0\quad(k>l).$$

Therefore

$$\begin{aligned}
&\lim_{m\to\infty}E[|\theta_m-\theta_m^{(\varepsilon)}|^2]\\
&=\lim_{m\to\infty}[\sum_K E(E[\{v(\xi_{m,k})-v^{(\varepsilon)}(\xi_{m,k})\}^2\cdot d(t_{m,k})^2/a_{t_{m,k}}])\\
&\qquad+2\cdot\sum_{k>l}E(E[\{v(\xi_{m,k})-v^{(\varepsilon)}(\xi_{m,k})\}\{v(\xi_{m,l})-v^{(\varepsilon)}(\xi_{m,l})\}\cdot d(t_{m,k})\cdot d(t_{m,l})/a_{t_{m,l}}])]\\
&=\lim_{m\to\infty}[\sum_K E(E[\{v(\xi_{m,k})-v^{(\varepsilon)}(\xi_{m,k})\}^2\cdot(\langle f\rangle_{t_{m,k+1}}-\langle f\rangle_{t_{m,k}})/a_{t_{m,k}}])]\\
&=\lim_{m\to\infty}E[\sum_K\{v(\xi_{m,k})-v^{(\varepsilon)}(\xi_{m,k})\}^2\cdot(\langle f\rangle_{t_{m,k+1}}-\langle f\rangle_{t_{m,k}})]\\
&=E[\int_o^t|v(s)-v^{(\varepsilon)}(s)|^2 d\langle f\rangle_s]<\varepsilon^2
\end{aligned}$$

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Added in proof. From the proof of Convergence Theorem ([1], pp. 128-129) for $p=2$ it holds that $\int_o^t v(s)\,dB(s)=$ Itô integral (=a martingale) so that the weak L^1-inequality of ([4], p. 858) holds without "predictable" as follows.

Let $v = (v_1, v_2, \cdots)$ be a stochastic process (i. e. a sequence of measurable functions), let $f = (f_1, f_2, \cdots)$ be a martingale with difference sequence $d = (d_1, d_2, \cdots)$ and let $g = (g_1, g_2, \cdots)$, defined by $g_n = \sum_{K=1}^{n} v_k \cdot d_k$, be the transform of the martingale f under v. The L^1-norm of f is $\|f\|_1 = \sup_n \|f_n\|_1$ and the maximal function of g is defined by $g^*(\omega) = \sup_n |g_n(\omega)|$.
Then $\lambda \cdot P(g^* > \lambda) \leqslant c \cdot \|f\|_1$, $\lambda > 0$, ($c > 0$ is constant) as [4] so that

$\|\sum_{K=1}^{n} v_k \cdot d_k\|_p \leqslant c_p \cdot \|\sum_{K=1}^{n} d_k\|_p$ ($c_p > 0$ is constant), $1 < p < \infty$ as ([4] and [5], pp. 1000-1002).

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Let

(Ω, a, P) : a probability space

$\{a_t, 0 \leqslant t \leqslant 1\}$: an increasing family of sub-σ-fields of a

$\{f(t), 0 \leqslant t \leqslant 1\}$: an L^p martingale adapted to $\{a_t, 0 \leqslant t \leqslant 1\}$ ($p \geqslant 1$)

$\{v(t), 0 \leqslant t \leqslant 1\}$: a stochastic process with $\sup_t |v(t)| < \infty$ a. e.

$\Delta = \{0 = t_0 < t_1 < \cdots < 1\}$: any partition of [0, 1], $|\Delta| = \max_K (t_{k+1} - t_k)$, $\xi_k \in [t_k, t_{k+1}]$.

Then we have the followings:

1) For $\{f(t), 0 \leqslant t \leqslant 1\}$ and $\varepsilon > 0$ there is a uniformly bounded L^∞ martingale $\{f^{(\varepsilon)}(t), 0 \leqslant t \leqslant 1\}$ such that $\|f(t) - f^{(\varepsilon)}(t)\|_p < \varepsilon$ ($p \geqslant 1$),
2) Let $\{A_t, 0 \leqslant t \leqslant 1\}$ be an increasing continuous process. For $\{v(t), 0 \leqslant t \leqslant 1\}$ and $\varepsilon > 0$ there is a continuous stochastic process $\{v^{(\varepsilon)}(t), 0 \leqslant t \leqslant 1\}$ such that $E[\int_0^1 |v(t) - v^{(\varepsilon)}(t)|^p dA_t] < \varepsilon$ ($p \geqslant 1$),
3) Doob-Meyer decomposition ($p = 2$),
4) the weak L^1-inequality for martingale transform (without predicatable) ($p = 1$),
5) L^p-inequality (without predictable):

$$\|\sum_n v(\xi_n) \cdot [f(t_{n+1}) - f(t_n)]\|_p \leqslant c_p \cdot \|\sum_n [f(t_{n+1}) - f(t_n)]\|_p \qquad (p > 1).$$

Here the convergence problem is solved, i. e., the following convergence theorem is established:

THEOREM $\lim_{|\Delta| \to 0} \sum_n v(\xi_n) \cdot [f(t_{n+1}) - f(t_n)]$ exists in the convergence in L^p and a. e. for $p > 1$ and it exists in almost everywhere convergence for $p = 1$.
The limit defines a stochastic integral that is denoted by $\int_0^1 v(t)\, df(t)$.