Stochastic Integral

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Abstract

Let $\{v(t)\}$ be a stochastic process and let $\{f(t)\}$ be an L^2 integrable martingale. Then the stochastic integral $\int_0^1 v(t)\,df(t)$ is defined.

1. Introduction

Let (Ω, a, P) be a probability space and let $\{a_t\}_{t\geq 0}$ be an increasing family of sub-σ-fields of a. E denotes expectation and $E(X/\beta)$ denotes conditional expectation of a random variable X with respect to a sub-σ-field β of a. $f=\{f(t),\ 0\leq t\leq 1\}$ is an L^2 integrable martingale on a probability space $(\Omega, a, \{a_t\}, P)$, i. e., f is L^2 integrable and $E(f(t)/a_s)=f(s)$ for $t>s\geq 0$. Let $v=\{v(t),\ 0\leq t\leq 1\}$ be a stochastic process on $(\Omega, a, \{a_t\}, P)$ such that $v(t)$ is measurable a_t for each $t\geq 0$. Suppose $v^*=\sup_t|v(t)|<\infty$ a. e.. Let $\Delta=\{\Delta_m\}$, where $\Delta_m=\{t_{m,k}:0=t_{m,0}<\cdots<t_{m,s}\leq 1\}$, be a sequence of partitions of $[0, 1]$ with $|\Delta_m|=\operatorname{Max}_k(t_{m,k+1}-t_{m,k})\to 0$ as $m\to\infty$ and $d(t_{m,k})=f(t_{m,k+1})-f(t_{m,k})$ denotes the increments for f. $\|f\|_2$ is the L^2-norm of f.

In this paper the following theorem is proved.

Theorem. $\lim_{m\to\infty}\sum_k v(t_{m,k})\cdot d(t_{m,k})$ exists in the convergence in L^2. The limit defines the stochastic integral $\int_0^1 v(t)\,df(t)$.

2. Proof of the Theorem

Lemma 1. *Let* $\{A_t,\ 0\leq t\leq 1\}$ be an integrable, increasing and contiouous process.
For $\{v(t),\ a_t,\ 0\leq t\leq 1\}$ and $\varepsilon>0$ there is a *continuous* stochastic process $\{v^{(\varepsilon)}(t),\ a_t,\ 0\leq t\leq 1\}$ such that

$E\left(\int_0^1\left|v(t)-v^{(\varepsilon)}(t)\right|^p dA_t\right)<\varepsilon$ $(p\geq 1)$. The lemma is known.

Lemma 2. (Doob-Meyer decomposition) Let $\{f(t)\}$ be an L^2 integrable martingale then there is an increasing and integrable process $\langle f\rangle_t$ such that $E((f(t)-f(s))^2/a_s)=E(\langle f\rangle_t-\langle f\rangle_s/a_s)$ for $t>s\geq 0$.
Let $\varepsilon>0$. For $\{v(t)\}$, there is a continuous stochastic process $\{v^{(\varepsilon)}(t)\}$ such that

$$E\left(\int_0^1\left|v(t)-v^{(\varepsilon)}(t)\right|^2 d\langle f\rangle_t\right)<\varepsilon^2.$$

Let $\theta_m=\sum_k v(t_{m,k})\cdot d(t_{m,k})$ and let $\theta_m^{(\varepsilon)}=\sum_k v^{(\varepsilon)}(t_{m,k})\cdot d(t_{m,k})$.

Since $E(d(t_{m,k})^2/a_{t_{m,k}})=E(\langle f\rangle_{t_{m,k+1}}-\langle f\rangle_{t_{m,k}}/a_{t_{m,k}})$ and $E(d(t_{m,k})\cdot d(t_{m,l})/a_{t_{m,l}})=0$ $(k>l)$,

$$\lim_{m\to\infty}E[|\theta_m-\theta_m^{(\varepsilon)}|^2]=\lim_{m\to\infty}E(|\sum_k\{v(t_{m,k})-v^{(\varepsilon)}(t_{m,k})\}\cdot d(t_{m,k})|^2)$$

$$=\lim_{m\to\infty}\Big[\sum_k E(E(\{v(t_{m,k})-v^{(\varepsilon)}(t_{m,k})\}^2\cdot d(t_{m,k})^2/a_{t_{m,k}}))$$

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$$+2\cdot\sum_{k>l}E(E(\{v(t_{m,k})-v^{(\varepsilon)}(t_{m,k})\}\{v(t_{m,l})-v^{(\varepsilon)}(t_{m,l})\}\cdot d(t_{m,k})\cdot d(t_{m,l})/a_{t_{m,l}}))\Big]$$

$$=\lim_{m\to\infty}\Big[\sum_k E(E(\{v(t_{m,k})-v^{(\varepsilon)}(t_{m,k})\}^2\cdot(\langle f\rangle_{t_{m,k+1}}-\langle f\rangle_{t_{m,k}})/a_{t_{m,k}}))+0\Big]$$

$$=\lim_{m\to\infty}E(\sum_k\{v(t_{m,k})-v^{(\varepsilon)}(t_{m,k})\}^2\cdot(\langle f\rangle_{t_{m,k+1}}-\langle f\rangle_{t_{m,k}}))$$

$$=E\Big(\int_0^1\Big|v(t)-v^{(\varepsilon)}(t)\Big|^2 d\langle f\rangle_t\Big)<\varepsilon^2.$$

$$\lim_{m,n\to\infty}E\Big[\Big|\theta_m^{(\varepsilon)}-\theta_n^{(\varepsilon)}\Big|^2\Big]$$

$$=\lim_{m,n\to\infty}E\Big[\Big|\sum_k\{v^{(\varepsilon)}(t_{m',k})-v^{(\varepsilon)}(t_{n',k})\}\cdot d(t_{\cdot,k})\Big|^2\Big]\text{(Here }\Delta.\text{ is the union of }\Delta_m\text{ and }\Delta_n)$$

$$=\lim_{m,n\to\infty}E\Big[\sum_k\{v^{(\varepsilon)}(t_{m',k})-v^{(\varepsilon)}(t_{n',k})\}^2\cdot d(t_{\cdot,k})^2\Big]$$

$$=\lim_{m,n\to\infty}E\Big[\sum_k\{v^{(\varepsilon)}(t_{m',k})-v^{(\varepsilon)}(t_{n',k})\}^2\cdot(\langle f\rangle_{t_{\cdot,k+1}}-\langle f\rangle_{t_{\cdot,k}})\Big]$$

$$\le\lim_{m,n\to\infty}E\Big[\sup_k\Big|v^{(\varepsilon)}(t_{m',k})-v^{(\varepsilon)}(t_{n',k})\Big|^2\cdot\sum_k(\langle f\rangle_{t_{\cdot,k+1}}-\langle f\rangle_{t_{\cdot,k}})\Big]$$

$$=\lim_{m,n\to\infty}E\Big[\sup_k\Big|v^{(\varepsilon)}(t_{m',k})-v^{(\varepsilon)}(t_{n',k})\Big|^2(\langle f\rangle_1-\langle f\rangle_0)\Big]$$

$$=0.$$

In fact, it is shown as follows. Let $m\to\infty$ then $m'\to\infty$ and $|\Delta_{m'}|\ge|\text{union of }\Delta_{m'}\text{ and }\Delta_{n'}|\to 0$ so if $m\to\infty$ then $\sup_k|t_{m',k}-t_{n',k}|\le\sup_k|t_{m',k}-t_{m',k+1}|+\sup_k|t_{m',k+1}-t_{n',k}|\to 0$ since $t_{m',k}$, $t_{m',k+1}\in\Delta_{m'}$ and $t_{m',k+1}$, $t_{n',k}\in$ union of $\Delta_{m'}$ and $\Delta_{n'}$. So if $m\to\infty$ then $\sup_k\Big|v^{(\varepsilon)}(t_{m',k})-v^{(\varepsilon)}(t_{n',k})\Big|^2\to 0$ since $v^{(\varepsilon)}(t)$ is uniformly continuous. Suppose $|v^{(\varepsilon)}|\le c$ uniformly then there is a constant $C>0$ such that $\sup_k\Big|v^{(\varepsilon)}(t_{m',k})-v^{(\varepsilon)}(t_{n',k})\Big|^2\le C$ and notice that $\langle f\rangle_t$ is integrable.

By Lebesgue's dominated convergence theorem

$$\lim_{m,n\to\infty}E\Big[\sup_k\Big|v^{(\varepsilon)}(t_{m',k})-v^{(\varepsilon)}(t_{n',k})\Big|^2\cdot(\langle f\rangle_1-\langle f\rangle_0)\Big]$$

$$=\lim_{m\to\infty}E\Big[\sup_k\Big|v^{(\varepsilon)}(t_{m',k})-v^{(\varepsilon)}(t_{n',k})\Big|^2\cdot(\langle f\rangle_1-\langle f\rangle_0)\Big]$$

$$=0.$$

This holds on $\{|v^{(\varepsilon)}|\le c\}$ for *all* $c>0$ so it holds on $\Big\{\sup_t\Big|v^{(\varepsilon)}(t)\Big|<\infty\Big\}=\Omega$ since $v^{(\varepsilon)}(t)$ is continuous.

So

$$\lim_{m,n\to\infty}\Big\|\theta_m-\theta_n\Big\|_2\le\lim_{m\to\infty}2\cdot\Big\|\theta_m-\theta_m^{(\varepsilon)}\Big\|_2+\lim_{n,m\to\infty}\Big\|\theta_m^{(\varepsilon)}-\theta_n^{(\varepsilon)}\Big\|_2$$

$$<2\varepsilon+0\text{ for }all\ \varepsilon>0$$

on $\{|v|\le c\}$ for *all* $c>0$ so it holds if $v^*<\infty$ a.e..

This completes the proof of theorem.

References

1) D. L. Burkholder, Martingale transforms. Ann. Math. Statist. 37(1966), 1494-1504.

2) J. L. Doob, Stochastic processes. Wiley, New York 1953.

3) K. Itô, Stochastic integral. Proc. Imp. Acad. Tokyo 20(1944), 519-524.

4) H. Kunita and S. Watanabe, On square integrable martingales. Nagoya Math. J., 30(1967), 209-245.

5) T. Shintani and T. Ando, Best approximants in L^1 space. Z. Wahrscheinlichkeitstheorie verw. Gebiete 33(1975), 33-39.

Notes on "Stochastic integral"

1) $E(\{v(t_{m,k})-v^{(\varepsilon)}(t_{m,k})\}^2\cdot d(t_{m,k})^2/a_{t_{m,k}})$
$=\{v(t_{m,k})-v^{(\varepsilon)}(t_{m,k})\}^2\cdot E(d(t_{m,k})^2/a_{t_{m,k}})$
$=\{v(t_{m,k})-v^{(\varepsilon)}(t_{m,k})\}^2\cdot E(\langle f\rangle_{t_{m,k+1}}-\langle f\rangle_{t_{m,k}}/a_{t_{m,k}})$
$=E(\{v(t_{m,k})-v^{(\varepsilon)}(t_{m,k})\}^2\cdot(\langle f\rangle_{t_{m,k+1}}-\langle f\rangle_{t_{m,k}})/a_{t_{m,k}})$

2) $k>l$
$E(\{v(t_{m,k})-v^{(\varepsilon)}(t_{m,k})\}\{v(t_{m,l})-v^{(\varepsilon)}(t_{m,l})\}\cdot d(t_{m,k})\cdot d(t_{m,l})/a_{t_{m,l}})$
$=E(\{v(t_{m,k})-v^{(\varepsilon)}(t_{m,k})\}\{v(t_{m,l})-v^{(\varepsilon)}(t_{m,l})\}\cdot E(d(t_{m,k})/a_{t_{m,k}})\cdot d(t_{m,l})/a_{t_{m,l}})$
$=0$

3) Since $\{a_t\}_{t\geq 0}$ is an increasing family, without loss of generality, we may suppose that a continuous function $v^{(\varepsilon)}$ is measurable with respect to every sub-σ-field a_t.

4) $\left|\sum_k\{v(t_{m,k})-v^{(\varepsilon)}(t_{m,k})\}^2\cdot(\langle f\rangle_{t_{m,k+1}}-\langle f\rangle_{t_{m,k}})\right|$
$\leq\sup_k\left|v(t_{m,k})-v^{(\varepsilon)}(t_{m,k})\right|^2\cdot\sum_k(\langle f\rangle_{t_{m,k+1}}-\langle f\rangle_{t_{m,k}})$
$\leq 4c^2(\langle f\rangle_1-\langle f\rangle_0)\in L^1$ on $\{|v|\leq c\}$

By Lebesgue's dominated convergence theorem
$\lim_{m\to\infty}E(\sum_k\{v(t_{m,k})-v^{(\varepsilon)}(t_{m,k})\}^2\cdot(\langle f\rangle_{t_{m,k+1}}-\langle f\rangle_{t_{m,k}})$
$=E(\lim_{m\to\infty}\sum_k\{v(t_{m,k})-v^{(\varepsilon)}(t_{m,k})\}^2(\langle f\rangle_{t_{m,k+1}}-\langle f\rangle_{t_{m,k}}))$
$=E\left(\int_0^1\left|v(t)-v^{(\varepsilon)}(t)\right|^2 d\langle f\rangle_t\right)$