A Boundary Element Formulation of Semi-Infinite Elastic Body with Gravitational Losds

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Abstract

A boundary element formulation of two dimensional semi-infinite elastic body with constant gravitational loads is introduced precisely according to the procedure presented by Jiang[1] to obtain explicit form for computer program.

1. Notation

u_j : displacement vector
p_j : traction vector
b_j : body force vector
u^*_{ij}, p^*_{ij} : tensors corresponding to the fundamental solutions
Γ : boundary of the body and tunnel
Ω : domain of the body and tunnel
n, n_k, n_k' : outward normal vector
G_{ij} : Galerkin tensor
B_i : integral part of body force
B_{ij} : integral part of body force (matrix form)
u^f_j, p^f_j : displacement and traction due to body force
u^d_j, p^d_j : displacement and traction due to disturbance
G : shear modulus
ν : Poisson' s ratio
γ : unit weight of ground
ρ : radius of boundary Γ_o

2. Introduction

Boundary element method has many advantages derived from that the governing differential equation is transformed into the boundary integrals defined only over the surface which simply require discretization of the boundary.

In practical applications, however, geotechnical problems are in the presence of body forces such as constant gravitational loads. It is obvious that domain integral term have to be computed when body forces are considered. Unfortunately, this requires whole domain to be divided into internal cells for integrations in generally which increases the amount of date preparation

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and loses much of BEM's advantages over the domain type analysis such as FEM. Thus, it is of fundamental significance to deal with domain integrals for further development of boundary element method.

In this paper the transformation procedure for two dimensional half-plane problem is introduced to have an explicit form for computer program primarily according to the method proposed by Jiang[1] who presented a procedure to transfer the domain integrals onto the boundary for the half-plane problem making use of Galerkin tensor originated from Danson[2].

3. Transformation of Body Force Integral

Boundary intergral formulation including body force term is

$$u_i + \int_\Gamma p^*_{ij} u_j \, d\underline{\Gamma} = \int_\Gamma u^*_{ij} p_j \, d\underline{\Gamma} + \int_\Omega u^*_{ij} b_j \, d\underline{\Omega} \tag{1}$$

The second term on the right-hand-side requires discretization of whole domain to evaluate the integral. This treatment, however, increases the number of elements extremely and is also hard to apply to infinite or semi-infinite problem. Transformation of this domain integral to the boundary in terms of Galerkin tensor is the end of this section.

In what follows gravitational loads are considered as the body force denoted by b_j in equation (1). Now the b_j is taken out of the integral as constant gravitation field is assumed. The body force term of equation (1) is

$$B_j = b_j \int_\Omega u^*_{ij} \, d\underline{\Omega} \tag{2}$$

where

$$b_1 = \gamma$$
$$b_2 = o \quad . \tag{3}$$

γ is unit weight of ground. The coordinates for the problem are defined in Fig. 1. Fundamental solution using Galerkin potential function are described by Saada[3] as follows

$$u^*_{ij} = \frac{1}{2G}\,[2(1-\nu)G_{ij,kk} - G_{ik,kj}] \quad . \tag{4}$$

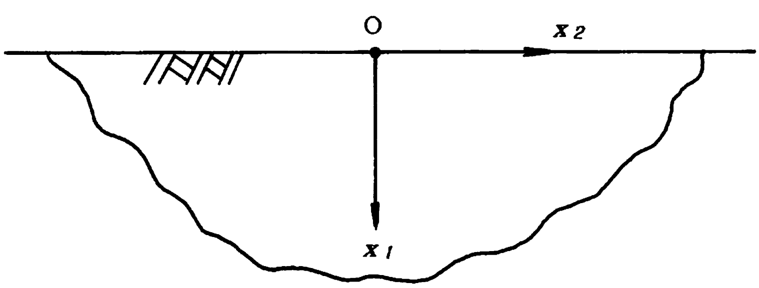

Fig. 1 Coordinates for half-plane problem

To substitute equation (4) into equation (2) and apply Gauss' theorem leads

$$B_i = \frac{b_j}{2G} \int_\Gamma [2(1-\nu)G_{ij,k} - G_{ik,j}]\, n_k \, d\Gamma \tag{5}$$

where n_k is direction cosines of outward normal at the point on the boundary. Domain integrals are transformed to the boundary by equation (5).

Corresponding Galerkin tensor for Melan's half-plane solution is deduced by Jiang as follows

$$G_{11}=K_s\{-\frac{r^2 lnr}{2}-\frac{[8(1-\nu)^2-(3-4\nu)]\,R^2 lnR}{2\,(3-4\nu)}+\frac{cr_2\,\theta}{2(1-\nu)}\}$$
$$G_{12}=K_s\{\frac{(3-4\nu)c\bar{x}\,\theta-c^2\,\theta}{2(1-\nu)}-\frac{2(1-2\nu)(1-\nu)R^2\,\theta}{(3-4\nu)}\}$$
$$G_{21}=K_s\{\frac{(3-4\nu)c\bar{x}\,\theta+c^2\,\theta}{2(1-\nu)}+\frac{2(1-2\nu)(1-\nu)R^2\,\theta}{(3-4\nu)}\}$$
$$G_{22}=K_s\{-\frac{r^2 lnr}{2}-\frac{[8(1-\nu)^2-(3-4\nu)]\,R^2 lnR}{2\,(3-4\nu)}+\frac{cr_2\,\theta}{(1-2\nu)}\}$$
$$K_s=\frac{1}{4\pi\,(1-\nu)}$$

where the definition of variables in the above expressin is shown in Fig. 2.

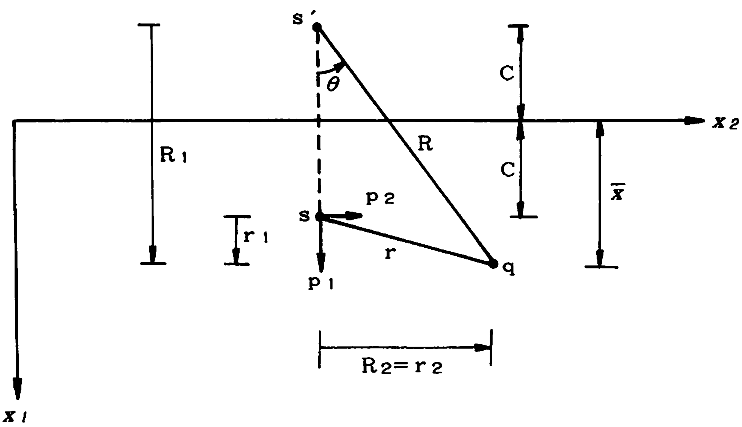

Fig. 2 Definition of variables in fundamental solutions and Galerkin tensors

Melan' s displacement solution will be obtained by substituting equation (6) into (4). These results for displacement, however, show difference on u^*_{11} and u^*_{22} by constants from those of Danson' s or Telles' [4]. Corresponding Melan' s solution are shown below for completeness

$$u^*_{11}=K_d\{\,-(3-4\nu)lnr\;+\frac{{r_1}^2}{r^2}-2\,(1-\nu)(5-4\nu)-[\,8\,(1-\nu)^2-(3-4\nu)]\;lnR$$
$$+\frac{[(3-4\nu)R_1{}^2-2\,c\bar{x}]}{R^2}+\frac{4\,c\bar{x}R_1{}^2}{R^4}\}$$
$$u^*_{12}=K_d\{\frac{r_1r_2}{r^2}+\frac{(3-4\nu)r_1r_2}{R^2}+\frac{4\,c\bar{x}R_1r_2}{R^4}-4\,(1-\nu)(1-2\nu)\,\theta\}$$
$$u^*_{21}=K_d\{\frac{r_2r_1}{r^2}+\frac{(3-4\nu)r_1r_2}{R^2}-\frac{4\,c\bar{x}R_1r_2}{R^4}+4\,(1-\nu)(1-2\nu)\,\theta\;\}$$
$$u^*_{22}=K_d\{\,-(3-4\nu)lnr\;+\frac{{r_1}^2}{r^2}-2\,(1-\nu)(5-4\nu)-[\,8\,(1-\nu)^2-(3-4\nu)]\;lnR$$
$$+\frac{[(3-4\nu)r_2{}^2+2\,c\bar{x}]}{R^2}-\frac{4\,c\bar{x}r_2^2}{R^4}\}$$
$$K_d=\frac{1}{8\,\pi\,(1-\nu)G} \qquad (7)$$

These differences do not have any important effects on the prpblem since these constants correspond to a rigid body motion and disappear when differential treatment is carried out for strain and stress computation. Equation(7) is the fundamental solution through this procedure.

4. Boundary Integral Formulation for half－plane problem

In what follows, boundary integral equation is obtained assuming tunneling problem in the half-plane.

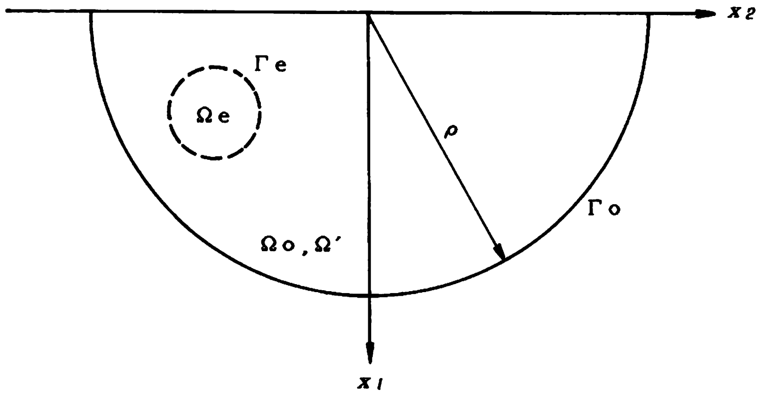

Fig. 3 Boundary and domain due to excavation in the semi-infinite ground ($\Omega' = \Omega_o - \Omega_e$)

Let us consider the domain Ω_o bounded by Γ_o with adequately large radius ρ in the half-plane changes to Ω' due to tunnel Ω_e bounded by Γ_e as shown in Fig. 3. The integral equation as to interior displacement can be written as follows,

$$u_i + \int_{\Gamma_o} p^*_{ij} u_j d\Gamma + \int_{\Gamma_e} p^*_{ij} u_j d\Gamma = \int_{\Gamma_o} u^*_{ij} p_j d\Gamma + \int_{\Gamma_e} u^*_{ij} p_j d\Gamma + \int_{\Omega'} u^*_{ij} b_j d\Omega \qquad . (8)$$

Note that no integral part on horizontal surface is in the above expressin since the traction-free condition of fundamental solution is included.

The displacement and tranction can be separated into two parts as

$$u_j = u^f_j + u^d_j$$
$$p_j = p^f_j + p^d_j \qquad (9)$$

where the superscript of f and d denote gravitational force prior to excavation and disturbance due to tunneling respectively.

The body force term can be also expressed by the difference of two domain integrals as

$$B_i = \int_{\Omega'} u^*_{ij} b_j d\Omega = \int_{\Omega_o} u^*_{ij} b_j d\Omega - \int_{\Omega_e} u^*_{ij} b_j d\Omega \qquad . (10)$$

The expression of integral equation (1) is rewritten as follows according to the variables in Fig. 3.

$$u^f_i + \int_{\Gamma_o} p^*_{ij} u^f_j d\Gamma = \int_{\Gamma_o} u^*_{ij} p^f_j d\Gamma + \int_{\Omega_o} u^*_{ij} b_j d\Omega \qquad . (11)$$

Substituting equation (9) and (10) into equation (8) and taking account of equation (11), leads to

$$u_i + \int_{\Gamma_o} p^*_{ij} u^d_j d\Gamma + \int_{\Gamma_e} p^*_{ij} u_j d\Gamma = \int_{\Gamma_o} u^*_{ij} p^d_j d\Gamma + \int_{\Gamma_e} u^*_{ij} p_j d\Gamma - \int_{\Omega_e} u^*_{ij} b_j d\Omega + u^f_i \qquad . (12)$$

Applying Saint-Venant's principle－ two distributions of forces, when statically equivalent, have the same resultant force and moment (stresses) at large distances －, following equation is obtained

$$\lim_{\rho \to \infty} \int_{\Gamma_o} p^*_{ij} u^d_j d\Gamma = \lim_{\rho \to \infty} \int_{\Gamma_o} u^*_{ij} p^d_j d\Gamma \qquad . (13)$$

Above equation means that displacements and tractions due to tunneling in the half-plane are the same as those due to an equivalent concentrate load (fundamental solutions) at sufficiently large ρ.

Hence equation (12) is simply written as follows

$$u_i+\int_{\Gamma_e} p^*_{ij}u_j d\underline{\Gamma}=\int_{\Gamma_e} u^*_{ij}p_j d\underline{\Gamma}-\int_{\Omega_e} u^*_{ij}b_j d\underline{\Omega}+u^f_i \tag{14}$$

which implies that displacements at any interior points can be obtained by the boundary and domain integrals for the excavated parts and free term of analytical solution without tunnel. This free term is given by the elastic stress-strain relationship, which is

$$u^f_1=-\frac{(1-2\nu)}{4G(1-\nu)}b_1x_1{}^2 \qquad u^f_2=0 \tag{15}$$

The domain integral on the right-hand-side of equation (14) can be transformed into the following boundary integral by using equation (5), and all the integrals in the equation (14) are, then, represented merely on the boundary.

$$B_i=\int_{\Omega_e} u^*_{ij}b_j d\underline{\Omega} =\frac{b_j}{2G}\int_{\Gamma_e}[2(1-\nu)G_{ij,k}-G_{ik,j}]\ n'_k d\underline{\Gamma} \tag{16}$$

where $n_k{}'$ in the above equation is outward normal direction cosines just the opposite to that in equation (5).

Equation (16) can be rewritten as follows

$$B_i=B_{ij}b_j \tag{17}$$

The complete forms of B_{ij} are listed below (B_{12} and B_{22} are omitted because b_2 is always equal to zero)

$$B_{11}=\frac{1}{2G}\int_{\Gamma_e}\{[(1-2\nu)G_{11,1}]\ n'_1+[2(1-\nu)G_{11,2}-G_{12,1}]\,n'_2\}\ d\Gamma$$

$$B_{21}=\frac{1}{2G}\int_{\Gamma_e}\{[(1-2\nu)G_{21,1}]\ n'_1+[2(1-\nu)G_{21,2}-G_{22,1}]\,n'_2\}\ d\Gamma$$

$$G_{11,1}=K_S\{-\frac{r_1(1+2\ln r)}{2}-A_1R_1(1+2\ln R)-\frac{A_2cr_2{}^2}{R^2}\}$$

$$G_{11,2}=K_S\{-\frac{r_2(1+2\ln r)}{2}-A_1r_2(1+2\ln R)+A_2c(\theta+\frac{R_1r_2}{R^2})\}$$

$$G_{12,1}=K_S\{A_3c(\theta-\frac{\bar{x}r_2}{R^2})+\frac{A_2c^2r_2}{R^2}-A_4(2R_1\theta-r_2)\}$$

$$G_{21,1}=K_S\{A_5c(\theta-\frac{\bar{x}r_2}{R^2})-\frac{A_6c^2r_2}{R^2}+A_4(2R_1\theta-r_2)\}$$

$$G_{21,2}=K_S\{\frac{A_5c\bar{x}R_1}{R^2}+\frac{A_6c^2R_1}{R^2}+A_4(2r_2\theta+R_1)\}$$

$$G_{22,1}=K_S\{-\frac{r_1(1+2\ln r)}{2}+(A_7+\frac{1}{2})R_1(1+2\ln R)-\frac{A_6cr_2{}^2}{R^2}\}$$

$$K_S=\frac{1}{4\pi(1-\nu)}$$

$$A_1=\frac{8(1-\nu)^2-(3-4\nu)}{2(3-4\nu)}$$

$$A_2=\frac{1}{2(1-\nu)}$$

$$A_3=\frac{3-4\nu}{2(1-\nu)}$$

$$A_4=\frac{2(1-\nu)(1-2\nu)}{3-4\nu}$$

$$A_5 = \frac{3-4\nu}{1-2\nu}$$

$$A_6 = \frac{1}{1-2\nu}$$

$$A_7 = -\frac{4(1-\nu)^2}{3-4\nu}$$

Above consideration, thus, gives a boundary element formulation of semi–infinite elastic body with gravitational loads as body force without domain integrals of the problem.

6. References

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