A note on L^1-bounded martingales

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Abstract

Let f be a martingale. If f is L^1-bounded then f is of bounded variation. Using this the martingale transform g of f converges a. e..

Let (Ω, A, P) be a probability space and $A_0, A_1, \ldots$ a nondecreasing sequences of sub-σ-fields of A. Let $f=(f_1, f_2, \ldots)$ be a martingale with norm $\| f \|_1 = \sup_n E | f_n | < \infty$. Let $v = (v_1, v_2, \ldots)$ be a predictable sequence, that is, $v_k : \Omega \rightarrow R$ is A_k-measurable, $k \geqslant 1$. Then $g=(g_1, g_2, \ldots)$, defined by $g_n = \sum_{k=1}^{n} v_k (f_{k+1} - f_k)$ with $| v | \leqslant 1$ in absolute value, is the transform of the martingale f by v. Write $\| f \|_p = \sup_n \| f_n \|_p$ and define the maximal function of g by $g^*(\omega) = \sup_n | g_n(\omega) |$.

THEOREM 1. If $\| f \|_1 < \infty$ then $\sum_{n=1}^{\infty} | f_{n+1} - f_n | < \infty$ a. e., that is, f is of bounded variation.

PROOF. Since f is L^1-bounded $f_n \rightarrow f_\infty$ a. e. in L^1 as $n \rightarrow \infty$. Now,

$$\sum_{n=1}^{\infty} | f_{n+1}(\omega) - f_n(\omega) | \leqq 2 \cdot \sum_{n=1}^{\infty} | f_n(\omega) - f_\infty(\omega) | .$$

Let $a_n(\omega) = | f_n(\omega) - f_\infty(\omega) |$ then $\lim_{n \rightarrow \infty} a_n(\omega) = 0$.

Thus, for $0 < \varepsilon < 1$, there exists a number $N = N(\varepsilon, \omega) > 0$ such that $0 \leqslant a_n(\omega) < \varepsilon < 1$ $(\forall n \geqslant N)$. So $0 \leqslant \sqrt[n]{a_n(\omega)} < 1$ $(\forall n \geqq N)$.

Let $x_n(\omega) = \sqrt[n]{a_n(\omega)}$ and $S = \{x_1(\omega), x_2(\omega), x_3(\omega), \ldots\}$.

Since S is a bounded infinite set of real numbers in general, by the Bolzano-Weierstrass' theorem, S has the accumulation point $\lambda = \lambda(\omega) \geqslant 0$ and $\lambda = \lim_{n \rightarrow \infty} x_n(\omega) < 1$.

In fact, if $\lim_{n \rightarrow \infty} x_n(\omega) = \lambda = 1$ then, for $0 \leqslant \forall \varepsilon < 1$, there exists a number $N = N(\varepsilon, \omega) > 0$ such that $| \sqrt[n]{a_n(\omega)} - 1 | \leqslant \varepsilon$ for all $n \geqslant N$. So $a_n(\omega) \geqslant (1-\varepsilon)^n \gneqq 0$ $(\forall n \geqq N)$.

Let $\varepsilon \downarrow 0$ so $n \uparrow \infty$ then $a_\infty(\omega) \geqq 1^\infty \gneqq 0$.

Thus it is not that $a_n(\omega) \rightarrow 0$ as $n \rightarrow \infty$. This contradicts to the fact that $a_n(\omega) \rightarrow 0$ as $n \rightarrow \infty$.

Thus $\lambda \neq 1$.

So, by Cauchy's result, $\sum_{n=1}^{\infty} a_n(\omega)$ converges for almost all ω. Therefore $\sum_{n=1}^{\infty} |f_{n+1} - f_n| < \infty$ a. e.

THEOREM 2. If $\| f \|_1 < \infty$ then the martingale transform g converges a. e..

In fact,

$| g_\infty(\omega) | \leqslant \sum_{n=1}^{\infty} | v_n(\omega) | \cdot | f_{n+1}(\omega) - f_n(\omega) | \leqslant \sum_{n=1}^{\infty} | f_{n+1}(\omega) - f_n(\omega) | < \infty$ for almost all ω.

THEOREM 3. Let $1 < p < \infty$. Then

$\lambda \cdot P(g^* > \lambda) \leqslant c \cdot \| f \|_1$, $\lambda > 0$, and $\| g \|_p \leqslant c_p \cdot \| f \|_p$ hold.

PROOF. By a result of Burkholder (THEOREM 1. 1 0f [2]), the following statements, each to hold for all such f and g, are equivalent :

$\| f \|_1 < \infty \Rightarrow$ g converges a. e.,

$\lambda \cdot P(g^* > \lambda) \leqslant c \cdot \| f \|_1$, $\lambda > 0$,

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$\|g\|_p \leqslant c_p \cdot \|f\|_p$.

Combine this result with THEOREM 2.

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