A note on L^1-bounded martingales. II

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Abstract.

Let f be an X-valued martingale when a Banach space X has the Radon-Nikodým Property. If f is L^1_X-bounded then f is of bounded variation. Using this the martingale transform g of f converges a. e. in X.

1. Notations.

Let (Ω , a, P) be a probability space and a_1, a_2,... a nondecreasing sequences of sub-σ-fields of a. Let X be a Banach space with norm $|\cdot|$ and the Radon-Nikodým property. Let f = (f_1, f_2,...) be an X- valued martingale with norm $\|f\|_1 = \sup_n E\ |f_n| < \infty$. Let v = (v_1, v_2,...) be a real-valued predictable sequence, that is, v_k : $\Omega \to R$ is a_k measurable, $k \geqq 1$. Then g = (g_1, g_2,...), defined by $g_n = \sum_{k=1}^{n} v_k\ (f_{k+1} - f_k)$ with $|v| \leqq 1$ in absolute value, is the transform of the martingale f by v. Write $\|f\|p = \sup_n \|f_n\|_p$ and define the maximal function g^* of g by $g^*(\omega) = \sup_n |g_n(\omega)|$.

2. Results and the proofs.

Theorem 1. If $\|f\|_1 < \infty$ then $\sum_{n=1}^{\infty} |f_{n+1} - f_n| < \infty$ a. e., that is, f is of bounded variation.

Proof. By Chatterji's result, $f_n \to f_\infty$ a. e. in L^1_X as $n \to \infty$.

Now, for almost all $\omega \in \Omega$

$$\sum_{n=1}^{\infty} |f_{n+1}(\omega) - f_n(\omega)| \leqq 2 \cdot \sum_{n=1}^{\infty} |f_n(\omega) - f_\infty(\omega)| .$$

Let $a_n(\omega) = |f_n(\omega) - f_\infty(\omega)|$ then $\lim_{n\to\infty} a_n(\omega) = 0$.

Thus, for $0 < \varepsilon < 1$, there exists a number $N = N(\varepsilon, \omega) > 0$ such that $0 \leqq a_n(\omega) < \varepsilon < 1$ ($\forall\ n \geqq N$).

So $0 \leqq \sqrt[n]{a_n(\omega)} < 1$ ($\forall\ n \geqq N$). Let $x_n(\omega) = \sqrt[n]{a_n(\omega)}$ and $S = \{ x_1(\omega), x_2(\omega), x_3(\omega), \ldots \}$.

Since S is a bounded infinite set of real numbers in general, by the Bolzano-Weierstrass' theorem, S has the accumulation point $\lambda = \lambda(\omega) \geqq 0$ and $\lambda = \lim_{n\to\infty} x_n(\omega) < 1$.

In fact, if $\lim_{n\to\infty} x_n(\omega) = \lambda = 1$ then, for $0 \leqq \forall\ \varepsilon < 1$, there exists a number $N = N(\varepsilon, \omega) > 0$ such that $|\sqrt[n]{a_n(\omega)} - 1| \leqq \varepsilon$ for all $n \geqq N$. So $0 \lneqq (1-\varepsilon)^n \leqq a_n(\omega) \leqq (1+\varepsilon)^n$ ($\forall\ n \geqq N$).

If ε does not near to 0 then $\lim_{n\to\infty} \sqrt[n]{a_n(\omega)} = 1$ does not hold so let $\varepsilon \to 0$ then N is nondecreasing and N = N (0, ω) is finite or infinite.

Since for $\varepsilon = 0$ the above inequality holds for all $n \geqq N$ when $N = N(0, \omega) \leqq \infty$, so

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$a_\infty(\omega) = \lim_{n\to\infty} 1^n = 1 \neq 0$, if $N < \infty$,

$= 1^\infty = 1^{\lim_{n\to\infty} n} = \lim_{n\to\infty} 1^n = 1 \neq 0$, if $N = \infty$.

Thus it is not that $a_n(\omega) \to 0$ as $n \to \infty$. This contradicts to the fact that $a_n(\omega) \to 0$ as $n \to \infty$. Thus $\lambda \neq 1$.

So, by Cauchy's result, $\sum_{n=1}^{\infty} a_n(\omega)$ converges for almost all ω. Therefore $\sum_{n=1}^{\infty} |f_{n+1} - f_n| < \infty$ a. e..

Theorem 2. If $\|f\|_1 < \infty$ then the martingale transform g converges a. e. in X.

In fact,

$$|g_\infty(\omega)| \leq \sum_{n=1}^{\infty} |v_n(\omega)| \cdot |f_{n+1}(\omega) - f_n(\omega)| \leq \sum_{n=1}^{\infty} |f_{n+1}(\omega) - f_n(\omega)| < \infty$$

for almost all ω.

Corollary. Let $v = (v_1, v_2, \ldots)$ be a sequence of any random variables with $|v| \leq 1$. Then $h_n = \sum_{k=1}^{n} v_k (f_{k+1} - f_k)$ converges a. e..

Theorem 3. Let $1 < p < \infty$. For a Banach space X with the Radon-Nikodým property, $\lambda \cdot P(g^* > \lambda) \leq c \cdot \|f\|_1$, $\lambda > 0$ and $\|g\|_p \leq c_p \cdot \|f\|_p$ hold.

Proof. For any Banach space X, by a result of Burkholder (Theorem 1.1 of [2]), the following statements, each to hold for all such f and g, are equivalent:

$\|f\|_1 < \infty \Rightarrow g$ converges a. e.,

$\lambda \cdot P(g^* > \lambda) \leq c \cdot \|f\|_1$, $\lambda > 0$,

$\|g\|_p \leq c_p \cdot \|f\|_p$.

Combine this result with Theorem 2.

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