L^p-convergence of an extended stochastic integral

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Abstract

Let $1 < p < \infty$. Let $f = \{f(t),\ 0 \leq t \leq 1\}$ be an L^p-integrable martingale and $v = \{v(t),\ 0 \leq t \leq 1\}$ a family of random variables with a continuous parameter t. Suppose $|v| \leq 1$ in absolute value and that $v(t)$ is continuous. Put

$$\theta_m = \sum_{K=0}^{s_m - 1} v(\xi_{m,k})\,[f(t_{m,k+1}) - f(t_{m,k})].$$

Here, $\forall \xi_{m,k} \in [t_{m,k},\ t_{m,k+1}]$, $k \geq 0$, and $\operatorname{Max}_k (t_{m,k+1} - t_{m,k}) \to 0$ as $m \to \infty$.

Then θ_m converges in L^p and θ_∞ defines a new stochastic integral $\int_0^1 v(t)\,df(t)$.

Let (Ω, a, P) be a probability space and $\{a_t\}_{t \geq 0}$ a nondecreasing family of sub-σ-fields of a. Let $f = \{f(t),\ 0 \leq t \leq 1\}$ be an L^p-integrable martingale where $1 < p < \infty$ on a probability space $(\Omega, a, \{a_t\}, P)$ and $v = \{v(t),\ 0 \leq t \leq 1\}$ a family of random variables with a continuous parameter t.
Suppose that $|v| \leq 1$ in absolute value, $v(t)$ is continuous and $v(t)$ is a_t-adapted.

Let $\Delta = \{\Delta_m\}$, where $\Delta_m = \{t_{m,k} : 0 = t_{m,0} < t_{m,1} < \cdots\cdots < t_{m,s} = 1\}$, be a sequence of partitions (i. e., subdivisions) of $[0, 1]$ with $|\Delta_m| = \operatorname{Max}_k (t_{m,k+1} - t_{m,k}) \to 0$ as $m \to \infty$. Here notice that if $m \uparrow \infty$ then $s \uparrow \infty$. So it may be that $s = s_m$ and $s_m \uparrow \infty$ as $m \uparrow \infty$.

Put $\theta_m = \sum_{K=0}^{s_m - 1} v(\xi_{m,k})\,[f(t_{m,k+1}) - f(t_{m,k})]$ $(\forall \xi_{m,k} \in [t_{m,k},\ t_{m,k+1}], k \geq 0)$

and $\tilde{\theta}_m = \sum_{K=0}^{s_m - 1} v(t_{m,k})\,[f(t_{m,k+1}) - f(t_{m,k})]$.

By the results of R. C. James [3] and G. Pisier [7],

Theorem. (G. Pisier [7, Theorem 1. 3, (iv)])
Let X be a Banach space and $f = (f_n)_{n \geq 0}$ an arbitrary X-valued martingale.
Then
(∗) X is super-reflexive (= super-Radon-Nikodým)
$\Longleftrightarrow$
(∗∗) $\sum_{n \geq 0} \|f_{n+1} - f_n\|_p \leq C \cdot \sup_n \|f_n\|_p$ $(1 < p < \infty)$.

(Here, C is a constant which does not depend on f.)
Since X = R is super-reflexive, (∗∗) holds.
(∗∗) shall be called by the name of Pisier's inequality.
In this paper, it is proved that the following theorem holds :

Theorem. θ_m converges in L^p and $\theta_\infty = \tilde{\theta}_\infty = \int_0^1 v(t)\,df(t)$.
θ_∞ defines a new stochastic integral.

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Proof. Let $1 < p < \infty$.

$$\|\theta_m\|_p = E^{1/p}[(|\sum_{K\geq 0} v(\xi_{m,k})[f(t_{m,k+1}) - f(t_{m,k})]|)^p]$$

$$\leqq E^{1/p}[(\sum_{K\geq 0}|v(\xi_{m,k})|\cdot|f(t_{m,k+1}) - f(t_{m,k})|)^p] \quad \text{(Here, } |v| \leq 1)$$

$$\leqq E^{1/p}[(\sum_{K\geq 0}|f(t_{m,k+1}) - f(t_{m,k})|)^p] \quad \text{(Since } L^p \text{ is a Banach lattice. See [8].)}$$

$$\leqq \sum_{K\geq 0}\|f(t_{m,k+1}) - f(t_{m,k})\|_p$$

$$\leqq C\cdot\sup_k\|f(t_{m,k})\|_p \quad (m = 0, 1, 2, \cdots\cdots) \text{ (By Pisier's inequality)}$$

(Since C does not depend on f, C does not depend on m.)

$$\leqq C\cdot\sup_{t\in[0,1]}\|f(t)\|_p \quad \text{(Since } t_{m,k}\in[0, 1].)$$

$$\leqq C\cdot\|f(1)\|_p$$

(Since $|f(t)|^p$ is a submartingale, $E\,|f(t)|^p \leqq E\,|f(1)|^p$ so $E^{1/p}[\,|f(t)|^p] \leqq E^{1/p}[\,|f(1)|^p] < \infty$).

Thus, $E[\,|\theta_m|^p] \leqq C^p\cdot\|f(1)\|_p{}^p$

and $E[\,|\theta_\infty|^p] = E[\lim_{m\to\infty}|\theta_m|^p]$

$$\leqq \liminf_{m\to\infty} E[\,|\theta_m|^p] \quad \text{(By Fatou's lemma.)}$$

$$\leqq C^p\cdot\|f(1)\|_p{}^p < \infty.$$

Therefore $\|\theta_\infty\|_p \leqq C\cdot\|f(1)\|_p < \infty$, i. e., $\theta_\infty \in L^p$.

Next, it is proved that the existence of θ_∞.

By a result of P. W. Millar [5], $\widetilde{\theta}_m$ converges in L^p,

that is, $\lim_{m,n\to\infty}\|\widetilde{\theta}_m - \widetilde{\theta}_n\|_p = 0$.

Take arbitrary $\varepsilon > 0$ and fix this.

$$\|\theta_m - \widetilde{\theta}_m\|_p \leqq \sum_{K=0}^{S_m-1}\|[v(\xi_{m,k}) - v(t_{m,k})][f(t_{m,k+1}) - f(t_{m,k})]\|_p$$

$$= \sum_{K=0}^{S_m-1} E^{1/p}[(|v(\xi_{m,k})(\omega) - v(t_{m,k})(\omega)|\cdot|f(t_{m,k+1})(\omega) - f(t_{m,k})(\omega)|)^p].$$

Here, since $v(t)$ is uniformly continuous on $[0, 1]$, for sufficiently large $m_0 = m_0(\varepsilon, \omega) = m_0(\omega)$

$|v(\xi_{m,k})(\omega) - v(t_{m,k})(\omega)| < \varepsilon \quad (\forall\ k \geq 0,\ \forall\ m \geq m_0)$

holds for every $\omega\in\Omega$. Therefore, for $m = m(\omega) \geq m_0$

$$|v(\xi_{m,k})(\omega) - v(t_{m,k})(\omega)|\cdot|f(t_{m,k+1})(\omega) - f(t_{m,k})(\omega)|$$
$$\leqq \varepsilon\cdot|f(t_{m,k+1})(\omega) - f(t_{m,k})(\omega)| \quad (\forall\, k \geqq 0).$$

Since L^p is a Banach lattice,

$$\sum_{K=0}^{S_m-1} E^{1/p}[(|v(\xi_{m,k}) - v(t_{m,k})|\cdot|f(t_{m,k+1}) - f(t_{m,k})|)^p]$$

$$\leqq \sum_{K=0}^{S_m-1} E^{1/p}[(\varepsilon\cdot|f(t_{m,k+1}) - f(t_{m,k})|)^p] \quad \text{(The right-hand side increases in m.)}$$

$$\leqq \sum_{K=0}^{S_{m'}-1} E^{1/p}[(\varepsilon\cdot|f(t_{m',k+1}) - f(t_{m',k})|)^p]$$

(Here, since $m = m(\omega) \in \{0, 1, 2, \cdots\cdots, \infty\}$ and $0 \leq m_0 \leq m(\omega) < \infty$,
$m' = \sup_{\omega\in\Omega} m(\omega) \in \{0, 1, 2, \cdots\cdots, \infty\}$.)

$$= \varepsilon\cdot\sum_{K=0}^{S_{m'}-1}\|f(t_{m',k+1}) - f(t_{m',k})\|_p$$

$$\leqq \varepsilon\cdot\sup_{m''}\sum_{K=0}^{S_{m''}-1}\|f(t_{m'',k+1}) - f(t_{m'',k})\|_p$$

($m' \in \{0, 1, 2, \cdots\cdots, m'', \cdots\cdots, \infty\}$. m'' does not depend on ω.)

$\leqq \varepsilon \cdot \sup_{m''}\{C \cdot \sup_{n} \|f(t_{m'',n})\|_p\}$ (By Pisier's inequality.)

$\leqq \varepsilon \cdot \sup_{m''}\{C \cdot \sup_{t\in[0,1]} \|f(t)\|_p\}$ (Since $t_{m'',n} \in [0, 1]$.)

$\leqq \varepsilon \cdot \sup_{m''}\{C \cdot \|f(1)\|_p\}$ (Since f is a martingale.)

$= \varepsilon \cdot C \cdot \|f(1)\|_p$.

Here, by the arbitrariness of ε, it can be that $\varepsilon \cdot C \cdot \|f(1)\|_p < \varepsilon'$ holds for any $\varepsilon' > 0$. So

$$\lim_{m\to\infty} \|\theta_m - \tilde{\theta}_m\|_p < \varepsilon' \quad \text{for all } \varepsilon' > 0.$$

From $\|\theta_m - \theta_n\|_p \leqq \|\theta_m - \tilde{\theta}_m\|_p + \|\tilde{\theta}_n - \theta_n\|_p + \|\tilde{\theta}_m - \tilde{\theta}_n\|_p$
it follows that

$$\lim_{m,n\to\infty} \|\theta_m - \theta_n\|_p \leqq 2 \cdot \lim_{m\to\infty} \|\theta_m - \tilde{\theta}_m\|_p + \lim_{m,n\to\infty} \|\tilde{\theta}_m - \tilde{\theta}_n\|_p$$
$$< 2 \cdot \varepsilon' + 0 = 2\varepsilon' \quad \text{for all } \varepsilon' > 0.$$

So, by the completeness of R, θ_m converges in L^p.
From this proof $\theta_\infty = \tilde{\theta}_\infty = \int_0^1 v(t)\, df(t)$ follows.

Remark. When $p > 1$ the Pisier's inequality implies the Burkholder's L^p-inequality in [1] so that the Millar's results [5] hold without that $v(t)$ is $\mathfrak{a}_t$-adapted. Therefore, it may be that v is any uniformly bounded and continuous random variable.

Corollary. $\int_0^1 v(t)\, dB(t)$ converges in L^2.
(The convergence of this integral cannot be proved by the method of R. L. Stratonovich [10].)

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