Guide to Applied Mathematics for Foreign Students I

Shin-ichi ISHI *

(Received 30 November, 1994)

ABSTRACT

This article is a self-study note for foreign students. The content is a part of lectures, particularly in Laplace transformation and its applications.

1 Laplace Transformations

1.1 Gamma function

In this chapter, we study Laplace transformation and its applications. The terms "Laplace transformation" and "Laplace integral" are sometimes used as same meaning. We shall consider Laplace integrals through Gamma function. Then, we shall review the definition and some properties of Gamma function and, in addition, Beta function, which is closely related with Gamma function. Let a be a positive parameter, $a > 0$. The improper integral $\lim_{T\to\infty}\int_0^T e^{-t}t^{a-1}\,dt$ may exist, and is written simply $\int_0^\infty e^{-t}t^{a-1}\,dt$, which is called Gamma function:

$$\Gamma(a)=\int_0^\infty e^{-t}t^{a-1}dt \tag{1}$$

Substituting $t = x^2$ in the definition (1), we obtain another form of Gamma function:

$$(1) \overset{t=x^2}{=} 2\int_0^\infty e^{-x^2}x^{2a-1}dx \tag{2}$$

Integrating by parts in the definition (1), we get

$$\Gamma(a)=\int_0^\infty e^{-t}t^{a-1}dt=\frac{1}{a}\Gamma(a-1)$$

Rewritting the above equation, we obtain a recurrence formula:

$$\Gamma(a+1)=a\,\Gamma(a) \tag{3}$$

If let a be a positive integer n ($a = n$), the reccurence formula (3) can be written

$$\Gamma(n+1)=n! \tag{4}$$

That is,

$$\Gamma(a+1)=a\,\Gamma(a)=a(a-1)\Gamma(a-1)=\cdots=a(a-1)(a-2)\cdots 2\cdot 1$$

where

$$\Gamma(1)=1 \tag{5}$$

From such a result, Gamma function is so-called factorial function.

Next, taking $a=\frac{1}{2}$ in the definitions (1) or (2), we have

$$\Gamma\left(\frac{1}{2}\right)=\int_0^\infty e^{-t}t^{-\frac{1}{2}}dt = 2\int_0^\infty e^{-x^2}dx = 2\frac{\sqrt{\pi}}{2}=\sqrt{\pi} \tag{6}$$

Using the reccurence formula (3), we have

$$\Gamma\left(\frac{3}{2}\right)=\Gamma\left(\frac{1}{2}+1\right)=\frac{1}{2}\Gamma\left(\frac{1}{2}\right)=\frac{1}{2}\sqrt{\pi}$$

* associate professor

$$\Gamma\left(\frac{5}{2}\right)=\Gamma\left(\frac{3}{2}+1\right)=\frac{3}{2}\Gamma\left(\frac{3}{2}\right)=\frac{3}{2}\frac{1}{2}\sqrt{\pi}$$

In the general case of $a=$ half integer, another reccurence formula can be obtained,

$$\Gamma\left(\frac{2n+1}{2}\right)=\frac{2n-1}{2}\frac{2n-3}{2}\cdots\frac{3}{2}\frac{1}{2}\Gamma\left(\frac{1}{2}\right)=\frac{(2n-1)!!}{2^n}\sqrt{\pi}\overset{(2n)!!=2^n n!}{=}\frac{(2n-1)!}{n!2^{2n}}\sqrt{\pi} \tag{7}$$

Example 1. Evaluate the following integrals.

(1) $\int_0^\infty e^{-x^2}x^{2n}dx$ (2) $\int_0^\infty e^{-x^2}x^{2n+1}dx$

Solution

(1) $\int_0^\infty e^{-x^2}x^{2n}dx \overset{x^2=t}{=} \frac{1}{2}\int_0^\infty e^{-t}t^{n-\frac{1}{2}}dt \overset{(7)}{=} \frac{(2n-3)!}{(n-1)!2^{2(n-1)}}\sqrt{\pi}$

(2) $\int_0^\infty e^{-x^2}x^{2n+1}dx \overset{x^2=t}{=} \frac{1}{2}\int_0^\infty e^{-t}t^n dt \overset{(2),(5)}{=} \frac{\Gamma(n+1)}{2}\left(=\frac{n!}{2}\right)$

Exercise 1. Verify the following formulas by the method of the change of variable.

(1) $\Gamma(a) \overset{e^{-t}=x}{=} \int_0^1 \log\left(\frac{1}{x}\right)^{a-1}dx$ (2) $\Gamma(a) \overset{t^a=x}{=} \frac{1}{a}\int_0^\infty e^{-x^{\frac{1}{a}}}dx$

We shall show the relation between Gamma and Beta functions :

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}=B(a,b) \tag{8}$$

Beta function is defined

$$B(a,b)=\int_0^1 t^{a-1}(1-t)^{b-1}dt \tag{9}$$

which is symmetric with respect to the parameters a and b ; $B(a,b)=B(b,a)$.

$$B(a,b) \overset{1-t=x}{=} -\int_1^0 (1-x)^{a-1}x^{b-1}dx=\int_0^1 x^{b-1}(1-x)^{a-1}dx \overset{(9)}{=} B(b,a)$$

Subsituting $t=\sin^2\theta$ in the definition (9), we obtain another form of Beta function :

$$(9) \overset{t=\sin^2\theta}{=} 2\int_0^{\frac{\pi}{2}}\sin^{2a-1}\theta\cos^{2b-1}\theta\, d\theta \tag{10}$$

Finally, by using these, it follows that

$$\Gamma(a)\Gamma(b)=\int_0^\infty t^{a-1}e^{-t}dt\int_0^\infty t^{b-1}e^{-t}dt \overset{(2)}{=} 2\int_0^\infty x^{2a-1}e^{-x^2}dt\cdot 2\int_0^\infty y^{2b-1}e^{-y^2}dy$$

$$=2^2\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}x^{2a-1}y^{2b-1}dxdy \overset{x=r\cos\theta,\ y=r\sin\theta}{=} 2^2\int_0^\infty\int_0^{\frac{\pi}{2}}e^{-r^2}r^{2a-1}\cos^{2a-1}\cdot r^{2b-1}\theta\sin^{2b-1}\theta\cdot rdrd\theta$$

$$=2\int_0^r e^{-r^2}r^{2(a+b)-1}dr\cdot 2\int_0^\infty\cos^{2a-1}\theta\sin^{2b-1}\theta\, d\theta \overset{(2),(10)}{=} \Gamma(a+b)B(b,a)=\Gamma(a+b)B(a,b)$$

The identity (8) is established.

Example 2. Evaluate the following integrals.

(1) $\int_0^{\frac{\pi}{2}}\sin^n\theta\, d\theta$ (2) $\int_0^{\frac{\pi}{2}}\cos^n\theta\, d\theta$ (3) $\int_0^1\frac{1}{\sqrt{t(t-1)}}dt$

Solution.

(1). From the formula (10), it follows that

$$\int_0^{\frac{\pi}{2}}\sin^{2a-1}\theta\cos^{2b-1}\theta\, d\theta=\frac{1}{2}\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

Here, let $a=\frac{n+1}{2}$ and $b=\frac{1}{2}$, then the equation reduces to

$$\int_0^{\frac{\pi}{2}} \sin^n \theta \, d\theta = \frac{1}{2} \frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n+2}{2}+1\right)} = \frac{1}{2} B\left(\frac{n+1}{2}, \frac{1}{2}\right)$$

In the case of $n = 2m$, the equation reduces to

$$\frac{(2m-1)(2m-3)\cdots 3\cdot 1}{2m\cdot(2m-2)\cdots 4\cdot 2}\frac{\pi}{2} = \frac{(2m-1)!!}{(2m)!!}\frac{\pi}{2}$$

In the case of $n = 2m + 1$,

$$\frac{2m\cdot(2m-2)\cdots 4\cdot 2}{(2m+1)(2m-1)\cdots 3\cdot 1} = \frac{(2m)!!}{(2m+1)!!}$$

(2). Putting $a = \frac{1}{2}$ and $b = \frac{n+1}{2}$ similarly in (1), we have the same results. From these results, we may write

$$\int_0^{\frac{\pi}{2}} \sin^n \theta \, d\theta = \int_0^{\frac{\pi}{2}} \cos^n \theta \, d\theta = \begin{cases} \frac{(n-1)!!}{n!!}\frac{\pi}{2} & (\text{n: even}) \\ \frac{(n-1)!!}{n!!} & (\text{n: odd}) \end{cases}$$

(3).

$$\int_0^1 \frac{1}{\sqrt{t(t-1)}} dt = B\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)} = (\sqrt{\pi})^2 = \pi$$

Exercise 2. Verify the following formulas by the method of the change or variable.

(1) $B(a, b) \overset{t=\frac{x}{1+x}}{=} \int_0^{\infty} \frac{x^{a-1}}{(1+x)^{a+b}} dx$

(2) $B(a, b) \overset{t=\frac{x+1}{2}}{=} \frac{1}{2^{a+b+1}} \int_{-1}^{1} (1+x)^{a+1}(1-x)^{b-1} dx$

1.2 Laplace Transformation (Laplace Integrals)

1.2.1 Definition or Laplace transformation

Replacing e^{-t} and t^{a-1} by e^{-st} and a general form of function $f(t)$ in the definition (1) respectively, and if $\lim_{T\to\infty}\int_0^T e^{-st} f(t)\,dt$ may exist, we write

$$L(f(t)) = F(s) \overset{\text{def}}{=} \int_0^{\infty} e^{-st} f(t)\,dt \tag{11}$$

which is called Laplace integral or Laplace transformation. Here the symbol L denotes Laplace's linear integral operator and e^{-st} is the kernel of its integral operator. The term "linear operator" means that

$$L(af(t) + bg(t)) = aL(f(t)) + bL(g(t)) \tag{12}$$

$L(f(t))$ (or $F(s)$) is called Laplace transform (or the image function) of the original function $f(t)$. It is convenient to use both the notations $L(f(t))$ and $F(s)$, which represent Laplace transform and the image function of the parameter s implicitely of the original function $f(t)$, respectively. Hereafter for simplicity, we assume that a one-to-one correspondence has been established between $f(t)$ and $F(s)$. Hence this statement denotes symbolically as

$$f(t) \underset{L^{-1}}{\overset{L}{\rightleftharpoons}} F(s) \quad \text{or} \; f(t) \Leftrightarrow F(s)$$

where L^{-1} represents the inverse L oparetor, which is defined as

$$LL^{-1} = L^{-1}L = I \tag{13}$$

where I denotes a unit element.

1.2.2 Examples of Laplace transforms for elementary functions

1) Now consider Laplace integral $L(t^n)$ for the original function $f(t) = t^n$, which may be calculated by the change of variable

$$L(t^n) = \int_0^\infty e^{-st} t^n dt \overset{st=x}{=} \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^n dx = \frac{\Gamma(n+1)}{s^{n+1}} = \frac{n!}{s^{n+1}} \tag{14}$$

where $n + 1 > 0$, that is, $n > -1$. When $n = 0$, the formula (14) reduces to

$$L(1) = \frac{1}{s} \quad (0! = 1) \tag{15}$$

The above formula (14) may be shown inductively as follows:

$$\frac{dL(1)}{dt} = \frac{d}{ds}\left(\frac{1}{s}\right) = -\frac{1}{s^2} = L(t) = L(1 \cdot (-t))$$

$$\frac{dL(t)}{dt} = -\frac{d}{ds}\left(\frac{1}{s^2}\right) = -\frac{2}{s^3} = -L(t^2) = L(t \cdot (-t))$$

$$\frac{dL(t^n)}{dt} = \frac{d}{ds}\left(\frac{(n-1)!}{s^n}\right) = (-1)^n \frac{n!}{s^{n+1}} = (-1)^n L(t^n) = (-1)^{n-1} L(t^{n-1} \cdot (-t))$$

These results show that the differentiation of image function $F'(s)$ corresponds to the multiplication of the original function by $(-t)$.

$$L(t^n \cdot 1) = \frac{n!}{s^{n+1}} = (-1)^n \frac{d^n}{ds^n} L\left(\frac{1}{s}\right)$$

Consider integral $\int_s^\infty L(t^n)\, ds$ with respect to $\int_s^\infty \frac{1}{s^n} ds = \frac{1}{n-1} \frac{1}{s^{n-1}}$:

$$\int_s^\infty L(t^n)\, ds = \int_s^\infty \frac{n!}{s^{n+1}} ds = \frac{(n-1)!}{s^n} = L(t^{n-1}) = L\left(\frac{t^n}{t}\right) \tag{16}$$

This result shows that the integration of the image function corresponds to the division of the original function by t.

When n is half-integer, we may obtain the following formulas

$$L(t^{-\frac{1}{2}}) = \frac{\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}} = \sqrt{\frac{\pi}{s}} \tag{17}$$

$$L(t^{\frac{1}{2}}) = \frac{\Gamma\left(1 + \frac{1}{2}\right)}{s^{\frac{3}{2}}} = \frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$$

On the basis of the formula (14), we shall investigate Laplace integrals with elementary functions, like exponential functions and trigonometrical functions, since they are expanded by power series.

2) Consider the integral:

$$L(e^{-at}) = \int_0^\infty e^{-st} e^{-at} dt$$

The exponential function e^{-at} is expanded into power series,

$$e^{-at} = 1 - at + \frac{(at)^2}{2!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} (at)^n$$

Then, the integral reduces to

$$L(e^{-at})=\sum_{n=0}^{\infty}(-1)^n\frac{a^n}{n!}(L(t))^n=\sum_{n=0}^{\infty}(-1)^n\frac{a^n}{n!}\frac{n!}{s^{n+1}}$$

$$=\sum_{n=0}^{\infty}(-1)^n\frac{1}{s}\left(\frac{a}{s}\right)^n=\frac{1}{s}\left[1-\frac{a}{s}+\left(\frac{a}{s}\right)^2-\left(\frac{a}{s}\right)^3+\cdots\right]=\frac{1}{s}\frac{1}{1-\left(-\frac{a}{s}\right)}=\frac{1}{s+a}$$

That is,

$$L(e^{-at})=\frac{1}{s+a} \tag{18}$$

Of course, we can easily found this formula (18) by the direct integration:

$$\int_0^{\infty}e^{-st}e^{-at}dt=\left[-\frac{1}{s+a}e^{-(s+a)t}\right]_0^{\infty}=\frac{1}{s+a}$$

In like manner, we have $L(e^{at})=\frac{1}{s-a}$. Then, we write

$$L(e^{\pm at})=\frac{1}{s\mp a} \tag{19}$$

Putting $a=\pm ia$ ($i=\sqrt{-1}$), we have

$$L(e^{\pm iat})=\frac{1}{s\mp ia} \tag{20}$$

3) Consider Laplace integrals for trigonometrical functions:

$$L\begin{pmatrix}\sin at\\ \cos at\end{pmatrix}=\int_0^{\infty}e^{-st}\begin{pmatrix}\sin at\\ \cos at\end{pmatrix}dt \tag{21}$$

A sine function is expanded by power series

$$\sin at=\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)!}(at)^{2n+1}$$

Then, we obtain in terms of the formula (14)

$$L(\sin at)=\sum_{n=0}^{\infty}(-1)^n\frac{a^{2n+1}}{(2n+1)!}L(t^{2n+1})=\sum_{n=0}^{\infty}(-1)^n\frac{a^{2n+1}}{(2n+1)!}\frac{(2n+1)!}{s^{2(n+1)}}=\sum_{n=0}^{\infty}(-1)^n\frac{1}{a}\left(\frac{a}{s}\right)^{2(n+1)}$$

$$=\frac{a}{s^2}-\frac{a^3}{s^4}+\frac{a^5}{s^6}-\cdots=\frac{a}{s^2}\left\{1-\left(\frac{a}{s}\right)^2+\left(\frac{a}{s}\right)^4-\cdots\right\}=\frac{a}{s^2}\frac{1}{1-\left(-\frac{a}{s}\right)^2}=\frac{a}{s^2+a^2} \tag{22}$$

In like manner, a cosine series is

$$\cos(at)=\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}(at)^{2n}$$

Then, we have

$$L(\cos(at))=\sum_{n=0}^{\infty}(-1)^n\frac{a^{2n}}{2n!}L(t^{2n})=\cdots=\frac{s}{s^2+a^2} \tag{23}$$

The formulas obtained (22) and (23) can be found easily by using the formula (7) with the help of the relationship; $e^{\pm iat}=\cos at\pm\sin at$. A sine function is expressed in terms of exponential funcutions

$$\sin at=\frac{1}{2i}(e^{iat}-e^{-iat})$$

Obviously, we get

$$L(\sin at)=\frac{1}{2i}\{L(e^{iat})-L(e^{-iat})\}=\frac{1}{2i}\left(\frac{1}{s-ia}-\frac{1}{s+ia}\right)$$

$$=\frac{s}{s^2+a^2}$$

Similarly,

$$\cos at = \frac{1}{2}(e^{iat} + e^{-iat})$$

Then, te get

$$L(\cos at) = \cdots = \frac{s}{s^2+a^2}$$

Example. Find Laplace transforms for the following functions.

(1) $\sin\sqrt{t}$ (2) $\dfrac{\cos\sqrt{t}}{\sqrt{t}}$ (3) $\sum_{n=0}^{\infty}(-1)^n \dfrac{1}{(n!)^2}\left(\dfrac{t}{2}\right)^{2n}$

Solution.

(1).

$$\sin\sqrt{t} = t^{\frac{1}{2}} - \frac{t^{\frac{3}{2}}}{3!} + \frac{t^{\frac{5}{2}}}{5!} - \frac{t^{\frac{7}{2}}}{7!}\cdots$$

Consequently, we have

$$L(\sin\sqrt{t}) = \frac{\Gamma\left(\frac{3}{2}\right)}{s^{\frac{3}{2}}} - \frac{\Gamma\left(\frac{5}{2}\right)}{3!s^{\frac{5}{2}}} - \frac{\Gamma\left(\frac{7}{2}\right)}{5!s^{\frac{7}{2}}}\cdots = \frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}\left\{1 - \frac{1}{2^2s} + \frac{1}{2!}\left(\frac{1}{2^2s}\right)^2 - \frac{1}{3!}\left(\frac{1}{2^3s}\right)^3 + \cdots\right\} = \frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}e^{-\frac{s}{4}}$$

(2).

$$\frac{\cos\sqrt{t}}{\sqrt{t}} = t^{-\frac{1}{2}}\left\{1 - \frac{\left(t^{\frac{1}{2}}\right)^2}{2!} + \frac{\left(t^{\frac{1}{2}}\right)^4}{4!} - \frac{\left(t^{\frac{1}{2}}\right)^6}{6!} + \cdots\right\} = t^{-\frac{1}{2}} - \frac{t^{\frac{1}{2}}}{2!} + \frac{t^{\frac{3}{2}}}{4!} - \frac{t^{\frac{5}{2}}}{6!} + \cdots$$

Hence, we have

$$L\left(\frac{\cos\sqrt{t}}{\sqrt{t}}\right) = \frac{\Gamma\left(\frac{3}{2}\right)}{s^{\frac{3}{2}}} - \frac{\Gamma\left(\frac{5}{2}\right)}{3!s^{\frac{5}{2}}} - \frac{\Gamma\left(\frac{7}{2}\right)}{5!s^{\frac{7}{2}}} + \cdots = \frac{\sqrt{\pi}}{s^{\frac{1}{2}}}\left\{1 - \frac{1}{2^2s} + \frac{1}{2!}\left(\frac{1}{2^2s}\right)^2 - \frac{1}{3!}\left(\frac{1}{2^3s}\right)^3 + \cdots\right\} = \frac{\sqrt{\pi}}{s^{\frac{1}{2}}}e^{-\frac{s}{4}}$$

(3).

$$L\left(\sum_{n=0}^{\infty}(-1)^n\frac{1}{(n!)^2}\left(\frac{t}{2}\right)^{2n}\right) = \sum_{n=0}^{\infty}(-1)^n\frac{1}{(n!)^2}\left(\frac{1}{2}\right)^{2n}L(t^{2n})$$

$$= \sum_{n=0}^{\infty}(-1)^n\frac{1}{(n!)^2}\left(\frac{1}{2}\right)^{2n}\frac{\Gamma(2n+1)}{s^{2n+1}} = \sum_{n=0}^{\infty}(-1)^n\frac{(2n)!}{2^{2n}(n!)^2}\frac{1}{s^{2n+1}}$$

$$= \sum_{n=0}^{\infty}(-1)^n\frac{(2n-1)!!}{(2n)!!}\frac{1}{s^{2n+1}} = \frac{1}{s} - \frac{1}{2}\left(\frac{1}{s^3}\right) + \frac{3\cdot1}{4\cdot2}\left(\frac{1}{s^5}\right) - \frac{5\cdot3\cdot1}{6\cdot4\cdot2}\left(\frac{1}{s^7}\right) + \cdots$$

$$= \frac{1}{s}\left\{1 - \frac{1}{2}\left(\frac{1}{s^2}\right) + \frac{3\cdot1}{4\cdot2}\left(\frac{1}{s^2}\right)^2 - \frac{5\cdot3\cdot1}{6\cdot4\cdot2}\left(\frac{1}{s^2}\right)^3 + \cdots\right\} = \frac{1}{s}\left(1 + \frac{1}{s^2}\right)^{-\frac{1}{2}} = \frac{1}{\sqrt{s^2+1}}$$

This expression denotes Bessel function of digree zero of the first kind.

4) Consider Laplace integrals for hyperboric functions :

$$L\begin{pmatrix}\sinh at\\ \cosh at\end{pmatrix} = \int_0^{\infty} e^{-st}\begin{pmatrix}\sinh at\\ cosh\ at\end{pmatrix}dx = \begin{pmatrix}\dfrac{a}{s^2-a^2}\\ \dfrac{s}{s^2-a^2}\end{pmatrix} \tag{24}$$

where

$$\begin{pmatrix}\sinh at\\ \cosh at\end{pmatrix} = \frac{1}{2}(e^{at} \mp e^{-at})$$

Exercise. Verify the formula (24).

1.3 Some Properties of Laplace Transformations

1.3.1 Linearity (Scalar Product)

Consider the integral,

$$L(f(at))=\int_0^{\infty}e^{-st}f(at)\,dt$$

Substituting $at=x$, we have

$$L(f(at))=\int_0^{\infty}e^{-s\frac{x}{a}}f(x)\frac{dx}{a}=\frac{1}{a}F\left(\frac{s}{a}\right) \tag{25}$$

Using this formula and $L(e^{\pm t})=\frac{1}{s\mp 1}$, we may find

$$L(e^{\pm at})=\frac{1}{a}\frac{1}{\left(\frac{s}{a}\mp 1\right)}=\frac{1}{s\mp a}$$

Similarly, we have

$$L(e^{\pm iat})=\frac{1}{s\mp ia}$$

In terms of the formula (25), we may show the following formulas given.

$$L(\sin t)=\frac{1}{s^2+1}\Rightarrow L(\sin at)=\frac{1}{a}\frac{1}{\left(\frac{s}{a}\right)^2+1}=\frac{a}{s^2+a^2}$$

$$L(\cos t)=\frac{s}{s^2+1}\Rightarrow L(\cos at)=\frac{1}{a}\frac{1}{\left(\frac{s}{a}\right)^2+a}=\frac{s}{s^2+a^2}$$

$$L(\sinh t)=\frac{1}{s^2+1}\Rightarrow L(\sinh at)=\frac{1}{a}\frac{1}{\left(\frac{s}{a}\right)^2-1}=\frac{a}{s^2-a^2}$$

$$L(\cosh t)=\frac{s}{s^2-1}\Rightarrow L(\cosh at)=\frac{1}{a}\frac{1}{\left(\frac{s}{a}\right)^2-a}=\frac{s}{s^2-a^2}$$

Here we wish to consider the meaning to the following formula:

$$L(1)=\frac{1}{s}\text{ and }L(e^{\pm at})=L(e^{\pm at}\cdot 1)=\frac{1}{s\mp a} \tag{26}$$

That is to say, "To multiplication the original function $f(t)=1$ by $e^{\pm at}$ corresponds to shift s by $s\mp a$ on the image's variable."

The general type of $L(e^{\pm at}f(t))$ may be given:

$$L(e^{\pm at}f(t))=\int_0^{\infty}e^{-st}\{e^{\pm at}f(t)\}\,dt=\int_0^{\infty}e^{-(s\mp a)t}f(t)\,dt$$
$$=F(s\mp a) \tag{27}$$

Example. Show the following formulae.

(1) $L(t^n e^{\pm at})=\dfrac{n!}{(s\mp a)^{n+1}}$

(2) $$L\begin{pmatrix} e^{\pm at}\sin bt \\ e^{\pm at}\cos bt\end{pmatrix}=\begin{pmatrix}\dfrac{b}{(s\mp a)^2+b^2} \\ \dfrac{s\mp a}{(s\mp a)^2+b^2}\end{pmatrix} \tag{28}$$

Solution.

(1) $L(t^n)=\dfrac{n!}{s^{n+1}}$ and $L(e^{\pm st})=\dfrac{1}{s\mp a} \Longrightarrow L(t^n e^{\pm at})=\dfrac{n!}{(s\mp a)^{n+1}}$

(2) $L(e^{\pm st})=\dfrac{1}{s\mp a}$ and $\begin{pmatrix} L(\sin bt)=\dfrac{b}{s^2+b^2} \\ L(\cos bt)=\dfrac{s}{s^2+b^2} \end{pmatrix} \Longrightarrow \begin{pmatrix} \dfrac{b}{(s\mp a)^2+b^2} \\ \dfrac{s\mp a}{(s\mp a)^2+b^2} \end{pmatrix}$

1.3.2 Shifting (Translation)

Consider the integral:

$$L(f(t-b))=\int_0^\infty e^{-st}f(t-b)\,dt$$

On replacing $t-b$ by x, the integral reduces to

$$L(f(t-b))=\int_0^\infty e^{-s(x-b)}f(x)\,dx=e^{-bs}\int_0^\infty e^{-st}f(x)\,dx=e^{-bs}F(s) \tag{29}$$

From this result we may state that "shifing to b on the t-axis corresponds to the multiplation of the image function by e^{-bs}". This means that the role of e^{bs} in the image space is played as "shifting operator" or "translation operator"

Example. Find the following formulas.

(1) $L(f(at-b))=e^{-\frac{b}{a}s}\dfrac{1}{a}F\left(\dfrac{s}{a}\right)$

(2) $L(f(at+b))=\dfrac{1}{a}e^{\frac{b}{a}s}\left\{F\left(\dfrac{s}{a}\right)-\int_0^b e^{-\frac{s}{a}x}f(x)\,dx\right\}$

Solution.

(1) $L(f(at-b))\overset{at-b=x}{=}\dfrac{1}{a}e^{-s\frac{b}{a}}\int_0^\infty e^{-\frac{s}{a}x}f(t)\,dt=e^{-\frac{b}{a}s}\dfrac{1}{a}F\left(\dfrac{s}{a}\right)$

(2) $L(f(at+b))\overset{at+b=x}{=}\dfrac{1}{a}e^{\frac{b}{a}s}\int_b^\infty e^{-\frac{s}{a}x}f(x)\,dx$

$=\dfrac{1}{a}e^{\frac{b}{a}s}\left\{\int_0^\infty e^{-\frac{s}{a}x}f(x)\,dx-\int_0^b e^{-\frac{s}{a}x}f(x)\,dx\right\}=\dfrac{1}{a}e^{\frac{b}{a}s}\left\{F\left(\dfrac{s}{a}\right)-\int_0^b e^{-\frac{s}{a}x}f(x)\,dx\right\}$

Here we introduce the Heaviside's unit step function, or simply Heaviside's function $H(t)$ defined as

$$H(t-b)=\begin{cases}1 & (t-b>0)\\ 0 & (t-b<0)\end{cases} \tag{30}$$

Consider the integral:

$$L(H(t-b))=\int_0^\infty e^{-st}H(t-b)\,dt=\int_b^\infty e^{-st}\,dt=e^{-bs}\frac{1}{s} \tag{31}$$

The Heaviside's function plays the role of a transtional operator. Taking $b=0$, that is,

$H(t)=\begin{cases}1 & (t>0)\\ 0 & (t<0)\end{cases}$, then, we have

$$L(H(t))=L(1)=\frac{1}{s} \tag{32}$$

1.3.3 Laplace integrals for functions $t^n\cdot f(t)$ and $\dfrac{f(t)}{t^n}$

1) Consider the integral:

$$L(t\cdot f(t))=\int_0^\infty e^{-st}(t\cdot f(t))\,dt=-\int_0^\infty\frac{\partial}{\partial s}(e^{-st})f(t)\,dt=-\frac{d}{ds}\int_0^\infty e^{-st}f(t)\,dt=-\frac{dF(s)}{dt}=-(F(s))'$$

Further, generally it follows that

$$L(t^n f(t))=(-1)^n \frac{d^n}{ds^n}F(s) \tag{33}$$

Example. Find Laplace transforms for the following functions.

(1) $t^n e^{-at}$ (2) $t^2 \sin at$ (3) $te^{-bt}\cos at$

Solution.

(1) $L(t^n e^{-at})=(-1)^n \frac{d^n}{ds^n}\left(\frac{1}{s+a}\right)=(-1)^n \frac{(-1)^n n!}{(s+a)^{n+1}}=\frac{n!}{(s+a)^{n+1}}$

(2) $L(t^2 \sin at)=(-1)^2 \frac{d^2}{ds^2}\left(\frac{a}{s^2+a^2}\right)=\frac{2s(s^2-a^2)}{(s^2+a^2)^3}$

(3) $L(te^{-bt}\cos at)=(-1)\frac{d}{ds}\left(\frac{x+b}{(s+b)^2+a^2}\right)=\frac{a^2-(s+b)^2}{(a^2+(s+a)^2)^2}$

2) Consider the integral:

$$L\left(\frac{f(t)}{t}\right)=\int_0^\infty e^{-st}\frac{f(t)}{t}dt=\int_0^\infty\left(\int_s^\infty e^{-st}ds\right)f(t)\,dt$$

$$=\int_s^\infty\left(\int_0^\infty e^{-st}f(t)\,dt\right)ds=\int_s^\infty F(s)\,ds$$

Further, in general form we have

$$L\left(\frac{f(t)}{t^n}\right)=\underbrace{\int_s^\infty\cdots\int_{s_{n-1}}^\infty}_{n} F(s)\,ds ds_1\cdots ds_{n-1} \tag{34}$$

Example. Find Laplace transforms for the following functions.

(1) $\frac{\sin at}{t}$ (2) $\frac{1-\cos at}{t}$ (3) $\frac{e^{-at}-e^{-bt}}{t}$ (4) $e^t\frac{d^n}{dt^n}(e^{-t}t^n)$

Solution.

(1) $L\left(\frac{\sin st}{t}\right)=\int_s^\infty \frac{a}{x^2+a^2}dx=\left[\tan^{-1}\frac{x}{a}\right]_s^\infty=\frac{\pi}{2}-\frac{\tan^{-1}s}{a}$

(2) $L\left(\frac{1-\cos at}{t}\right)=\int_s^\infty\left(\frac{1}{x}-\frac{x}{x^2+a^2}dx\right)=\left[\log x-\frac{1}{2}\log(x^2+a^2)\right]_s^\infty=\frac{1}{2}\log\frac{s^2+a^2}{s^2}=\frac{1}{2}\log\left(1+\frac{a^2}{s^2}\right)$

(3) $L\left(\frac{e^{-at}-e^{-bt}}{t}\right)=\int_s^\infty\left(\frac{1}{x+a}-\frac{1}{x+b}\right)dx=\left[\log(s+a)-\log(s+b)\right]_s^\infty=\log\frac{s+a}{s+b}$

(4) $L\left(e^t\frac{d^n}{dt^n}(e^{-t}t^n)\right)=\int_0^\infty e^{-st}\left(e^t\frac{d^n}{dt^n}(e^{-t}t^n)\right)dt=\int_0^\infty e^{-(s-1)t}\left(\frac{d^n}{dt^n}(e^{-t}t^n)\right)dt$

By integrating by parts, we get

$$=(s-1)\int_0^\infty e^{-(s-1)t}\left(\frac{d^{n-1}}{dt^{n-1}}(e^{-t}t^n)\right)dt$$

By repeated integration by parts, we obtain

$$=(s-1)^n\int_0^\infty e^{-(s-1)t}(e^{-t}t^n)\,dt=(s-1)^n\int_0^\infty e^{-(s-1)t}t^n dt=(s-1)^n\frac{n!}{s^{n+1}}=n!\left(1-\frac{1}{s}\right)^n\frac{1}{s}$$

Here the expression $Ln(t)=\frac{1}{n!}e^t\frac{d^n}{dt^n}(e^{-t}t^n)$ denotes Laguerre's polynomials.

1.3.4. Laplace integrals for derivatives and integrals

1) Condider the integral for the first derivative of function $f(t)$. Integrating by parts, we obtain

$$L\left(\frac{df}{dt}\right)=L(f'(t))=\int_0^\infty e^{-st}f'(t)\,dt=\left[e^{-st}f(t)\right]_0^\infty+s\int_0^\infty e^{-st}f(t)\,dt=-f(0)+sF(s)$$

That is ;

$$L(f'(t))=sF(s)-f(0) \qquad (35)$$

where $f(0)$ means $f(+0)$, which denotes the right limit.

In a similar manner, we have for the second derivative :

$$L(f''(t))=s^2F(s)-sf(0)-f'(0) \qquad (36)$$

We get repeated integrating by parts

$$L(f^n(t))=s^nF(s)-s^{n-1}f(0)-s^{n-2}f'-(0)\cdots-f^{n-1}(0) \qquad (37)$$

Example. Verify the following formulas.

(1) $\lim_{s\to\infty} sF(s)=\lim_{t\to 0} f(t)$ (2) $\lim_{s\to 0} sF(s)=\lim_{t\to\infty} f(t)$

where $F(s)=L(f(t))$ and the limit of the function $f(t)$ is assumed to exist as $t\to 0$ or $t\to\infty$, namely, $\lim_{t\to\infty} f(t)=$ const. or $\lim_{t\to\infty} f(t)=$ const. These equalities are called 'Initial value theorem' and 'Final value theorem' respectively.

Solution.

(1) From the definition, it follows that

$$\lim_{s\to\infty} L(f')=\lim_{s\to\infty}\int_0^\infty e^{-st}f'(t)\,dt=0$$

Consequently, $\lim_{s\to\infty}(sF(s)-f(0))=0$. Then, we get

$$\lim_{s\to 0} sF(s)=f(0)$$

(2) From the formula (35),

$$L(f')=sF(s)-f(0)$$

$$\lim_{s\to 0} L(f')=\lim_{s\to\infty}\int_0^\infty e^{-st}f'(t)\,dt=\int_0^\infty f'(t)\,dt=f(\infty)-f(0)=\lim_{t\to\infty} f(t)-f(0)$$

Consequently, we get

$$\lim_{s\to 0} sF(s)=\lim_{t\to\infty} f(t)$$

2) Consider the following integral :

$$L\left(\int_0^t f(x)\,dx\right)=\int_0^\infty e^{-st}\left(\int_0^t f(x)\,dx\right)dx=\left[-\frac{1}{s}e^{-st}\int_0^t f(x)\,dx\right]_0^\infty+\frac{1}{s}\int_0^\infty e^{-st}f(t)\,dt=\frac{1}{s}F(s)$$

In a similar manner, we have

$$L\left(\int_0^t\int_0^{t_1} f(x)\,dxdx_1\right)=\frac{1}{s^2}F(s)$$

In general, we may expect

$$L\left(\underbrace{\int_0^t\cdots\int_0^{t_{n-1}}}_{n} f(x)\,dxdx_1\cdots dx_{n-1}\right)=\frac{1}{s^2}F(s) \qquad (38)$$

Example. Verify the following formulas.

(1) $L\left(\int_0^t \frac{f(x)}{x}dx\right)=\frac{1}{s}\int_s^\infty F(s)\,ds$

(2) $L\left(\int_t^\infty \frac{f(x)}{x}dx\right)=\frac{1}{s}\int_0^s F(s)\,ds$

Solution.

(1) $L\left(\frac{f(t)}{t}\right) \overset{(34)}{=} \int_s^\infty F(s)\,ds \Longrightarrow L\left(\int_0^t \frac{f(x)}{x}dx\right) \overset{(38)}{=} \frac{1}{s}\int_s^\infty F(x)\,ds$

(2) $L\left(\int_t^\infty \frac{f(x)}{x}dx\right) = L\left(\int_0^\infty \frac{f(x)}{x}dx\right) - L\left(\int_0^t \frac{f(x)}{x}dx\right)$

where

$$\int_t^\infty \frac{f(x)}{x}dx = \lim_{t\to\infty}\int_0^t \frac{f(x)}{x}dx = \lim_{s\to\infty} s\left[\frac{1}{s}\int_0^\infty F(s)\,ds\right] = \int_0^\infty F(s)\,ds$$

Consequently, we have

$$L\left(\frac{f(t)}{t}\right) = L\left(\int_0^\infty F(s)\,ds\right) - \frac{1}{s}\int_s^\infty F(s)\,ds = \frac{1}{s}\int_0^\infty F(s)\,ds - \frac{1}{s}\int_s^\infty F(s)\,ds = \frac{1}{s}\int_0^s F(s)\,ds$$

Exercise. Find Laplace transforms for the following functions.

(1) $\int_0^t \frac{\sin t}{t}\,dt$ (2) $\int_t^\infty \frac{\cos t}{t}\,dt$ (3) $\int_t^\infty \frac{e^{-t}}{t}\,dt$

Solution.

(1) $\frac{1}{s}\left(\frac{\pi}{2} - \tan^{-1}s\right)$ (2) $\frac{\log(s^2+1)}{2s}$ (3) $\frac{\log(s+1)}{s}$

1.3.5. Laplace transforms for periodic functions

A periodic function is written

$$f(t) = f(t+T) \tag{39}$$

where T denotes a period. Consider the integral:

$$L(f(t)) = \int_0^\infty e^{-st}f(t)\,dt = \int_0^T e^{-st}f(t)\,dt + \int_T^\infty e^{-st}f(t)\,dt$$

where

$$\int_T^\infty e^{-st}f(t)\,dt \overset{t-T=x}{=} \int_0^\infty e^{-s(x+T)}f(x+T)\,dx \overset{f(x+T)=f(x)}{=} e^{-sT}\int_0^\infty e^{-sx}f(x)\,dx = e^{-sT}L(f)$$

Consequently, it follows that

$$(1-e^{-sT})L(f) = \int_0^T e^{-st}f(t)\,dt$$

Then, the equation reduces to

$$L(f) = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\,dt \tag{40}$$

Example. Find Laplace transforms for the following periodic functions.

(1) $f(t) = f(t+T) = t$ (2) $f(t) = f(t+T) = \begin{cases} c & 0 < t < \frac{T}{2} \\ -c & \frac{T}{2} < t < T \end{cases}$

Solution.

(1) $L(f) = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}\,dt = \frac{1}{1-e^{-sT}}\left\{\frac{1}{s^2} - \frac{e^{-sT}}{s}\left(T + \frac{1}{s}\right)\right\}$

(2) $L(f) = \frac{1}{1-e^{-sT}}\left\{\int_0^{\frac{T}{2}} e^{-st}\cdot c\,dt + \int_{\frac{T}{2}}^\infty e^{-st}\cdot -c\,dt\right\} = \frac{c}{s}\tanh\left(\frac{Ts}{4}\right)$

1.4 Convolution and its Laplace transformation

Let's begin the definition of convolution or " * product". We shall write that of function of $f(t)$ and $g(t)$ to be

$$f*g \overset{der}{=} \int_0^t f(t-x)g(x)dx \tag{41}$$

Note that the symbol * denotes the convolution in distinction from the scalar product or " · product" of functions, that is, $f*g \neq f\cdot g = fg$.

Example 1. Let $f = g = t$. Evaluate the integral for the convolution $t*t$ and compare the result with the scalar product $t\cdot t$.

Solution.

$$t*t = \int_0^t (t-x)xdx = \left[tx - \frac{1}{2}x^2\right]_0^t = \frac{1}{6}t^3$$

$$t\cdot t = t^2$$

The difference between $t*t$ and $t\cdot t$ is clear.

Example 2. Evaluate the following convolutions, where m and n are integer or real mumber.

(1) $t^m * t^n$　　(2) $e^{mt} * \sin nt$　　(3) $\sin mt * \cos nt$

Solution.

$$(1) = \int_0^t (t-x)^m x^n dx = t^m \int_0^t \left(1 - \frac{x}{t}\right)^m x^n dx$$

$$\overset{\frac{x}{t}=T}{=} t^m \int_0^t (1-T)^m (tT)^n tdT = t^{m+n+1}\int_0^1 (1-T)^m T^n dT = t^{m+n+1}B(m+1, n+1)$$

This result shows that

$$\Gamma(m+1)\Gamma(n+1) = \Gamma(m+n+2)B(m+1, n+1)$$

which is the relation between Γ and B functions given in (8):

$$B(m+1, n+1) = \frac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(m+n+2)}$$

$$(2) = \int_0^t e^{-m(t-x)} \sin nx\, dx = e^{-mt}\int_0^t e^{mt} \sin nx\, dx = e^{-mt}\cdot(2)'$$

Then, the expression $(2)'$ reduces to

$$(2)' = \frac{1}{m^2+n^2}\{me^{mt}\sin nt - ne^{mt}\cos nt + n\}$$

Hence, we have

$$(2) = \frac{1}{m^2+n^2}\{m\sin nt - n\cos nt + ne^{-mt}\}$$

$$(3) = \int_0^t \sin m(t-x)\cos nx\, dx = \frac{1}{2}\int_0^t \{\sin(mt - mx + nx) + \sin(mt - mx - nx)\}dx$$

$$= \frac{m}{m^2-n^2}(\cos nt - \cos mt)$$

By setting $f = 1$ in the definition (41), the convolution is written

$$1*g = \int_0^t g(x)dx \tag{42}$$

Note that 1 means a constant-function, butt differs from a constant-value as a unit element. It is seen from the equation (42) that the role of 1 in the convolution is plyed as an integral operator.

Example 3. Show that

(1) $\underbrace{1*\cdots*1}_{n}=\frac{1}{(n-1)!}t^{n-1}$

(2) $\underbrace{1*\cdots*1}_{n}*g=\frac{1}{(n-1)!}\int_0^t(t-x)^{n-1}g(x)\,dx$

Solution.

$1*1=\int_0^t dx=t,\ 1*1*1=\int_0^t(t-x)\,dx=\frac{1}{2!}t^2$

We prove the equation (1) by mathematical induction on n. Taking $n=2$, we have $1*1=t$. On taking $n=k-1$, the equation (1) is assumed to be correct. Then, taking $n=k$, we may show the equation is correct as follows.

$$\underbrace{1*\cdots*1}_{k}=1*1^{k-1}=\int_0^t\frac{1}{(k-2)!}x^{k-2}dx=\frac{t^{k-1}}{(k-1)!} \tag{43}$$

It is seen from the result (1) and the definition of the convolution that

$$\underbrace{1*\cdots*1}_{n}*g=\frac{1}{(n-1)!}=\int_0^t(t-x)^{n-1}g(x)\,dx \tag{44}$$

The following are the fundamental properties of the convolution (41):

(a) $f*g=g*f$ (commutative law)

(b) $f*(g+h)=f*g+f*h$ (distributive law)

(c) $f*(g*h)=(f*g)*h$ (associative law)

In algebraic words, the convolution makes the ring.

Now, consider Laplace transformation for the convolution

$$L(f*g)=\int_0^\infty e^{-st}\left\{\int_0^t f(t-x)g(x)\,dx\right\}dt \tag{45}$$

In order to perform the integration in the equation (45), we insert the unit step function $H(t-x)$ into the integrand.

$$L(f*g)=\int_0^\infty e^{-st}\left\{\int_0^\infty f(t-x)H(t-x)g(x)\,dx\right\}dt=\int_0^\infty g(x)\left\{\int_0^\infty e^{-st}f(t-x)H(t-x)\,dt\right\}dx$$

By substitution $t-x=T$ we get

$$=\int_0^\infty e^{-sx}g(x)\,dx\int_0^\infty e^{-sT}f(T)H(T)\,dt=G(s)F(s)$$

we write this result as

$$L(f*g)=L(f)L(g)=F(s)G(s)$$

or as the same symbols

$$f*g\ \underset{L^{-1}}{\overset{L}{\rightleftharpoons}}\ F(s)G(s)\ \text{ or } f*g\Leftrightarrow F(s)G(s)$$

The above formula shows that Laplace transformation for the convolution is the product of each Laplace transform of f and g.

By using the convolution, the formula (38) is directly shown

$$L(1*f(t))=L(1)L(f(t))=\frac{1}{s}F(s)$$

In a similar manner, we have

$$L\left(\int_0^t\int_0^{t_1}f(x)\,dxdx_1\right)=L(1*1*f)=\frac{1}{s^2}F(s)$$

In general, we get

$$L\left(\underbrace{\int_0^t \cdots \int_0^{t_{n-1}}}_{n} f(x)\, dx dx_1 \cdots dx_{n-1}\right)=\frac{1}{s^n}F(s)$$

1.5 Delta function and Its Laplace transformation

This section gives Laplace transformation for the delta function, which is denoted by δ. The theory of the delta function, which is often called generalized function, was given a rigorous foundation from distribution and hyperfunction once. We treat it by means of the simplified model, but, the results obtained hold correct even in the regorous theories.

We consider a following rectangle function

$$f(t)=\begin{cases} c & (a<t<b) \\ 0 & (t<a,\ t>b) \end{cases}, \quad \int_a^b f(t)\,dt=\int_{-\infty}^{+\infty} f(t)\,dt=1$$

A problem is that evaluate $L(f(t))$ and $\lim_{h\to 0}\left(\frac{L(f(t))}{h}\right)$. In order to find $L(f(t))$, we write

$$f(t)=c\{H(t-a)-H(t-b)\}$$

At once, $L(f(t))$ may be given from the formula (32)

$$L(f(t))=c\frac{1}{s}(e^{-sa}-e^{-bs})$$

Substituting $b=a+h$ and making $h\to 0$, we have

$$\lim_{h\to 0}\left(\frac{L(f(t))}{h}\right)=c\frac{1}{hs}e^{-sa}\{1-(1-hs+\cdots)\}=ce^{-sa}$$

Here we assume that the operations of the limit $\lim_{h\to 0}$ and the Laplace transformation L are commutative.

Then, we have

$$\lim_{h\to 0}\left(\frac{L(f(t))}{h}\right)=L\left(\lim_{h\to 0}\frac{f(t)}{h}\right)=cL\left(\lim_{h\to 0}\frac{H(t-a)-H(t-b)}{h}\right)=cL\left(\frac{dH(t-a)}{dt}\right)=ce^{-sa}$$

Then, we define $\frac{dH(t)}{dt}$ by δ:

$$\frac{dH(t-a)}{dt}=\delta(t-a) \tag{46}$$

In words, the differentiation of the step function gives the delta function. Then, we get

$$L(\delta(t-a))=\int_0^\infty e^{-st}\delta(t-a)\,dt=e^{-sa} \tag{47}$$

Putting $a=0$, them, we have

$$L\left(\frac{dH(t)}{dt}\right)=L(\delta(t))=\int_0^\infty e^{-st}\delta(t)\,dt=e^{-s0}=1 \tag{48}$$

Example. Find Laplace transforms for derivatives, $\frac{d^{n+1}H(t-a)}{dt^{n+1}}=\delta^{(n)}(t-a)$. For $n=0$, $\delta^{(0)}$ means δ.

Solution.

$$L(\delta(t-a))=e^{-sa},\ L\left(\frac{d}{dt}\delta(t-a)\right)=\frac{d}{dt}(e^{-st})\Big|_{t=a}=se^{-sa},\ L\left(\frac{d^2}{dt^2}\delta(t-a)\right)=-\frac{d^2}{dt^2}(e^{-st})\Big|_{t=a}=s^2_e{}^{-sa}$$

$$\vdots$$

$$L\left(\frac{d^n}{dt^n}t-a)\right)=(-1)^n s^n e^{-sa}$$

When $a = 0$, then, it follows that

$$L(\delta^{(n)}) = s^n$$

and

$$L(\delta^{(0)}) = L(\delta) = s^0 = 1$$

1.6 Error function and its related Laplace transformations

Error function is defined

$$\mathrm{Erf}t = \frac{2}{\sqrt{\pi}}\int_0^t e^{-x^2}dx \tag{49}$$

and complementary error function is defined

$$\mathrm{Cert}t = 1 - \mathrm{Erf}t = \frac{2}{\sqrt{\pi}}\int_t^\infty e^{-x^2}dx \tag{50}$$

1) Consider the integral;

$$L(\mathrm{Erf}t) = \frac{2}{\sqrt{\pi}}\int_0^\infty\left(\int_0^t e^{-x^2}dx\right)e^{-st}dt = \frac{1}{s}F(s)$$

where

$$F(s) = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-t^2}e^{-st}dt = e^{\left(\frac{s}{2}\right)^2}\frac{2}{\sqrt{\pi}}\int_0^\infty e^{-\left(t+\frac{s}{2}\right)^2}dt = e^{\left(\frac{s}{2}\right)^2}\frac{2}{\sqrt{\pi}}\int_t^\infty e^{-x^2}dx = e^{\left(\frac{s}{2}\right)^2}\mathrm{Cerf}\,t$$

Hence, as the formula it is given

$$L(\mathrm{Erf}t) = \frac{1}{s}\left\{e^{\left(\frac{s}{2}\right)^2}\mathrm{Cerf}\,t\right\} \tag{51}$$

2) Consider the integral

$$L(\mathrm{Erf}\sqrt{t}) = \frac{2}{\sqrt{\pi}}\int_0^\infty\left(\int_0^{\sqrt{t}} e - {}^{x^2}dx\right)e^{-st}dt$$

By integration by parts, it follows that

$$= \frac{2}{\sqrt{\pi}}\left[\int_0^{\sqrt{t}} e^{-x^2}dx \cdot \frac{-e^{sx}}{s}\right]_0^\infty + \frac{2}{\sqrt{\pi}}\frac{1}{s}\int_0^\infty \frac{1}{2\sqrt{t}}e^{-t}e^{-st}dt$$

$$= \frac{1}{s\sqrt{\pi}}\int_0^\infty e^{-st}\left(e^{-t}\frac{1}{\sqrt{t}}\right)dt = \frac{1}{s\sqrt{\pi}}\frac{\sqrt{\pi}}{\sqrt{s+1}} = \frac{1}{s\sqrt{s+1}}$$

Then, we have

$$L(\mathrm{Erf}\sqrt{t}) = \frac{1}{s\sqrt{s+1}} \tag{52}$$

Form these results, the following formulas may be obtained

$$L(e^{-t}\mathrm{Erf}\sqrt{t}) = \frac{1}{(s+1)\sqrt{s+2}} \tag{53}$$

$$L(e^{t}\mathrm{Erf}\sqrt{t}) = \frac{1}{(s-1)\sqrt{s}} \tag{54}$$

By the use of the definition, we get

$$L(\mathrm{Cerf}\sqrt{t}) = L(1 - \mathrm{Erf}\sqrt{t}) = \frac{1}{s} - \frac{1}{s\sqrt{s+1}} = \frac{1}{\sqrt{s+1}(\sqrt{s+1}+1)}$$

From this, the following formulas are given

$$L(e^{-t}\mathrm{Cerf}\sqrt{t}) = \frac{1}{\sqrt{s+2}(\sqrt{s+2}+1)} \tag{55}$$

$$L(e^{t}\mathrm{Cerf}\sqrt{t}) = \frac{1}{\sqrt{s}(\sqrt{s+1})} = \frac{1}{s+\sqrt{s}} \tag{56}$$

3) Consider the integral

$$\frac{\sqrt{\pi}}{2}L\left(\mathrm{Erf}\frac{1}{\sqrt{t}}\right)=\int_0^\infty\left(\int_0^{\frac{1}{\sqrt{t}}}e^{-x^2}dx\right)e^{-st}dt=\left[\int_0^{\frac{1}{\sqrt{t}}}e^{-x^2}dx\cdot\frac{-e^{-st}}{s}\right]_0^\infty-\frac{1}{2s}\int_0^\infty\frac{1}{t^{\frac{3}{2}}}e^{-\frac{1}{t}}e^{-st}dt$$

$$\overset{t^{-\frac{1}{2}}=u}{=}\frac{1}{s}\int_0^\infty e^{-x^2}dx-\frac{1}{2s}\int_0^\infty e^{-\left(u^2+\frac{s}{u^2}\right)}du=\frac{1}{s}\frac{\sqrt{\pi}}{2}-\frac{1}{2s}\frac{\sqrt{\pi}}{2}e^{-2\sqrt{s}}$$

where the following integral formula is used

$$\int_0^\infty e^{-c\left(\frac{u^2}{a^2}+\frac{b^2}{u^2}\right)}du=\frac{a}{2}\sqrt{\frac{\pi}{c}}e^{-\frac{2bc}{a}}\quad(a,\ b,\ c>0)$$

Consequently,

$$L\left(\mathrm{Erf}\frac{1}{\sqrt{t}}\right)=\frac{1}{s}\left(1-\frac{1}{2}e^{-2\sqrt{s}}\right)$$

Then, we have

$$L\left(\mathrm{Cerf}\frac{1}{\sqrt{t}}\right)=L\left(1-\mathrm{Erf}\frac{1}{\sqrt{t}}\right)=\frac{1}{s}-\frac{1}{s}\left(1-\frac{1}{2}e^{-2\sqrt{s}}\right)=\frac{1}{2s}e^{-2\sqrt{s}}$$

In a similar manner, we have

$$L\left(\mathrm{Erf}\frac{a}{2\sqrt{t}}\right)=\frac{1}{s}\left(1-\frac{1}{s}e^{-a\sqrt{s}}\right)\tag{57}$$

By differentiating with respect to the parameter a, the followings may be obtained

$$L\left(\frac{1}{\sqrt{\pi t}}e^{-\frac{a^2}{4t}}\right)=\frac{1}{s}e^{-a\sqrt{s}}\tag{58}$$

and

$$L\left(\frac{1}{2\sqrt{\pi}\,t^{\frac{3}{2}}}e^{-\frac{a^2}{4t}}\right)=e^{-a\sqrt{s}}\tag{59}$$

2 Applications of Laplace Transformations

2.1 Ordinary Differential Equation

We consider here an application that is in particular related to the solving differential and integral equations. It is convenient for the solving to utilize the Laplace transformation of the convolution and the delta function.

An important differential equation of general typa, which is known motion of equation or Newton's equation, is

$$m\frac{d^2y(t)}{dt^2}=F(t)\tag{1}$$

where unknown function $y(t)$ and given function $F(t)$ are a displacement and external force dependent on time-variable t respectively and m, mass of a particle, is a constant.

In order to solve the differential equation (1) by the method of Laplace transformation, we may proceed as follows:

step (1). Make the differential equation Laplace transformation

step (2). Solve algebraic equation as a function of the parameter s

step (3). Find solution by inverse Laplace transformation

We shall illustrate the above procedure by some examples. First let us take equation (1) as an example:

step (1):

$$m\left(s^2Y(s)-sy(0)-y'(0)\right)=F(s)$$

where $Y(s)$ and $F(s)$ are image functions of $y(t)$ and $F(t)$ respectively and $y(0)$ and $y'(0)$ are constants of the initial condition.

step (2):

$$Y(s)=\frac{y(0)}{s}+\frac{y'(0)}{s^2}+\frac{F(s)}{m}\frac{1}{s^2} \tag{2}$$

step (3):

$$L^{-1}(Y(s))=y(t)=y(0)=y'(0)t+\frac{1}{m}t*F(t) \tag{3}$$

We shall distinguish between $F(t)=0$ and $F(t)\neq 0$, which are called the homogeneous system under differential equations and the non-homogeneous one, respectively:

(a) $F(t)=0$: From (3) a solution is written

$$y(t)=y(0)+y'(0)t$$

Then, the velocity of a particle $v(t)$ is the derivative

$$y'(t)=v(t)=y'(0)$$

which represents the motion of uniform velocity or the law of inertia.

(b) $F(t)\neq 0$: When $F(t)=-mg$, where g is what is called the gravitational constant, a solution is written from the equation (3)

$$y(t)=y(0)+y'(0)t-\frac{g}{2}t^2 \tag{4}$$

If $y(0)=y'(0)=0$, then,

$$y(t)=-\frac{g}{2}t^2 \tag{5}$$

which is well-known the formula for free fall. Then, the accelation is the derivative of the velocity

$$y''(0)=v'(t)=-g$$

which represents uniformly accelerated motion.

As is seen from the above procedure, a merit of the method of solving differential equations by Laplace transformation is that if nessesary, either of the general solution and the special solution with the initial condition may be chosen in solving process.

Example 1. Solve the following differential equation:

$$m\frac{d^2y(t)}{dt^2}=-mg-mc\frac{dy(t)}{dt} \tag{5}$$

where the term $mc\frac{dy}{dt}$ is due to medium resistance, which is assumed to be propotional to the velocity of a particle and mc is a constant coefficient of damping force.

Solution

step (1):

$$m(s^2Y(s)-sy(0)-y'(0))=-mg\frac{1}{s}-mc(sY(s)-y(0))$$

$$s(s+c)Y(s)=sy(0)+y'(0)+cy(0)-g\frac{1}{s}$$

step (2):

$$Y(s)=\frac{y(0)}{s+c}+\frac{y'(0)+cy(0)}{s(s+c)}-g\frac{1}{s^2(s+c)} \tag{6}$$

step (3):

$$L^{-1}(Y(s))=y(t)=y(0)e^{-ct}+(y'(0)+cy(0))\times(1*e^{-ct})-g(t*e^{-ct}) \tag{7}$$

If $y(0)=y'(0)=0$, then,

$$y(t) = -g(t * e^{-ct}) = -g\int_0^t (t-x)e^{-cx}dx = -\frac{g}{c}\left(t - \frac{1}{c}(1-e^{-ct})\right) \tag{8}$$

It is obvious that when $c = 0$, the solution refers to the formula for free fall, since $t*1 = 1*t = \frac{t^2}{2}$.

The velocity $v(t)$ is the derivative:

$$\frac{dy(t)}{dt} = y'(t) = v(t) = -\frac{g}{c}(1+e^{-ct})$$

so that

$$v(t \to \infty) = -\frac{g}{c} = \text{const.}$$

which is called "terminal velocity".

As a second example, we shall solve the following differential equation:

$$m\frac{d^2y(t)}{dt^2} + ky(t) = F(t) \tag{9}$$

which is called a differential equation for forced vibration. The term $ky(t)$ means restoring force and k is a spring constant. Upon rewriting, the equation (9) reduces to

$$\frac{d^2y(t)}{dt^2} + w_0^2 y(t) = F(t) \tag{10}$$

where $\sqrt{\frac{k}{m}} = w_0$, which is called proper frequency of the system and $F(t)\left(F(t) = \frac{1}{m}F(t)\right)$ is a renewed function. Let's solve the equation (9) as in proceeding procedure

step (1):

$$\{s^2Y(s) - sy(0) - y'(0)\} + w_0^2 Y(s) = F(s)$$

step (2):

$$Y(s) = \frac{sy(0)}{s^2+w_0^2} + \frac{y'(0)}{s^2+w_0^2} + \frac{1}{s^2+w_0^2}F(s) \tag{11}$$

step (3):

$$L^{-1}(Y) = y(t) = y(0)\cos w_0 t + \frac{y'(0)}{\sqrt{w_0}}\sin w_0 t + \frac{1}{\sqrt{w_0}}\sin w_0 t * F(t) \tag{12}$$

(a) $F(t) = 0$: A solution is

$$y(t) = y(0)\cos w_0 t + \frac{y'(0)}{\sqrt{w_0}}\sin w_0 t \tag{13}$$

which represents the motion of simple harmonic vibration (oscillation).

Example 2. Solve the differential equation (10) under the following restricted condition, $y(0) = y(a) = 0$, which is called the boundary condition.

Solution. With the use of the restricted conditions and the solution (13), it follows that

$$y(0) = 0$$

$$y(a) = \frac{y'(0)}{\sqrt{w_0}}\sin w_0 a = 0$$

Consequently, we obtain

$$w_0 = \frac{n\pi}{a}$$

which is called proper or eigen frequency. In a limited sense, the integer n is called discrete eigen value. Then, the solution is written

$$y(t)=\frac{y'(0)}{\sqrt{w_0}}\sin\frac{n\pi}{a}t=c\sin\frac{n\pi}{a}t$$

which is called an eigen function for the eigen value n and the constant c is determined under the normalization condition.

(b) $F(t)\neq 0$: If $y(0)=y'(0)=0$ and let $F(t)$ be $F\cos wt$, where F is a constant, a solution due to the external force is written

$$y(t)=\frac{1}{\sqrt{w_0}}\sin w_0 t*F\cos wt=\frac{F}{\sqrt{w_0}}\int_0^t \sin w_0(t-x)\cos wx dx=\frac{F}{\sqrt{w_0}}\frac{2w_0}{w^2-w_0^2}\sin\left(\frac{w+w_0}{2}t\right)\sin\left(\frac{w-w_0}{2}t\right)$$

When $w\to w_0$, which brings about resonance phenomina, a solution reduces to

$$y(t)\overset{w\to w_0}{=}\frac{F}{2\sqrt{w_0}}t\sin w_0 t\xrightarrow{t\to\infty}\infty$$

In order avoiding divergent solution, a damping force may be taken into account in the equation (10):

$$\frac{d^2y(t)}{dt^2}+2c\frac{dy(t)}{dt}+w_0^2y(t)=F(t) \tag{14}$$

where a factor 2 is meaningless, only for convenience. By ordinaly procedure, a solution may be written

$$y(t)=y(0)e^{-ct}\cos\sqrt{w_0^2-c^2}t+\frac{y'(0)+2cy(0)-cy(0)}{\sqrt{w_0^2-c^2}}$$

$$\times e^{-ct}\sin\sqrt{w_0^2-c^2}t+\frac{1}{\sqrt{w_0^2-c^2}}e^{-ct}\sin\sqrt{w_0^2-c^2}t*F(t)$$

A solution due to the external force $F(t)=F\cos wt$ may be written

$$y(t)=\frac{F}{\sqrt{w_0^2-c^2}}e^{-ct}\sin\sqrt{w_0^2-c^2}t*\cos wt=\frac{F}{\sqrt{w_0^2-c^2}}\int_0^t e^{-cx}\sin\sqrt{w_0^2-c^2}x\cdot\cos w(t-x)dx$$

when $t\to\infty$, the terms with damping factor e^{-ct} vanish. The remaining terms are

$$y(t)\xrightarrow{t\to\infty}\frac{c}{c^2+(w-w_0)^2}\sin wt+\frac{w-w_0}{c^2+(w-w_0)^2}\cos wt=\frac{1}{\sqrt{c^2+(w-w_0)^2}}\sin(wt+\phi)$$

where

$$\tan\phi=\frac{c}{w-w_0}$$

2.2 Solving method by the use of delta function

In the previous section, we have practiced solving differential equations by the method of Laplace transformation. In this section, we like to find solving differential equations more systematically by Laplace transformation with the help of the delta function. From here, ordinary differential operators $\frac{d^n}{dt^n}$ ($n=0, 1, 2,$ ......) are denoted by D^n for simple desription and in particular D^0 means I, which denotes the unit operator $D^0 f=If=f$.

Putting $F(t)=\delta(t)$, which represents an impulsive force at $t=0$, in the motion of equation (1), it reduces to

$$mD^2u(t)=\delta(t) \tag{15}$$

According to the same procedure as before, we shall solve it:

step (1):

$m\{s^2U(s)-su(0)-u'(0)\}=1$ where $L(\delta)=1$

step (2):

$$U(s)=\frac{1}{m}\left(\frac{u(0)}{s}+\frac{u'(0)}{s^2}\right)+\frac{1}{ms^2}$$

step (3):

$$L^{-1}(U(s))=u(t)=\frac{1}{m}(u(0)+u'(0)t)+\frac{1}{m}t$$

Formally, when $u(0)=u'(0)=0$, then, a solution is written

$$u(t)=\frac{1}{m}t \tag{16}$$

Such a solution due to an impulsive force is called "impulse response" or " δ response" for simplicity. It should be noted that a similar solution may be obtained for the differential equation,

$$mD^2u(t)=0$$

with the intial conditions: $u(0)$ and $u'(0)=1$.

Example 1. Find solutions for the following differential equations with initial conditions.

(1) $Du=\delta$, $u(0)$ (2) $Du=0$, $u(0)=1$

Solution.

(1) $sU(s)-u(0)=sU(s)=1,\ U(s)=\frac{1}{s},\ u(t)=H(t)$

(2) $sU(s)-u(0)=sU(s)-1=0,\ U(s)=\frac{1}{s},\ u(t)=H(t)$

Note that the general solution for $Du=\delta$ is $u(t)=H(t)+c$ ($c=$ const.).

Example 2. find solutions for the following differential equations:

(1) $Dy+y=\delta$ (2) $Dy+y=H(t)$ (3) $Dy+y=t$

Solution.

(1) $sY-y(0)+Y=1,\ Y=\frac{y(0)}{s+1}+\frac{1}{s+1},\ y(t)=y(0)e^{-t}+e^{-t}$

(2) $sY-y(0)+Y=\frac{1}{s},\ Y=\frac{y(0)}{s+1}+\frac{1}{s(s+1)},\ y(t)=y(0)e^{-t}+e^{-t}*1$

(3) $sY-y(0)+Y=\frac{1}{s^2},\ Y=\frac{y(0)}{s+1}+\frac{1}{s^2(s+1)},\ y(t)=y(0)e^{-t}+e^{-t}*t$

Now, consider the relation between the solutions $u(t)$ and $y(t)$, which is the solution for $mD^2y(t)=F(t)$ with the initial condetions: $y(0)=y'(0)=0$ (see the previous section). From a comparison between $u(t)$ and $y(t)$, the solution $y(t)$ may be written

$$y(t)=u(t)*F(t)=\int_0^t u(t-x)F(x)\,dx \tag{17}$$

As an example, putting $u(t)=\frac{1}{m}t$ and $F(t)=-mg$ in (17), then, we have

$$y(t)=\frac{1}{m}t*(-mg)=-g(t*1)=-\frac{g}{2}t^2 \tag{18}$$

which corresponds to the solution for free fall given in (5).

The statement (17) may hold in general: If a solution for $\sum_{k=0}^{n} a_k D^k u(t)=\delta(t)$ has been known, a solution for $\sum_{k=0}^{n} a_k D^k y(t)=F(t)$ may be verified in terms of the convolution

$$\sum_{k=0}^{n} a_k D^k y=\sum_{k=0}^{n} a_k D^k(u*F)=\sum_{k=0}^{n}(a_k D^k u*F)=\delta*F=F(t)$$

As an alternative way of statement, a solution for the non-homogenious system of differential equations may be represented formally if the solution due to an impulsive force, namely, the impulse response has been obtained.

According to the above statement, we shall find the solution for $mD^2y+mcDy=-mg$ with $y(0)=y'$

$(0)=0$. First, solve the differential equation $mD^2y + mcDy = \delta$ with $u(0)=u'(0)=0$. Its solution is written

$$U(s)=\frac{1}{m}\frac{1}{s(s+c)},\ u(t)=\frac{1}{m}1 * e^{-ct}$$

Then, it follows that

$$y(t)=u(t)*F(t)=\frac{1}{m}\{1 * e^{-ct} * (-mg)\}=-g(1 * e^{-ct} * 1) \qquad (19)$$

$$=-g(t * e^{-ct}) \qquad (20)$$

Example 3. Find solutions (δ response) for the following differential equations with the inital conditions: $u(0)=u'(0)=0$

(1) $D^2u + w_0^2 u = \delta$ (2) $D^2u + 2cDu + w_0^2 u = \delta$

Solution.

(1) $(s^2U(s)-su(0)-u'(0))+w_0^2U(s)=(s^2+w_0^2)U(s)=1\ U(s)=1,$

$$U(s)=\frac{1}{s^2+w_0^2},\ u(t)=\frac{1}{\sqrt{w_0}}\sin w_0t$$

(2) $(s^2U(s)-su(0)-u'(0))+2c(sU(s)-u(0))+w_0^2U(s)=(s^2+2cs+w_0^2)U(s)=1$

$$U(s)=\frac{1}{(s+c)^2+(\sqrt{w_0^2-c^2})^2},\ u(t)=\frac{1}{\sqrt{w_0^2-c^2}}e^{-ct}\sin\sqrt{w_0^2-c^2}t$$

By using these solutions and the formula (17), we can easily give the solutions for the differential equations,

$D^2y + w_0^2y = F\cos wt$ or $D^2y + 2cDy + w_0^2y = F\cos wt$

Here, we shall introduce new functions and simplify the description. As is seen from the Examples 2 and 3, an algebraic equation solved as a function of the parameter s after Laplace transformations of differential equations in quastion, may be written

step (1)

$$G(s)Y(s)-G_0(s)=F(s)$$

where the terms $G(s)$ and $G_0(s)$ are polynomials of the parameter s. The degree of the polynomial $G(s)$ is always heigher than that of $G_0(s)$ in general. The term "$G_0(s)$" depends on values of the initial condition. When G_0, the solution $Y(s)$ for s reduces to

step (2)

$$Y(s)=\frac{1}{G(s)}\{G_0(s)+F(s)\}$$

Hence, in terms of L^{-1} transformation the desired solution $y(t)$ is

step (3)

$$y(t)=L^{-1}\left(\frac{1}{G(s)}\right)*\{L^{-1}(G_0(s))+L^{-1}(F(s))\}=u(t)*\{L^{-1}(G_0(s))+L^{-1}(F(s))\} \qquad (21)$$

When $G_0(s)=0$, the solution may be written again

$$y(t)=u(t)*F(t)=\int_0^t u(t-x)F(x)\,dx \qquad (22)$$

That is to say, "to find the solution $y(t)$ is to write $G(s)$ down". The from of "$G(s)$" is determined by the proper claracter only in the differential equation in quastion and the equation about $G(s)=0$ is called "characteristic equation" in the ordinary linear differential equation theory. $\frac{1}{G(s)}$ is called transfer function, denoted as $W(s)$, and $L^{-1}\left(\frac{1}{G(s)}\right)$ is "impulse response" (δ response) as known.

As a summary until here, We shall solve the general type of ordinary linear differential equations of the second order

$$aD^2y(t)+bDy(t)+cy(t)=f(t) \qquad (23)$$

where the coefficents a, b, c, are real. As in preceding procedure, a desired algebraic equation as a function of s may be written

$$G(s)Y(s)-G_0(s)=F(s),\ Y(s)=\frac{1}{G(s)}\{G_0(s)+F(s)\}$$

where

$$G(s)=as^2+bs+c;\ G_0(s)=ay(0)s+ay'(0)+by(0)$$

It is said that it is enough for the solving equation to evaluate $\frac{1}{G(s)}=T(s)$

According to conditions of the discriminant $D=b^2-4ac$ of the quadratic function $G(s)$ we have to consider solutions for three cases, that is, $D>0$, $D=0$, $D<0$.

(1) $D>0$. The quadratic function $G(s)$ may be factored

$$G(s)=a(x-\alpha)(x-\beta)$$

where α, β are two different roots of $G(s)=0$:

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-b\pm\sqrt{D}}{2a}$$

Hence, the solution $\frac{1}{G(s)}$ may be written

$$u(t)=L^{-1}\left(\frac{1}{G(s)}\right)=AL^{-1}\left(\frac{1}{s-\alpha}\right)+BL^{-1}\left(\frac{1}{s-\beta}\right)$$
$$=Ae^{\alpha t}+Be^{\beta t} \tag{24}$$

where A and B are new constants.

(2) $D=0$. As in (1),

$$G(s)=a(x-\alpha)^2$$

where $\alpha=-\frac{b}{2a}$ are equal roots of $G(s)=0$. Then, we have

$$u(t)=Ate^{\alpha t} \tag{25}$$

where A is a renewed constant.

(3) $D<0$. The $G(s)$ may be factored as

$$G(s)=a(x-\alpha')(x-\beta')$$

where α', β' are two imaginary roots of $G(s)=0$.

$$\begin{pmatrix}\alpha'\\ \beta'\end{pmatrix}=\frac{-b\pm i\sqrt{b^2-4ac}}{2a}=\frac{-b\pm i\sqrt{D}}{2a}$$

Then, the solution $u(t)$ is

$$u(t)=Ae^{\alpha' t}+Be^{\beta' t} \tag{26}$$

where A and B are renewed constants. In the real form of the function $u(t)$, it follows that

$$u(t)=A'e^{-\frac{b}{2a}\cdot t}\sin\sqrt{b^2-4ac}t+B'e^{-\frac{b}{2a}t}\cos\sqrt{b^2-4ac}t$$

Examples 4. Solve the following differential equations with initial conditions; $y(0)=y'(0)=0$.

(1) $2D^2y+3Dy+y=\delta$ (2) $4D^2y-4Dy+y=\delta$

(3) $2D^2y+Dy+3y=\delta$

Solution.

(1) $G(s)=(2s+1)(s+1)$, $y(t)=2e^{\frac{1}{2}t}+e^{-t}$

(2) $G(s)=(2s+1)^2$, $y(t)=te^{-\frac{1}{2}t}$

(3) $G(s)=\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{11}}{2}\right)^2=\left(s+\frac{1}{2}\right)^2-\left(i\frac{\sqrt{11}}{2}\right)^2$

$$y(t)=Ae^{-\frac{1}{2}t+i\frac{\sqrt{11}}{2}t}+Be^{-\frac{1}{2}t-i\frac{\sqrt{11}}{2}t}=e^{-\frac{1}{2}t}\left(A'\sin\frac{\sqrt{11}}{2}t+B'\cos\frac{\sqrt{11}}{2}t\right)$$

2.3 Simultaneous Differential Equations

Instead of the solving procedure of the differential equation (1), by introducing new function v (t) we shall solve the following system of differential equations of first order.

$$\begin{cases} mDy(t)=v(t) \\ Dv(t)=F(t) \end{cases} \tag{27}$$

Of course, the solution obtained is the same one of the equation (1) since two equations are identical. As previously,

$$m\{sY(s)-y(0)\}=V(s)$$
$$sV(s)-v(0)=F(s)$$

These equations reduce to

$$msY(s)-V(s)=my(0)$$
$$sV(s)=v(0)+F(s)$$

Eliminating $V(s)$ from these equations, we obtain the relation

$$Y(s)=\frac{y(0)}{s}+\frac{v(0)}{m}\frac{1}{s^2}+\frac{F(s)}{m}\frac{1}{s^2} \tag{28}$$

This equation is formally identical with the equation (2), since $v(0)=my'(0)$.

It is more convenient for finding solutions to write the equation (27) in the matrix form:

$$\begin{pmatrix} ms & -1 \\ 0 & s \end{pmatrix}\begin{pmatrix} Y(s) \\ V(s) \end{pmatrix}=A\begin{pmatrix} Y(s) \\ V(s) \end{pmatrix}=\begin{pmatrix} my(0) \\ v(0)+F(s) \end{pmatrix}$$

where the matrix and its determinant are

$$A=\begin{pmatrix} ms & -1 \\ 0 & s \end{pmatrix} \text{ and } \det A = |A| = \begin{bmatrix} ms & -1 \\ 0 & s \end{bmatrix} = ms^2$$

The solutions $Y(s)$ and $V(s)$ are given simultaneously by using Cramel's formula

$$Y(s)=\frac{\begin{bmatrix} my(0) & -1 \\ v(0)+F(s) & s \end{bmatrix}}{\det A}=\frac{y(0)}{s}+\frac{v(0)}{m}\frac{1}{s^2}+\frac{F(s)}{m}\frac{1}{s^2}$$

and

$$V(s)=\frac{\begin{bmatrix} ms & my(0) \\ 0 & v(0)+F(s) \end{bmatrix}}{\det A}=\frac{1}{s}\{v(0)+F(s)\}$$

we shall further examine the above solving method in the following examples.

Example 1.

$$(1)\begin{cases} mDy(t)=v(t) \\ Dv(t)=-mg-mcDy(t) \end{cases} \qquad (2)\begin{cases} Dy(t)=v(t) \\ Dv(t)=-w_0^2y(t)+F(t) \end{cases}$$

These equations (1) ans (2) correspond to the equations (5) and (10) respectively.

Solution.

(1) A desired simultaneous equation may be written

$$\begin{cases} msY(s)-V(s)=my(0) \\ mcsY(s)+sV(s)=v(0)+mcy(0)-mg\frac{1}{s} \end{cases}$$

In the matrix form,

$$\begin{pmatrix} ms & -1 \\ mcs & s \end{pmatrix}\begin{pmatrix} Y(s) \\ V(s) \end{pmatrix}=A\begin{pmatrix} Y(s) \\ V(s) \end{pmatrix}=\begin{pmatrix} my(0) \\ v(0)+mcy(0)-mg\frac{1}{s} \end{pmatrix}$$

where

$$A=\begin{pmatrix} ms & -1 \\ mcs & s \end{pmatrix} \text{ and } \det A = |A| = ms(s+c)$$

The solutions $Y(s)$ and $V(s)$ are given by Cramel's formula

$$Y(s)=\frac{\begin{bmatrix} my(0) & -1 \\ v(0)+mcy(0)-mg\frac{1}{s} & s \end{bmatrix}}{\det A}=\frac{y(0)}{s+c}+\frac{v(0)+mcy(0)}{ms(s+c)}-\frac{g}{s^2(s+c)} \qquad (29)$$

and

$$V(s)=\frac{\begin{bmatrix} ms & my(0) \\ mcs & v(0)+mcy(0)-mg\frac{1}{s} \end{bmatrix}}{\det A}=\frac{v(0)}{s+c}-mg\frac{1}{s(s+c)}$$

(2)

$$\begin{cases} sY(s)-V(s)=y(0) \\ w_0^2Y(s)+sV(s)=v(0)+F(s) \end{cases}$$

Here,

$$A=\begin{pmatrix} s & w_0^2 \\ w_0^2 & s \end{pmatrix} \text{ and } \det A=|A|=s^2+w_0^2$$

The solutions $Y(s)$ and $V(s)$ are given

$$Y(s)=\frac{\begin{bmatrix} y(0) & v(0)+F(s) \\ -1 & s \end{bmatrix}}{\det A}=\frac{1}{s^2+w_0^2}\{y(0)s+v(0)+F(s)\}$$

and

$$Y(s)=\frac{\begin{bmatrix} s & y(0) \\ w_0^2 & v(0)+F(s) \end{bmatrix}}{\det A}=\frac{1}{s^2+w_0^2}\{v(0)+F(s)-y(0)w_0^2\}$$

Consider an interacting system of simple harmonic oscillations, which is called "coupled oscillation". Each equation of motion for the particles, m_1 and m_2 is

$$\begin{cases} m_1D^2y_1(t)=-k_1y_1(t)+K(y_2(t)-y_1(t)) \\ m_2D^2y_2(t)=-k_2y_2(t)-K(y_2(t)-y_1(t)) \end{cases} \qquad (30)$$

where $K\,|\,y_2-y_1\,|$ is the interaction term, which is assumed to be proportional to the difference between the displacements of two particles m_1 and m_2 (K = const.).

In the following, we assume for simplicity that $m_1=m_2=m$ and $k_1=k_2=k$. Then, the equation (30) reduces to

$$\begin{cases} Dy_1(t)=-w_0^2y_1(t)+k^2(y_2(t)-y_1(t)) \\ Dy_2(t)=-w_0^2y_2(t)-k^2(y_2(t)-y_1(t)) \end{cases} \qquad (31)$$

where $w_0^2=\frac{k}{m}$ and $k^2=\frac{K}{m}$. As usual,

$$\begin{cases} (s^2+w_0^2+k^2)Y_1(s)-k^2Y_2(s)=y_1(0)s-y'_1(0) \\ -k^2Y_1(S)+(s^2+w_0^2+k^2)Y_2(s)=y_2(0)s-y'_2(0) \end{cases}$$

where $L(y_1(t))=Y_1(s)$ and $L(y_2(t)=Y_2(s)$. Then,

$$A=\begin{pmatrix} s^2+w_0^2+k^2 & -k^2 \\ -k^2 & s^2+w_0^2+k^2 \end{pmatrix}$$

and

$$\det A=|A|=(s^2+w_0^2)(s^2+w_0^2+2k^2)$$

Then, it follows that

$$G(s)=L^{-1}\left(\frac{1}{|A|}\right)=\frac{1}{2k^2}\left(\frac{1}{s^2+w_0^2}-\frac{1}{s^2+w_0^2+2k^2}\right) \qquad (32)$$

The solutions $Y(s)$ is written

$$Y_1(s)=\frac{\begin{bmatrix} y_1(0)-y'_1(0) & -k^2 \\ y_2(0)-y'_2(0) & s^2+w_0^2+k^2 \end{bmatrix}}{\det A}=\frac{\{s^2+w_0^2+k^2\}\{y_1(0)-y'_1(0)\}+k^2\{y_2(0)s-y'_2(0)\}}{\det A}$$

Hence, the required solution is

$$y_1(t)=L^{-1}\left(\frac{1}{|A|}\right)*(As^2+Bs+C)$$

where the constants A, B and C are

$$A=y_1(0)-y'_1(0),\ B=k^2y_2(0),\ C=(w_0^2+k^2)\{y_1(0)-y'_1(0)\}-y'_2(0)k^2$$

Another solution is

$$Y_2(s)=\frac{\begin{bmatrix} s^2+w_0^2+k^2 & y_1(0)s-y'_1(0) \\ -k^2 & y_2(0)s-y'_2(0) \end{bmatrix}}{\det A}$$

$$=-\frac{\{s^2+w_0^2+k^2\}\{y_1(0)-y'_1(0)\}+k^2\{y_2(0)s-y'_2(0)\}}{\det A}=-Y_1(s)$$

2.4 Differential Equations with Variable Coefficients

Consider the solving following differential equation of the second order with polynomial coefficients

$$(at+b)D^2y(t)+(ct+d)Dy(t)+(et+f)y(t)=f(t) \tag{33}$$

where a, ..., f are constants.

By solving differential equations in terms of Laplace transformation, the degree of polynomials of variable coefficients has to be less than (or equal to) the order of derivatives of differential equations because of transformation formula for derivatives.

As in proceding procedure of solving differential equations, the equation transformed may be written

$$\left(-a\frac{d}{ds}+b\right)(s^2Y(s)-sy(0)-y'(0))+\left(-c\frac{d}{ds}+d\right)(sY(s)-y(0))+\left(-e\frac{d}{ds}+f\right)Y(s)=F(s)$$

and is reduced to

$$-(as^2+cs+e)Y'(s)+\{bs^2-(2a-d)s-c+f\}Y(s)$$
$$+\{by(0)s+ay(0)-by'(0)+dy(0)\}=F(s)$$

This is the differential equation of the first order with respect to the function Y (s) , the solution of which may be wuitten

$$Y(s)=e^{-\int^t e^{-\left\{\frac{bs^2-(2a-d)s-c+f}{as^2+cs+e}\right\}ds}}\left\{\int e^{\int^t e^{-\left\{\frac{bs^2-(2a-d)s-c+f}{as^2+cs+e}\right\}ds}}\right.$$
$$\left.\times\left(F(s)-\frac{by(0)s+ay(0)-by'(0)+dy(0)}{as^2+cs+e}\right)ds+c\right\} \tag{34}$$

where assuming that $(as^2+cs+e)\neq 0$. Hence, the solution of the original differential equation is given

$$y(t)=L^{-1}(Y(s))$$

Example 1. Solve the following differential equation with the initial condition, $y(0)=1$.

$$tD^2y(t)+Dy(t)+ty(t)=0$$

Solution.

$$-\frac{d}{ds}(s^2Y(s)-sy(0)-y'(0))+sY(s)-y'(0)-\frac{d}{ds}Y(s)=0,\ -(s^2+1)Y'(s)=sY(s)$$

$$Y(s)=\frac{c}{\sqrt{s^2+1}},\ y(t)=L^{-1}(Y(s))\overset{\text{Example 1.2}}{=}J(t)$$

where $c=1$ by considering the initial condition, $y(0)=1$.

Example 2. Solve

$tD^2y(t)+(1-t)Dy(t)+ny(t)=0$ (n interger)

Solution.

$$-\frac{d}{ds}(s^2Y(s)-sy(0)-y'(0))+\left(1+\frac{d}{ds}\right)(sY(s)-y'(0))+nY(s)=0$$

$$-(s^2-s)Y'(s)+(-s+1+n)Y(s)=0;\ Y(s)=\left(1-\frac{1}{s}\right)^n\cdot\frac{1}{s}\Longleftrightarrow e^t\frac{d^n}{ds^n}(e^{-t}t^n)$$

2.5 Partial Differential Equations

Let x and y or t be variables and let $u(x, t)$ or $u(x, t)$ be an unknown function. The standard form of linear differential equations of the second order with constant coefficients will be taken to be

$$\left(a\frac{\partial^2}{\partial x^2}+2b\frac{\partial^2}{\partial x\partial y}+c\frac{\partial^2}{\partial y^2}+d\frac{\partial}{\partial x}+e\frac{\partial}{\partial y}+f\right)u(x,y)=F(x,y) \tag{35}$$

where the coefficients a, ..., f are constant and $F(x, y)$ represents a non-homogeneous term (an external force). This equation (35) may be classified according to the discrinant $D=b^2-4ac$, which is similar that to the quadratic equation:

(1) When $D<0$, it is called "partial differential equation of elliptic type" (2) When $D>0$, it is called "partial differential equation of hyperbolic type" (3) When $D=0$, it is called "partial differential equation of parabilic type"

Hereafter, we shall treat three typical differential equations in the field of mathematical science nad engineering. That is;

(1) $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)u(x,y)=0 \quad (D<0)$

(2) $\left(\frac{\partial^2}{\partial t^2}-c^2\frac{\partial^2}{\partial x^2}\right)u(x,t)=0 \quad (D>0)$

(3) $\left(\frac{\partial}{\partial t}-c^2\frac{\partial^2}{\partial t^2}\right)u(x,y)=0 \quad (D=0)$

which is called (two-demensional) potential, (one-dimensional) wave and (one-dimensional) heat equations, respectively. We introduce here the notations for differential operators for convinience in writting;

(1) $\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$; (2) $\Box=\frac{\partial^2}{\partial t^2}-c^2\frac{\partial^2}{\partial x^2}$; (3) $\Diamond=\frac{\partial}{\partial t}-c^2\frac{\partial^2}{\partial t^2}$

The notations $\triangle$, $\Box$, and $\Diamond$ for differential operators are called Laplacian, d'Alembertian and Fourierian, respectively.

1) Solve the following one-dimensional heat equation

$$\frac{\partial u(x,y)}{\partial t}-c^2\frac{\partial^2 u(x,y)}{\partial x^2}=0$$

with boundary and initial conditions

$$u(0,t)=u(L,t)=0 \tag{36}$$

and

$$u(x,0)=\sin\frac{\pi}{L}x \tag{37}$$

Physically, the solution $u(x, t)$ represents the heat distribution at any place and time, (x, t). These conditions (36) and (37) mean isothermal conditions at both sides, $x=0$ and $x=L$ and initial distribution of heat at $t=0$, respectively.

The solving procedures are the following:

step 1: Transform the parital differential equation to the ordinary differential equation by Laplace transformation with respect to the variable t.

$$L_t\left(\frac{\partial u(x,y)}{\partial t}-c^2\frac{\partial^2 u(x,t)}{\partial x^2}\right)=0$$

where

$$L_t(u(x,t))=U(x,\eta)=\int_0^\infty e^{-\eta t}u(x,t)\,dt \tag{38}$$

Consequently, the equation reduces to

$$\eta\,U(x,\eta)-u(x,0)-c^2\frac{d^2U(x,\eta)}{dx^2}=0$$

Then, we get

$$\eta\,U(x,\eta)-c^2\frac{d^2U(x,\eta)}{dx^2}=\sin\frac{\pi}{L}x$$

step 2: Find the solution of the ordinary differential equation with respect to the variable x, regarding η as the parameter. As in proceding procedure,

$$L_x\left(\eta\,U(x,\eta)-c^2\frac{d^2U(x,\eta)}{dx^2}=L_x\left(\sin\frac{\pi}{L}x\right)\right.$$

The algebraic equation with respect to the parameter ξ may be written

$$\eta\,U(\xi,\eta)-c^2\{\xi^2U(\xi,\eta)-\xi\,U(\xi,\eta)-U'_x(0,\eta)\}=\frac{\frac{\pi}{L}}{\xi^2+\left(\frac{\pi}{L}\right)^2}$$

The solution of the algebraic equation may be written

$$U(\xi,\eta)=\frac{c^2U'_x(0,\eta)}{\eta-c^2\xi^2}+\frac{1}{\eta-c^2\xi^2}\frac{\frac{\pi}{L}}{\xi^2-\left(\frac{\pi}{L}\right)^2}=\frac{c^2U'_x(0,\eta)}{\eta-c^2\xi^2}+\frac{\pi}{L}\frac{1}{\eta^2+\left(\frac{\pi c}{L}\right)^2}$$

$$\left\{\frac{c^2}{\eta-c^2\xi^2}+\frac{1}{\xi^2-\left(\frac{\pi}{L}\right)^2}\right\}$$

Here the solution of the ordinary differential equation may be written by inverse Laplace transformation

$$L_x^{-1}(U(\xi,\eta))=U(x,\eta)=-U'_x(0,\eta)\frac{\sqrt{\eta}}{c}\sinh\frac{\sqrt{\eta}}{c}+\frac{\pi}{L}\frac{1}{\left(\frac{\pi c}{L}\right)^2+\eta}$$

$$\left\{-\frac{c}{\sqrt{\eta}}\sinh\frac{\sqrt{\eta}}{c}+\frac{L}{\pi}\sin\left(\frac{\pi}{L}x\right)\right\}$$

Here the constant coefficient $U'x(0,\eta)$ may be determined by considering the boundary condition at $x=L$,

$$U'_x(0,\eta)=-\frac{\pi}{L}\frac{1}{\left(\frac{\pi c}{L}\right)^2+\eta}\sin\frac{\pi}{L}x$$

Consequently, the solution of the ordinary differential equation reduces to

$$U(x,\eta)=\frac{1}{\left(\frac{\pi c}{L}\right)^2+\eta}\sin\frac{\pi}{L}x$$

step 3: The desired solution of the partial differential equation may be obtained by inverse Laplace transformation

$$L_t^{-1}(U(x,\eta))=u(x,y)=e^{-\left(\frac{\pi c}{L}\right)^2t}\sin\frac{\pi}{L}x$$

2) Solve the following one-dimensional wave equation.

$$\frac{\partial^2 u(x,t)}{\partial t^2} - c^2 \frac{\partial^2 u(x,t)}{\partial x^2} = 0$$

with boundary and initial conditions

$$u(0,t) = u(L,t) = 0 \tag{39}$$

and

$$u(x,0) = \sin\frac{\pi}{L}x \ \text{and} \ \left.\frac{du(x,t)}{dt}\right|_{t=0} = u'_x(x,0) = 0 \tag{40}$$

Physically, for example, the solution $u(x,t)$ represents the vibration of string fixed at both sides with initial displacement and intial velocity.

step 1 : Transform the partial differential equation to the ordinary differential equation by Laplace transformation with respect to the variable t.

$$L_t\left(\frac{\partial^2 u(x,t)}{\partial t^2} - c^2 \frac{\partial^2 u(x,t)}{\partial x^2}\right) = 0$$

Consequently, the equation reduces to

$$\eta^2 U(x,\eta) - c^2 \frac{d^2U(x,\eta)}{dx^2} = \eta \sin\frac{\pi}{L}x$$

step 2 : Find the solution of the ordinary differential equation with respect to the variable x, regarding η as the parameter. The solution of the algebraic equation may be written

$$U(\xi,\eta) = \frac{1}{\eta^2 - c^2\xi^2}\left\{-c^2U'_x(0,\eta) + \frac{\pi\eta}{L}\frac{1}{\xi^2 + \left(\frac{\pi\eta}{L}\right)^2}\right\}$$

Here the constant coefficient $U'x(0,\eta)$ may be determined by considering the boundary condition at $x = L$,

$$U'_x(0,\eta) = \frac{\pi\eta}{L}\frac{1}{\left(\frac{\pi c}{L}\right)^2 + \eta^2}$$

Consequently, the solution of the ordinary differential equation may be written

$$U(x,\eta) = \eta\frac{1}{\left(\frac{\pi c}{L}\right)^2 + \eta^2}\sin\frac{\pi}{L}x$$

step 3 : The desired solution of the partial differential equation may be obtained by inverse Laplace transformation

$$L_t^{-1}(U(x,\eta)) = u(x,y) = \cos\frac{\pi c}{L}t\sin\frac{\pi}{L}x$$

3) Solve the following two-dimensional potential (Laplace) equation.

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0$$

with boundary and initial conditions

$$u(0,y) = u(L,y) = 0 \tag{41}$$

and

$$u(x,0) = \sin\frac{\pi}{L}x \ \text{and} \ \left.\frac{du(x,y)}{dy}\right|_{y=0} = u'_y(x,0) = 0 \tag{42}$$

step 1 : Transform the partial differential equation to the ordinary differential equation by Laplace transformation with respect to the variable t.

$$L_y\left(\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2}\right) = 0$$

Consequently, the equation reduces to

$$\eta^2 U(x, \eta) + \frac{d^2U(x, \eta)}{dx^2} = \eta \sin\frac{\pi}{L}x$$

step 2 : Find the solution of the ordinary differential equation with respect to the variable x, regarding η as the parameter. The image solution of the algebraic equation may be written

$$U(\xi, \eta) = \frac{1}{\eta^2 + \xi^2}\left\{ - U'x(0, \eta) + \frac{\pi}{L}\frac{\eta}{\xi^2 + \left(\frac{\pi}{L}\right)^2}\right\}$$

Here the constant coefficient $U'x(0, \eta)$ may be determined by considering the boundary condition at $x = L$,

$$U'_x(0, \eta) = \frac{\pi\eta}{L}\frac{1}{-\left(\frac{\pi}{L}\right)^2 + \eta^2}$$

Consequently, the solution of the ordinary differential equation may be written

$$U(x, \eta) = \eta\frac{1}{-\left(\frac{\pi}{L}\right)^2 + \eta^2}\sin\frac{\pi}{L}x$$

step 3 : The desired solution of the partial differential equation may be obtained by inverse Laplace transformation

$$Ly^{-1}(U(x, y)) = u(x, y) = \cosh\frac{\pi}{L}y\sin\frac{\pi}{L}x$$

2.6 Integral Equations

Return to the definition of Laplace transformation

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

As for this equation, given $F(s)$ and determining an unknown function $f(t)$, the equation is called integral equation, that is to say, an integral equation is one which involves an unknown function under the symbol of integration. The process of determining the unknown function is called solving the integral equation. A solution of the above integral equation is given formally by

$$f(t) = \frac{1}{2\pi i}\lim_{p\to\infty}\int_{a-ip}^{a+ip} e^{st}F(s)\, ds \quad (a > 0)$$

This is the inversion formula of Laplace transformation or called Bromwich's integral formula.

Forms of integral equations treated here are the followings :

$$\int_0^t k(t-x)y(x)\,\mathrm{dx} = f(t) \tag{43}$$

and

$$y(t) - \int_0^t k(t-x)f(x)\,\mathrm{dx} = f(t) \tag{44}$$

where the notation $k(t-x)$ is called a kernel of integral equation, simply, an integral kernel, and $y(t)$ and $f(t)$ are unknown and given functions, respectively. These equations are integral equations of convolutional types or are called integral equations of Volterra's types of the first and the second kinds respectively. The solvings of two integral equations (43) and (44) will be considered together.

Solving procedure, as in solving differential equations, can be illustrated below.

step 1. Render both members on the equations Laplace transforms

$$L\left(\int_0^t k(t-x)y(x)\,dx\right) = L(f(t)) \,\Big\|\, L\left(y(t) - \int_0^t k(t-x)f(x)\,dx\right) = L(f(t))$$

By convolutional formula, it follows that

$$K(s)Y(s) = F(s) \;||\; \{1 - K(s)\}Y(s) = \mathrm{F}(s)$$

step 2. The solutions of algerbaic equations are

$$Y(s) = \frac{F(s)}{K(s)} \,\Big\|\, Y(s) = \frac{F(s)}{1 - K(s)}$$

step 3. Then, the solutions of integral equations may be obtained by inverse Laplace transfomation

$$y(t)=L^{-1}\left(\frac{F(s)}{K(s)}\right)=f(t)*L^{-1}\left(\frac{1}{K(s)}\right)$$

$$y(t)=L^{-1}\left(\frac{F(s)}{1-K(s)}\right)=f(t)*L^{-1}\left(\frac{1}{1-K(s)}\right)$$

Example 1. Solve the following integral equations.

(1) $\int_0^t \frac{y(s)}{\sqrt{t-x}}dx = t$ (2) $y(t)-\int_0^t (t-x)\,y(x)\,dx = t$

Solution.

(1)

$$\sqrt{\frac{\pi}{s}}Y(s)=\frac{1}{s^2}$$

$$Y(s)=\frac{1}{\sqrt{\pi}s^{\frac{3}{2}}}$$

$$y(t)=L^{-1}(Y(s))=\frac{2}{\sqrt{\pi}}t^{\frac{1}{2}}$$

(2)

$$Y(s)-\frac{1}{s^2}Y(s)=\frac{1}{s^2}$$

$$Y(s)=\frac{1}{s^2-1}$$

$$y(t)=L^{-1}(Y(s))=\sinh t$$

Example 2. Solve the following differential-integral equations under the initial condition, $y(0)=0$.

(1) $\int_0^t \frac{y'(x)}{\sqrt{t-x}}dx = t$ (2) $y(t)-\int_0^t (t-x)\,y'(x)\,dx = t$

Solution.

(1)

$$\sqrt{\frac{\pi}{s}}(sY(s)-y(0))=\frac{1}{s^2}$$

$$Y(s)=\frac{4}{\sqrt{\pi}s^{\frac{5}{2}}}$$

$$y(t)=L^{-1}(Y(s))=\frac{4}{3\sqrt{\pi}}t^{\frac{3}{2}}$$

(2)

$$Y(s)-\frac{1}{s^2}(sY(s)-y(0))=\frac{1}{s^2}$$

$$Y(s)=\frac{1}{s(s-1)}$$

$$y(t)=L^{-1}(Y(s))=1*e^{-1}=e^{-t}-1$$

Exercise. Find the solutions for the following integral and differential-integral equations.

(1) $\int_0^t \frac{y(x)}{\sqrt{t-x}}dx = t^n$ (2) $\int_0^t \cos(t-x)\,y(x)\,dx = f(t)$

(3) $\int_0^t \frac{y'(x)}{\sqrt{t-x}}dx = t^n$ (4) $\int_0^t \cos(t-x)\,y'(x)\,dx = f(t)$

Solution.

(1) $y(t)=\frac{1}{\sqrt{\pi}}\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)}t^{n+\frac{1}{2}}$ (2) $y(t)-f'(t)+\int_0^t f(x)\,dx$

(3) $y(t)=\frac{1}{\sqrt{\pi}}\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)}t^{n-\frac{3}{2}}$ (4) $y(t)=f''(t)+f(t)$