L^p-convergence of an extended stochastic integral

by
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Abstract Let $1 < p < \infty$. Let $f = \{ f(t), 0 \leq t \leq 1 \}$ be an L^p-integrable martingale and $v = \{ v(t), 0 \leq t \leq 1 \}$ a family of random variables with a continuous parameter t. Suppose $|v| \leq 1$ in absolute value and that $v(t)$ is continuous. Put

$$\theta_m = \sum_{K=0}^{S_m-1} v(\xi_{m,k}) [f(t_{m,k+1}) - f(t_{m,k})].$$

Here, $\forall \xi_{m,k} \in [t_{m,k}, t_{m,k+1}]$, $k \geq 0$, and $\operatorname{Max}_k (t_{m,k+1} - t_{m,k}) \to 0$ as $m \to \infty$.

Then θ_m converges in L^p and θ_∞ defines a new stochastic integral $\int_0^1 v(t)\, df(t)$.

Let (Ω, a, P) be a probability space and $\{a_t\}_{t \geq 0}$ a nondecreasing family of sub-σ-fields of a. Let $f = \{ f(t), 0 \leq t \leq 1 \}$ be an L^p-integrable martingale where $1 < p < \infty$ on a probability space $(\Omega, a, \{a_t\}, P)$ and $v = \{ v(t), 0 \leq t \leq 1 \}$ a family of random variables with a continuous parameter t. Suppose that $|v| \leq 1$ in absolute value, $v(t)$ is continuous and $v(t)$ is a_t-adapted.

Let $\Delta = \{\Delta_m\}$, where $\Delta_m = \{ t_{m,k} : 0 = t_{m,0} < t_{m,1} < \cdots\cdots < t_{m,s_m} = 1 \}$, be a sequence of subdivisions of $[0, 1]$ with $|\Delta_m| = \operatorname{Max}_k (t_{m,k+1} - t_{m,k}) \to 0$ as $m \to \infty$. Here notice that if $m \uparrow \infty$ then $s_m \uparrow \infty$.

Put $\theta_m = \sum_{K=0}^{S_m-1} v(\xi_{m,k}) [f(t_{m,k+1}) - f(t_{m,k})]$ $(\forall \xi_{m,k} \in [t_{m,k}, t_{m,k+1}], k \geq 0)$

and $\widetilde{\theta}_m = \sum_{K=0}^{S_m-1} v(t_{m,k}) [f(t_{m,k+1}) - f(t_{m,k})]$.

By the results of R. C. James [4] and G. Pisier [8],

Theorem. (G. Pisier [8, Theorem 1. 3, (iv)])

Let X be a Banach space and $f = (f_n)_{n \geq 0}$ an arbitrary X-valued martingale.

Then

(*) X is super-reflexive (= super-Radon-Nikodým)

$\Longleftrightarrow$

(**) $\sum_{n \geq 0} \|f_{n+1} - f_n\|_p \leq C \cdot \sup_n \|f_n\|_p$ $(1 < p < \infty)$.

(Here, C is a constant which does not depend on f.)

Since $X = R$ is super-reflexive, (**) holds.

(**) will be called by the name of Pisier's inequality.

In this paper, it is proved that the following theorem holds :

Theorem. θ_m converges in L^p and $\theta_\infty = \widetilde{\theta}_\infty = \int_0^1 v(t)\, df(t)$.
θ_∞ defines a new stochastic integral.

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Proof. Let $1 < p < \infty$.

$\|\theta_m\|_p = E^{1/p}[(|\sum_{K\geqq 0} v(\xi_{m,k})[f(t_{m,k+1}) - f(t_{m,k})]|)^p]$

$\leqq E^{1/p}[(\sum_{K\geqq 0}|v(\xi_{m,k})|\cdot|f(t_{m,k+1}) - f(t_{m,k})|)^p]$ (Here, $|v| \leqq 1$)

$\leqq E^{1/p}[(\sum_{K\geqq 0}|f(t_{m,k+1}) - f(t_{m,k})|)^p]$

(Since L^p is a Banach lattice. See [9].)

$\leqq \sum_{K\geqq 0}\|f(t_{m,k+1}) - f(t_{m,k})\|_p$

$\leqq C\cdot\sup_k \|f(t_{m,k})\|_p$ $(m = 0, 1, 2, \cdots\cdots)$ (By Pisier's inequality)

(Since C does not depend on f, C does not depend on m.)

$\leqq C\cdot\sup_{t\in[0,1]}\|f(t)\|_p$ (Since $t_{m,k}\in[0, 1]$.)

$\leqq C\cdot\|f(1)\|_p$

(Since $|f(t)|^p$ is a submartingale, $E|f(t)|^p \leqq E|f(1)|^p$ so $E^{1/p}[|f(t)|^p] \leqq E^{1/p}[|f(1)|^p] < \infty$).

Thus, $E[|\theta_m|^p] \leqq C^p\cdot\|f(1)\|_p^p$.

If θ_∞ exists then

$\|\theta_\infty\|_p = \|s\text{-}\lim_{m\to\infty}\theta_m\|_p$

$= \lim_{m\to\infty}\|\theta_m\|_p$ (See [14].)

$\leqq C\cdot\|f(1)\|_p < \infty$.

Therefore $\theta_\infty \in L^p$.

Next, it is proved that the existence of θ_∞.

By a result of P. W. Millar [6], $\tilde{\theta}_m$ converges in L^p,

that is, $\lim_{m,n\to\infty}\|\tilde{\theta}_m - \tilde{\theta}_n\|_p = 0$.

Take arbitrary $\varepsilon > 0$ and fix this.

Since $v(t)$ is uniformly continuous on $[0,1]$, for sufficiently large

$m_0 = m_0(\varepsilon, \omega) = m_0(\omega) \geqq m$

$|v(\xi_{m',k})(\omega) - v(t_{m',k})(\omega)| \leqq \varepsilon$ $(\forall k \geqq 0, \ \forall m' \geqq m_0)$

(Because, since $v(t)$ is uniformly continuous, for $\varepsilon > 0$ there is a $\delta > 0$.

Since $|\Delta_m| \to 0$ as $m\to\infty$, for sufficiently large $m_0(\omega)$,

$\delta > |\Delta_{m'}| = \operatorname{Max}_k (t_{m',k+1} - t_{m',k})$ $(\forall m' \geqq m_0(\omega))$

$\geqq \xi_{m',k} - t_{m',k} \geqq 0$ $(\forall\ k \geqq 0)$

So $\varepsilon \geqq |v(\xi_{m',k})(\omega) - v(t_{m',k})(\omega)|$ $(\forall k \geqq 0)$.)

Therefore,

$\|\theta_{m'} - \tilde{\theta}_{m'}\|_p \leqq \sum_{K\geqq 0}\|[v(\xi_{m',k}) - v(t_{m',k})][f(t_{m',k+1}) - f(t_{m',k})]\|_p$

$\leqq \sum_{K\geqq 0} E^{1/p}[(|v(\xi_{m',k}) - v(t_{m',k})|\cdot|f(t_{m',k+1}) - f(t_{m',k})|)^p]$

$\leqq \sum_{K\geqq 0} E^{1/p}[(\varepsilon\cdot|f(t_{m',k+1}) - f(t_{m',k})|)^p]$

$\leqq \varepsilon\cdot\sup_{m',\,\omega\in\Omega}\sum_{K\geqq 0}\|f(t_{m',k+1}) - f(t_{m',k})\|_p$

$\leqq \varepsilon\cdot\sup_{m',\,\omega\in\Omega}\{C\cdot\sup_n\|f(t_{m',n})\|_p\}$

$\leqq \varepsilon\cdot\sup_{m',\,\omega\in\Omega}\{C\cdot\sup_{t\in[0,1]}\|f(t)\|_p\}$

$\leqq \varepsilon\cdot C\cdot\|f(1)\|_p$.

(Here, $\{f(t_{m',k})\}_{k\geqq 0}$ is a martingale.

In fact, take an any $\omega \in \Omega$ and fix an $m'(\omega)$.

In general, since $f=\{f(t)\}$ is a martingale, for any $m \geqq 0$

$\{f(t_{m,k})\}_{k\geqq 0}$ is a martingale. So for almost all $\omega' \in \Omega$

$$E(f(t_{m'(\omega),k(\omega)+1})/\mathfrak{a}_{t_{m'(\omega),k(\omega)}})(\omega')$$
$$= f(t_{m'(\omega),k(\omega)})(\omega'),\ k = k(\omega).$$

Here, let $\omega' = \omega$ then $\{f(t_{m'(\omega),k(\omega)})(\omega)\}_{k\geqq 0}$, that is, $\{f(t_{m',k})\}_{k\geqq 0}$ is a martingale.)

Thus,

$$\lim_{m\to\infty}\|\theta_m - \widetilde{\theta}_m\|_p = \lim_{m'\to\infty}\|\theta_{m'} - \widetilde{\theta}_{m'}\|_p \qquad (m \geqq \operatorname*{Max}_{\omega\in\Omega} m'\ (\geqq m'))$$

$$\leqq \varepsilon \cdot C \cdot \|f(1)\|_p \qquad \text{for all } \varepsilon > 0.$$

Therefore, from $\|\theta_m - \theta_n\|_p \leqq \|\theta_m - \widetilde{\theta}_m\|_p + \|\widetilde{\theta}_n - \theta_n\|_p + \|\widetilde{\theta}_m - \widetilde{\theta}_n\|_p$

it follows that

$$\lim_{m,n\to\infty}\|\theta_m - \theta_n\|_p$$

$$\leqq 2\cdot\lim_{m\to\infty}\|\theta_m - \widetilde{\theta}_m\|_p + \lim_{m,n\to\infty}\|\widetilde{\theta}_m - \widetilde{\theta}_n\|_p \qquad (m, n \geqq \operatorname*{Max}_{\omega\in\Omega} m'(\omega))$$

$$< 2\varepsilon \cdot C \cdot \|f(1)\|_p + 0 \qquad \text{for all } \varepsilon > 0.$$

So, by the completeness of L^p, θ_m converges in L^p so that θ_∞ exists.

From this proof $\theta_\infty = \widetilde{\theta}_\infty = \int_0^1 v(t)\, df(t)$ follows.

Remark. When $p > 1$ the Pisier's inequality implies the Burkholder's L^p-inequality in [1] so that the Millar's results [6] hold without that $v(t)$ is $\mathfrak{a}_t$-adapted. Therefore, it may be that v is any uniformly bounded and continuous random variable.

Corollary. $\int_0^1 v(t)\, dB(t)$ converges in L^2.

(The convergence of this integral cannot be proved by the method of R. L. Stratonovich. See [2] and [13].)

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