# MARTINGALE TRANSFORMS IN A BANACH SPACE

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**ABSTRACT**. If $f=(f_1,f_2,\cdots)$ is a real $L^1$-bounded martingale then $\sum_{n=1}^{\infty} | f_{n+1}-f_n | <\infty$ a.e. The same result holds for X-valued martingales, where X is a Banach space, provided X has the Radon-Nikodým property. Using this the martingale transform g of f by v converges almost everywhere without assuming that v is predictable.

**1. Notations**. Let $(\Omega,\alpha,P)$ be a probabity apace and $\alpha_1, \alpha_2, \cdots$ a nondecreasing sequence of sub-$\sigma$-fields of $\alpha$. Let X be a Banach space with norm $|\cdot|$ and the Radon-Nikodým property. Let $f=(f_1,f_2,\cdots)$ be an X-valued martingale with norm $\| f \|_1=\sup_n E|f_n|<\infty$.
Let $v=(v_1,v_2,\cdots)$ be a real-valued predictable sequence, that is, $v_k : \Omega\to R$ is $\alpha_k$-measurable, $k\geqslant 1$. Then $g=(g_1,g_2,\cdots)$, defined by $g_n=\sum_{k=1}^{n} v_k(f_{k+1}-f_k)$ with $|v|\leqslant 1$ in absolute value, is the transform of the martingle f by v. Write $\| f \|_p=\sup_n \| f_n \|_p$ and define the maximal function $g^*$ of g by $g^*(\omega)=\sup_n |g_n(\omega)|$.

**2. Real-valued case**. Let $\beta$ be a sub-$\sigma$-field of $\alpha$. If Z is a random variable with finite mean, by the Radon-Nikodým theorem, for Z there is a $\beta$-measurable function $\varphi$ which is satisfying

$$\int_A Z(\omega)dP=\int_A \varphi(\omega)dP \quad \text{for every } A\in\beta$$

and which decides the correspondence $Z\to\varphi$ (i.e., $Z(\omega)\mapsto\varphi(\omega)$).
This function $\varphi$ is unique up to a set of P-measure zero, and any such function, denoted by $E(Z/\beta)$, is called the conditional expectation of Z relative to $\beta$. Therefore, the above correspondence is written by

$$E(Z/\beta)(\omega)=E(Z(\omega)/\beta)=\varphi(\omega) \quad \text{for almost all } \omega\in\Omega.$$

If $f=(f_1,f_2,\cdots)$ is a martingale then, for almost all $\omega$,

$$E(f_{n+1}(\omega)/\alpha_n)=f_n(\omega) \quad (n=1,2,\cdots).$$

Let $X=R$, that is, let $f=(f_1,f_2,\cdots)$ be an $L^1$-bounded and real-valued martingale.
Then $|\cdot|$ denotes the absolute value.

**Theorem 1**. If $\| f \|_1<\infty$ then $\sum_{n=1}^{\infty} | f_{n+1}-f_n | <\infty$ a.e., that is, f is of bounded variation.

**Proof**. Suppose that there exists a subset M of $\Omega$ such that $P(M)\neq 0$ and
$\sum_{n=1}^{\infty} | f_{n+1}(\omega)-f_n(\omega) | =\infty$ for all $\omega\in M$.
Then, for any $G=G(\omega)>0$ there is a number $N=N(G,\omega)>0$ such that
$\sum_{k=1}^{n} | f_{k+1}(\omega)-f_k(\omega) | >G$ on M $(\forall n\geqslant N)$.
So there are a number $k=k(\omega)\leqslant n$ and a positive real number $G'=G'(\omega)$ such that $| f_{k+1}(\omega)-f_k(\omega) | =G'>0$ for each $\omega\in M$.
Here, set
$G'=G'(\omega')= | f_{k(\omega)+1}(\omega')-f_{k(\omega)}(\omega') |$ for each $\omega\in M$ $(\omega'\in\Omega,\ M\subset\Omega)$.
G' is well-defined on $\Omega$ and $G'>0$ when $\omega'=\omega$, i.e., $G'>0$ on M.
Now, when $\omega'=\omega$ $| f_{k+1}(\omega)-f_k(\omega) |$ is defined on M.

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By the definition of the absolute value

$$|f_{k+1}(\omega)-f_k(\omega)| = \begin{cases} f_{k+1}(\omega)-f_k(\omega) & \text{on } A \overset{\text{def.}}{=\!=} \{\omega ; f_{k+1}(\omega) \geqq f_k(\omega)\} \ (\subset M) \\ -(f_{k+1}(\omega)-f_k(\omega)) & \text{on } M \setminus A. \end{cases}$$

Since $k(\omega)=k<\infty$, $\{k(\omega) ; \omega \in M\} \subset \{1,2,\cdots,n,\cdots\}$.

Thus,

$$E|f_{k(\omega)}(\omega')| \leqslant \sup_{\lambda \in \{k(\omega) ; \omega \in M\}} E|f_\lambda| \leqslant \sup_{\lambda \in \{1,2,\cdots,n,\cdots\}} E|f_\lambda| = \sup_n E|f_n| < \infty.$$

So $|f_{k+1}-f_k| \in L^1$.

For almost all $\omega \in A$

$$\begin{aligned} E(|f_{k+1}-f_k|/\alpha_k)(\omega) &= E(|f_{k+1}-f_k|(\omega)/\alpha_k) \\ &= E(((f_{k+1}-f_k)^+ + (f_{k+1}-f_k)^-)(\omega)/\alpha_k) \\ &= E(\{(f_{k+1}-f_k)^+(\omega) + (f_{k+1}-f_k)^-(\omega)\}/\alpha_k) \\ &= E(|(f_{k+1}-f_k)(\omega)|/\alpha_k) \\ &= E(|(f_{k+1}(\omega)-f_k(\omega)|/\alpha_k) \\ &= E(f_{k+1}(\omega)-f_k(\omega)/\alpha_k) \\ &= E((f_{k+1}-f_k)(\omega)/\alpha_k) \\ &= E(f_{k+1}-f_k/\alpha_k)(\omega). \end{aligned}$$

In general, since f is a martingale $E(f_{k+1}/\alpha_k)=f_k$ a.e. for any k. Take any $\omega \in \Omega$ and fix this. Let $k=k(\omega)$.

Then $E(f_{k(\omega)+1}/\alpha_{k(\omega)})(\omega')=f_{k(\omega)}(\omega')$ for almost all $\omega' \in \Omega$.

Here take $\omega'=\omega$ then $E(f_{k(\omega)+1}/\alpha_{k(\omega)})(\omega)=f_{k(\omega)}(\omega)$

for almost all $\omega$. Thus, $E(f_{k+1}/\alpha_k)=f_k$ a.e..

So for almost all $\omega \in A$,

$$E(f_{k+1}-f_k/\alpha_k)(\omega)=(f_k-f_k)(\omega)=f_k(\omega)-f_k(\omega)=0.$$

That is, $E(|f_{k+1}-f_k|/\alpha_k)(\omega)=0$ for almost all $\omega \in A$.

For almost all $\omega \in M \setminus A$

$$\begin{aligned} E(|f_{k+1}-f_k|/\alpha_k)(\omega) &= E(|f_{k+1}(\omega)-f_k(\omega)|/\alpha_k) \\ &= E(f_k(\omega)-f_{k+1}(\omega)/\alpha_k) \\ &= E(f_k-f_{k+1}/\alpha_k)(\omega) \\ &= (f_k-f_k)(\omega) \\ &= f_k(\omega)-f_k(\omega) \\ &= 0. \end{aligned}$$

Therefore $E(|f_{k+1}-f_k|/\alpha_k)(\omega)=0$ for almost all $\omega \in M$.

On the other hand, for almost all $\omega' \in \Omega$

$$\begin{aligned} E(G'(\omega')/\{\phi,\Omega\}) &= E(G'/\{\phi,\Omega\})(\omega') \\ &= E(E(G'/\alpha_{k(\omega)})/\{\phi,\Omega\})(\omega') \\ &= E(E(G'/\alpha_{k(\omega)})(\omega')/\{\phi,\Omega\})) \\ &= E(E(G'(\omega')/\alpha_{k(\omega)})/\{\phi,\Omega\}). \end{aligned}$$

If $E(G'(\omega')/\alpha_{k(\omega)})=0$ $(k=k(\omega))$ for almost all $\omega' \in \Omega$

then

$$\begin{aligned} E(G'(\omega')) &= E(G'(\omega')/\{\phi,\Omega\}) \\ &= E(E(G'(\omega')/\alpha_{k(\omega)})/\{\phi,\Omega\}) \\ &= E(0/\{\phi,\Omega\}) \\ &= E(0) \\ &= 0. \end{aligned}$$

Thus, $G'=0$ a.e. This contradicts to $G'>0$ on M.

So $E(G'(\omega')/\alpha_k) \neq 0$ when $\omega' = \omega$ on M.

Then $0 = E(\,|f_{k+1}(\omega) - f_k(\omega)|\,/\alpha_k)$
$= E(G'(\omega)/\alpha_k)$
$\neq 0$ for some $\omega \in M$.

This is a contradiction on M. Thus there is not such M.

Therefore $\sum_{n=1}^{\infty} |f_{n+1}(\omega) - f_n(\omega)| < \infty$ for almost all $\omega \in \Omega$.

**Corollary 1.** If $f = (f_n)_{n \geq 1}$ is an $L^1$-bounded martingale then $E(\,|f_{n+1} - f_n|\,/\alpha_n) = 0$ a.e. and $\| f_{n+1} - f_n \|_1 = E\,|f_{n+1} - f_n| = 0$ for $n < \infty$.

In fact, let $M = \Omega$ in above proof.

**Corollary 2.** Under the above condition $\sum_{n=1}^{\infty} \| f_{n+1} - f_n \|_1 = \infty$.

In fact, $\lim_{n \to \infty} \| f_{n+1} - f_n \|_1 \neq 0$.

**3. Vector-valued case.** Let $Z(\omega)$ be a Bochner-integrable function on a probability space $(\Omega, \alpha, P)$ taking values in X.

Let $\beta$ be a sub-$\sigma$-field contained in $\alpha$. Then the conditional expectation $E(Z/\beta)$ of $z(\omega)$ relative to $\beta$ is defined as a Bochner-integrable function on $(\Omega, \alpha, P)$ such that $E(Z/\beta)$ is $\beta$-measurable and that

$$\int_A Z(\omega)\,dP = \int_A E(Z/\beta)(\omega)\,dP, \quad \forall A \in \beta,$$ where the integrals are Bochner-integrals.

Therefore, by above correspondence $Z(\omega) \mapsto E(Z/\beta)(\omega)$, similarly in the real-valued case $E(Z/\beta)(\omega)$ is written by $E(Z(\omega)/\beta)$ for almost all $\omega \in \Omega$.

(See [4], p.395 and p.396, Theorem 1. And also see [5], p.22.)

Let f be an X-valued and $L^1_X$-bounded martingale.

Then $E(f_{n+1}(\omega)/\alpha_n) = f_n(\omega)$ $(n = 1, 2, \cdots)$.

**Theorem 2.** If $\| f \|_1 < \infty$ then $\sum_{n=1}^{\infty} |f_{n+1} - f_n| < \infty$ a. e..

**Proof.** Suppose that there exists a subset M of $\Omega$ such that $P(M) \neq 0$ and $\sum_{n=1}^{\infty} |f_{n+1}(\omega) - f_n(\omega)| = \infty$ for all $\omega \in M$.

Then, for any $G = G(\omega) > 0$ there is a number $N = N(G, \omega) > 0$ such that $\sum_{k=1}^{n} |f_{k+1}(\omega) - f_k(\omega)| > G$ on M $(\forall n \geqslant N)$.

So there are a number $k = k(\omega) \leqslant n$ and a positive real number $G' = G'(\omega)$ such that $|f_{k+1}(\omega) - f_k(\omega)| = G' > 0$ for each $\omega \in M$.

Then, $\vec{g}(\omega') \stackrel{\text{def.}}{=\!=} f_{k(\omega)+1}(\omega') - f_{k(\omega)}(\omega')$ for each $\omega \in M$ $(\omega' \in \Omega,\ M \subset \Omega)$ such that $|\vec{g}(\omega)| = G'(\omega) > 0$ when $\omega' = \omega$, i. e., $\vec{g} = \vec{g}(\omega) \neq \vec{0}$ on M.

Since f is a martingale, for almost all $\omega' \in \Omega$

$E(f_{k(\omega)+1}(\omega') - f_{k(\omega)}(\omega')/\alpha_{k(\omega)}) = E(f_{k(\omega)+1} - f_{k(\omega)}/\alpha_{k(\omega)})(\omega') = \vec{0}$.

So $\int_M E(\vec{g}(\omega')/\alpha_{k(\omega)})\,dP(\omega') = \vec{0}$ and $\int_{\Omega \setminus M} E(\vec{g}(\omega')/\alpha_{k(\omega)})\,dP(\omega') = \vec{0}$.

Thus, $E(\vec{g}) = \int_\Omega E(\vec{g}(\omega')/\alpha_{k(\omega)})\,dP(\omega')$
$= \int_M E(\vec{g}(\omega')/\alpha_{k(\omega)})\,dP(\omega') + \int_{\Omega \setminus M} E(\vec{g}(\omega')/\alpha_{k(\omega)})\,dP(\omega')$
$= \vec{0}$ (Here E denotes the Bochner integral. See [5].)
$\stackrel{\text{def.}}{\Longleftrightarrow} E\,|\vec{g}| = 0$ (E is the Lebesgue integral)
$\Longleftrightarrow |\vec{g}| = 0$ a. e.
$\Longleftrightarrow \vec{g}(\omega') = \vec{0}$ for almost all $\omega' \in \Omega$ and for each $\omega \in M$.

So $\vec{g}(\omega) = \vec{0}$ on M $(\subset \Omega)$.

This is a contradiction on M. Thus, there is not such M.
Therefore $\sum_{n=1}^{\infty} |f_{n+1}(\omega) - f_n(\omega)| < \infty$ for almost all $\omega \triangle \Omega$.

## 4. Martingale transforms.

**Theorem 3.** If $\|f\|_1 < \infty$ then the martingale transform g converges a. e. in X without the assumption that v is predictable.

In fact,

$$|g_\infty(\omega)| \leqslant \sum_{n=1}^{\infty} |v_n(\omega)| \cdot |f_{n+1}(\omega) - f_n(\omega)| \leqslant \sum_{n=1}^{\infty} |f_{n+1}(\omega) - f_n(\omega)| < \infty$$

for almost all $\omega$.

**Theorem 4.** Let $1 < p < \infty$ and $\|f\|_1 < \infty$. For a Banach space X with the Radon-Nikodým property, $\lambda \cdot P(g^* > \lambda) \leqslant c \cdot \|f\|_1$, $\lambda > 0$, and $\|g\|_p \leqslant c_p \cdot \|f\|_p$ hold under the assumption that v is predictable.

**Proof.** For any Banach space X, by a result of Burkholder (Theorem 1.1 of [2]), the following statements, each to hold for all such f and g are equivalent :

$\|f\|_1 < \infty \Rightarrow$ g converges a. e.,

$\lambda \cdot P(g^* > \lambda) \leqslant c \cdot \|f\|_1$, $\lambda > 0$,

$\|g\|_p \leqslant c_p \cdot \|f\|_p$.

Combine this result with Theorem 3.

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