A Microscopic Study of Thermal Resistance between ^{3}He and a Solid

Hatsuyoshi KATO*

Tomakomai National College of Technology, Nishikioka, Tomakomai, 059-12 JAPAN

(Received November 22, 1995)

Abstract

We study thermal resistance between the liquid ^{3}He and solids from the point of view with a microscopic theory. We derive Hamiltonian that expresses interactions for quasiparticles of the liquid ^{3}He and a solid. The thermal resistance is calculated with this Hamiltonian from the first principles theoretically. It contains both terms of individual excitations and zero sound excitations.

1 Introduction

In research of material properties at low temperatures, thermal exchange between different materials becomes an important problem. It is well known that there is a sharp leap of temperature ΔT at a boundary of two different materials. Thermal resistance R_k at the boundary is roughly proportional to the minus third power of temperature, T^{-3}. The thermal resistance becomes rapidly higher when the temperature decreases. The definition of the thermal resistance R_k is

$$R_k = \Delta T / \dot{Q}(T),$$

where ΔT is the leap of the temperature and $\dot{Q}(T)$ is a net heat flux at the boundary. This is discussed in section 1.1.

There are many works on researches of the thermal conductance between liquid ^{4}He and solids. They are important for a consideration of the system for absorbed Helium system. ^{3}He, a Fermi particle, is widely used as a medium for the thermal exchange at mK temperatures. Liquid ^{3}He is interesting theoretically because it has strong interactions.

There is a theory explaining thermal resistance semi-phenomenologically by Toobs et al. for the heat transfer between ^{3}He and solid by Fermi liquid theory. However, there are no theories from the point of view with microscopic interactions in the system. In this paper the thermal resistance between ^{3}He and solids is studied in Hamiltonian formalism from the point of view of a microscopic theory.

In chapter 1, properties of liquid Helium are summarized. In chapter 2, Fermi liquid theory and conservation laws are discussed. In chapter 3, the thermal resistance between liquid ^{3}He and solids is derived.

1.1 General Features of Liquid Helium

^{3}He exists in the air by 1.3×10^{-4}%, but the main production method is the β decay of tritium (^{3}H) that is able to be got from a nuclear reactor.[1)]

Range of temperatures for which the Fermi liquid theory of Landau can be applied is from 2.7 mK (the critical temperature of superfluidity) to 100mK. In these temperatures, ^{3}He can be treated as a quasi-particle. Some

---

* 助教授　一般教科

Table 1 Parameters of Helium

		^{4}He	^{3}He	unit
m	mass	6.65	5.01	$\times 10^{-24}$g
n	number density	2.18	1.64×10^{22}	atm/cm^3
ρ	mass density	0.145	0.0815	g/cm^3
c_1	sound velocity (1K)	238	182	m/sec
σ		2.556	2.556	Å
ε		10.22	10.22	K
μ	chemical potential	-7.20	-2.52	K/atm

parameters for ^{3}He and ^{4}He are listed in Table 1. In this table σ and ε are parameters for the Lennard-Jones potential as in the following formula:

$$V(r)=4\varepsilon\left[\left(\frac{\sigma}{r}\right)^{12}-\left(\frac{\sigma}{r}\right)^{6}\right],$$

where r is a distance between two particles.

Effective mass m* of ^{3}He in the normal liquid is three times of a mass of one particle. ^{4}He has the phase transition of superfluidity at a temperature 4.17K. At lower temperatures, there are two elementary excitations: phonon and roton. Their dispersion relations are as follows:

phonon: $\omega_q=c_1 q$;

roton: $\varepsilon_p=\Delta+\frac{(p-p_0)^2}{2m_r}$.

At a temperature 1.1K, $\Delta/k_B=8.67$K, $p_0/(h/2\pi)=1.94$ Å^{-1}, $m_4/m_r=0.15$, where m_4 is the mass of the particle ^{4}He, q is a wave number of phonon and p is a momentum of the roton.

^{3}He can melt into the superfluid of ^{4}He by 6.4% till the absolute zero. Energy of a quasi-particle of ^{3}He is given as follows:

$$\varepsilon_p=\varepsilon_0+\frac{p^2}{2m^*},$$

where $\varepsilon_0=-2.785$K and $m^*=1.17\times 10^{-23}$g. The quasi-particle of ^{3}He in the superfluid ^{4}He has an important feature. That makes a dilute Fermi liquid and is used for a dilution refrigerator. Excitations in this dilute solution are phonons of ^{4}He and individual excitations of ^{3}He. Specific heats of these excitations are expressed as C_4 and C_3, respectively, and they are expressed as follows:

$$C_4=\frac{2\pi^2 k_B^4}{15(h/2\pi)^3 c_1^3}T^3$$
$$=3.058\times 10^3 T^3 \text{ J/deg m}^3,$$

$$C_3=\frac{m\, k_F\, k_B^2}{3(h/2\pi)^2}T$$
$$=5.78\times 10^5 x^{1/3} T \text{ J/deg m}^3,$$

where x is a molar fraction of ^{3}He. Fermi wave number (k_F), Fermi temperature (T_F), and Fermi velocity (v_F) have the following values:

$k_F=(3\pi^2 n_4)^{1/3}x^{1/3}=0.858x^{1/3}$ Å^{-1}

$T_F=2.53x^{2/3}$ K

$v_F=77x^{1/3}$ m/sec.

1.2 Kapitza Thermal Resistance (Theory of Acoustic Mismatch)

In this section the theory of acoustic mismatch by Khalatnikov is briefly explained.[2] Kapitza resistance R_k is defined as follows:

$R_k=\Delta T/\dot{Q}$,

where ΔT is a defference of temperature between a solid and the superfluid helium, $\dot{Q}$ is a heat flux.

Qualitatively, it is explained easily why the thermal conduction between a solid and He II is so small. It stems from a large difference between sound velocities in the solid and He II. The velocity in He II is 238 m/sec, and it is about 1/10 of the velocity in the solid. From this fact, it is seldom that conservation laws of momentum and energy are simultaneously satisfied for phonons getting into the solid from the liquid He with the same temperature. The conservation laws strictly limit phonons within certain incident angles. (From the samereason, thermal conduction is not easy to occur by the rotons in the liquid that get into the solid as phonons.)

Quantitatively, we explain the thermal resistance in the rest of this section and get results (Eqs.(12) to (14)) which are useful for discussions in later sections. Velocity field v in the liquid and displacement u in the solid can be expressed as follows:

$$\mathbf{v}=\nabla\varphi_{in}+\nabla\varphi_{ref},$$
$$\mathbf{u}=\nabla\Phi+\nabla\times\boldsymbol{\Psi},$$

where φ is a potential of the velocity, φ_{in} and φ_{ref} are velocity potentials of incident wave and reflected wave, respectively. Φ is a scalar potential of logitudinal wave and Ψ is a vector potential of transverse wave. We assume that **k**, **k**′ $\mathbf{k}_t$ and $\mathbf{k}_l$ are wave vectors on the same plane as in Fig.1, and call it a z-x plane. Hence, the wave vectors can be expressed as follows:

$$\mathbf{k}=(k\sin\theta,\ 0,\ -k\cos\theta),$$
$$\mathbf{k}'=(k\sin\theta,\ 0,\ k\cos\theta),$$
$$\mathbf{k}_t=(k_t\sin\theta_t,\ 0,\ -k_t\cos\theta_t),$$
$$\mathbf{k}_l=(k_l\sin\theta_l,\ 0,\ -k_l\cos\theta_l),$$

$\mathbf{k}_t$ is on the z-x plane, so that the $\mathbf{u}_t=\nabla\times\Psi=i\mathbf{k}_t\times\Psi$ is on z-x plane. The problem we are now discussing is 2-dimensional on the z-x plane and Ψ can be expressed only with nonzero y-component as $\Psi=(0,\ \psi,\ 0)$. The potentials are expressed as follows:

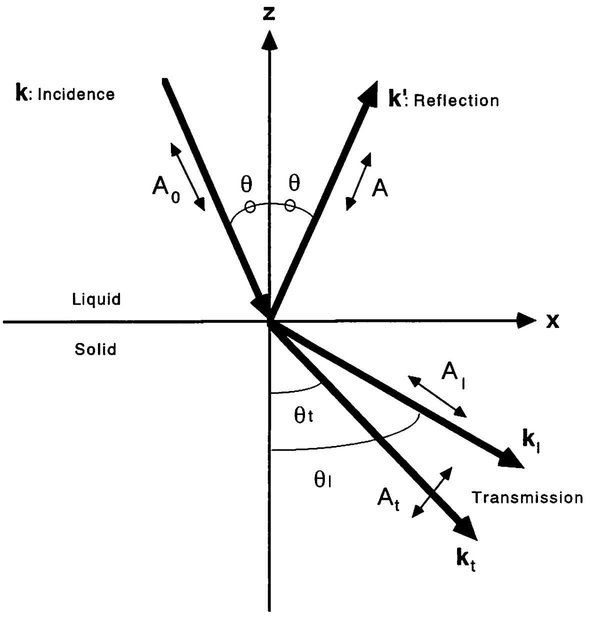

Fig.1

$$\varphi_{in}=A_0\exp(ikx\sin\theta-ikz\cos\theta-i\omega t),$$
$$\varphi_{ref}=A\exp(ikx\sin\theta+ikz\cos\theta-i\omega t),$$
$$\Phi=iA_l\exp(ik_lx\sin\theta_l-ik_lz\cos\theta_l-i\omega t),$$
$$\psi=iA_t\exp(ik_tx\sin\theta_t-ik_tz\cos\theta_t-i\omega t),$$

where

$$k=\frac{\omega}{c},\ k_l=\frac{\omega}{c_l},\ k_t=\frac{\omega}{c_t},$$

and c is the sound velocity in the liquid, c_t and c_l are sound velocities in the solid of transverse and logitudinal waves respectively. The velocities satisfy the following inequality:

$$c \ll c_t < c_l .$$

The system we are considering now has translational symmetry, i.e. ,

$$k\sin\theta=k_t\sin\theta_t=k_l\sin\theta_l,$$

or

$$\frac{\sin\theta}{c}=\frac{\sin\theta_t}{c_t}=\frac{\sin\theta_l}{c_l}. \tag{1}$$

The incident wave $\mathbf{v}_{in}$ and reflected wave $\mathbf{v}_{ref}$ in the liquid and transmitted wave $\mathbf{u}_l$ (longitudinal) and $\mathbf{u}_t$ (transverse) can be written as follows:

$$\mathbf{v}_{in}\equiv\nabla\varphi_{in}=i\mathbf{k}\varphi_{in}=i\mathbf{k}A_0\exp(i\mathbf{k}\cdot\mathbf{r}-i\omega t)$$
$$\mathbf{v}_{ref}\equiv\nabla\varphi_{ref}=i\mathbf{k}_{ref}\varphi_{ref}=i\mathbf{k}A\exp(i\mathbf{k}'\cdot\mathbf{r}-i\omega t),$$
$$\mathbf{u}_l\ \equiv\nabla\Phi=i\mathbf{k}_l\Phi=i\mathbf{k}_l iA\exp(i\mathbf{k}_l\mathbf{r}-i\omega t),$$
$$\mathbf{u}_t\ \equiv\nabla\times\boldsymbol{\Psi}=i\mathbf{k}_t\times\boldsymbol{\Psi}$$
$$=i(k_t\cos\theta_t,\ 0,\ k_t\sin\theta_t)iA_t\exp(i\mathbf{k}_t\cdot\mathbf{r}-i\omega t).$$

1.2.1 Boundary Conditions

(a) z-components of velocities in the liquid and the solid at boundary z=0 are the same:

$$\hat{z}\cdot\hat{v}=\hat{z}\cdot\frac{\partial u}{\partial t},$$

or

$$\hat{z}\cdot(\mathbf{v}_{in}+\mathbf{v}_{ref})=-i\omega\hat{z}\cdot(\mathbf{u}_t+\mathbf{u}_l).$$

From this condition we get the following equation:

$$k\cos\theta\ (A-A_0)$$
$$=-k_l\cos\theta_l\ \omega\ A_l+k_t\cos\theta_t\ \omega\ A_t. \tag{2}$$

(b) Continuity of stress normal to the interface in the liquid and the solid are expressed as follows:

$$\sigma^{(l)}_{ij}=p\delta_{ij}\ ;\ p=-\rho\frac{\partial\varphi}{\partial t},$$

$$\sigma_{ij}^{(s)} = 2Dc_t^2u_{ij} + D(c_l^2 - 2c_t^2)\nabla\cdot\mathbf{u}\delta_{ij},$$

where

$$u_{ij} = \frac{1}{2}\left(\partial_i u_j + \partial_j u_i\right) .$$

This is a component of the strain tensor. ρ is density of the liquid, D is density of the solid. Making use of these expressions, the boundary condition can be expressed as follows:

$$p\delta_{13} + 2Dc_t^2u_{13} + D(c_l^2 - 2c_t^2)\nabla\cdot\mathbf{u}\delta_{13} = 0. \tag{3}$$

Some terms of Eq.(3) can be expressed by potentials as

$$\nabla\cdot\mathbf{u} = -k_l^2\Phi,$$

$$u_{13} = \frac{1}{2}\,(k_l^2 \sin 2\theta_l\ \Phi + k_t^2 \cos 2\theta_t\ \varphi)\ ,$$

$$u_{23} = 0,$$

$$u_{33} = -k_l^2 \cos^2\theta_l\ \Phi + k_t^2 \sin\theta_t \cos\theta_t\ \psi\ ,$$

$$p = i\omega\rho\varphi\ .$$

The components of Eq.(3) for $i=1$ and 3 are expressed explicitly as follows.

$$i=1:\ k_l^2 \sin 2\theta_l\ A_l + k_t^2 \cos 2\theta_t\ A_t = 0, \tag{4}$$

$$i=3:\ \frac{\rho}{D}\,(A_0 + A) = \cos 2\theta_t\ \omega A_l - \sin 2\theta_t\ \omega A_t . \tag{5}$$

Making use of these equations, the boundary conditions can be gotten. We list the results again:

$$k\cos\theta\ (A - A_0) = -k_l \cos\theta_l\ \omega A_l + k_t \sin\theta_t\ \omega A_t, \tag{2'}$$

$$0 = k_l^2 \sin 2\theta_l\ A_l + k_t^2 \cos 2\theta_t\ A_t\ , \tag{4'}$$

$$\frac{\rho}{D}\,(A_0 + A) = \cos 2\theta_t\ \omega A_l - \sin 2\theta_t\ \omega A_t . \tag{5'}$$

1.2.2 Transmission Rates and Reflection Rate

Acoustic impedance can be defined as follows:

$$Z = \frac{\rho c}{\cos\theta}\ ,\ Z_t = \frac{D c_t}{\cos\theta_t},\ Z_l = \frac{D c_l}{\cos\theta_l}. \tag{6}$$

Making use of Eq.(1), the boundary conditions Eqs.(2'),(4') and (5') become

$$\cos\theta\ (A - A_0)$$
$$= -\sin\theta\,\frac{\cos\theta_l}{\sin\theta_l}\ \omega A_l + \sin\theta\ \omega A_t, \tag{7}$$

$$0 = \frac{c_t \cos\theta_l}{c_l \cos\theta_t}\ \sin 2\theta_t\ \omega A_l - \cos 2\theta_t\ \omega A_t\ , \tag{8}$$

$$\frac{\rho}{D}\,(A + A_0) = \cos 2\theta_t\ \omega A_l - \sin 2\theta_t\ \omega A_t . \tag{9}$$

The definition of Eq.(6) makes Eq.(8) in another way as follows:

$$Z_t \sin 2\theta_t\ \omega A_l + Z_l \cos 2\theta_t\ \omega A_t = 0. \tag{8'}$$

From Eqs. (9) and (8'), we get

$$\frac{\rho}{D}\,Z_l \cos 2\theta_t\ (A + A_0)$$
$$= (Z_l \cos^2 2\theta_t + Z_t \sin^2 2\theta_t)\ \omega A_t\ . \tag{10}$$

Also, Eq.(7) similarly becomes

$$\frac{\rho Z_l}{DZ}\,(A - A_0) = \frac{-1}{\cos 2\theta_t}\ \omega A_l\ . \tag{11}$$

After laborious calculations using the above equations, we get the following equations for ratios of the potential amplitudes.

$$\frac{A}{A_0} = \frac{Z_l \cos^2 2\theta_t + Z_t \sin^2 2\theta_t - Z}{Z_l \cos^2 2\theta_t + Z_t \sin^2 2\theta_t + Z}\ , \tag{12}$$

$$\frac{\omega A_l}{A_0} = \frac{\rho}{D}\,\frac{2Z_l \cos 2\theta_t}{Z_l \cos^2 2\theta_t + Z_t \sin^2 2\theta_t + Z}\ , \tag{13}$$

$$\frac{\omega A_t}{A_0} = -\frac{\rho}{D}\,\frac{2Z_t \sin 2\theta_t}{Z_l \cos^2 2\theta_t + Z_t \sin^2 2\theta_t + Z}\ . \tag{14}$$

1.2.3 Expression for the thermal resistance (Kapitza resistance)

Incident energy per unit time flowing normally to the boundary is defined as follows:

$$J_{inc} = \frac{1}{2}\,\rho k^2\,|A_0|^2\,c\cos\theta. \tag{15}$$

Work of the liquid to the boundary makes a flux which flows into the solid. Force exerted on the boundary can be written $f_z \equiv -p = i\omega\rho\varphi$, and the velocity of the motion of boundary is $v_z = \partial\varphi/\partial z$. Hence, the flux is expressed as follows:

$$J_{trans} = \overline{\mathrm{Re}[f_z]\ \mathrm{Re}[v_z]}$$
$$= -\frac{\omega\rho}{2}\,\mathrm{Re}\left[i\varphi\frac{\partial\varphi^*}{\partial z}\right]$$
$$= \frac{\omega\rho}{2}\,k\cos\theta\ \mathrm{Re}\left[(A_0 + A)(A_0^* - A^*)\right] \tag{16}$$

Making use of Eqs. (15) and (16), the transmission rate of energy w is expressed as follows:

$$w = J_{trans}/J_{inc}$$
$$= \mathrm{Re}\ [(1 + A/A_0)\ (1 - A^*/A_0^*)]. \tag{17}$$

Energy of the incident phonon is $h\omega/2\pi$ and the incident angle is θ. Hence, $J_{inc} = h\omega\ c \cos\theta / 2\pi$. Energy of phonons transmitted into the solid is expressed by making use of $w = w(\theta)$ in Eq. (17) as follows:

$$J_{trans} = w(\theta)\ J_{inc}$$
$$= (h\omega/2\pi)\ c\ w(\theta) \cos\theta\ . \qquad (18)$$

We assume that the phonons are in equilibrium expressed by Bose-Einstein distribution $n(h\omega/2\pi k_B T)$. Then the heat flux W from the liquid to the solid becomes

$$W = \int n(h\omega/2\pi k_B T)\, J_{trans} \frac{d^3k}{(2\pi)^3}$$
$$= \frac{\rho}{D} c \frac{4\pi^5}{15} \frac{(k_B T)^4}{(h c_t)^3} F\ . \qquad (19)$$

In derivation of Eq. (19), angular integration of wave vector k is considered only for the incident wave. F is an integral defined as follows:

$$F = \frac{1}{2} \frac{D}{\rho} \left(\frac{c_t}{c}\right)^3 \int_0^1 w(\theta) \cos\theta\, d(\cos\theta)\ . \qquad (20)$$

F depends on the velocities c, c_t, and c_l, but its value is about unity.

The above calculations are for a system in thermal equilibrium if there are no differences of temperature between the liquid and the solid. Therefore, there exists a heat flow from the solid to the liquid. The total heat flows must vanish. If there exists a difference of temperature δT and it is not large, the heat flux $\dot{Q}$ is derived as follows:

$$\dot{Q} = \frac{\partial W}{\partial T} \delta T = \frac{\rho}{D} c \frac{16\pi^5}{15} \frac{k_B{}^4\ T^3}{(h\ c_t)^3} F \delta T\ . \qquad (21)$$

Therefore, the reciprocal of Kapitza resistance R_k is given by

$$R_k{}^{-1} = \frac{2\pi^2}{15} \frac{\rho c}{D\ c_t{}^3} k_B \left(\frac{k_B T}{h/2\pi}\right)^3 F\ . \qquad (22)$$

The above expression is valid for the thermal resistance between ^{4}He and a solid. It is an important feature that $R_k\ T^3$ depends only on mass densities of liquid ^{4}He and the solid and the ratio of phonon velocities. For Cu and liquid ^{4}He, $R_k\ T^3 \approx 5 \times 10^2$ cm^{2}K^4/W. For ^{3}He and a solid, the same result could be gotten if we would consider the zero-sound instead of the phonon[3], i.e.

$$R_k{}^{-1} = \frac{16 \rho_L\ c_0 \pi^5\ k_B{}^4\ T^3}{15 D (h\ c_t)^3} F\ , \qquad (23)$$

where ρ_L is a mass density of liquid ^{3}He, c_0 is the velocity of of the zero-sound. For Cu and liquid ^{3}He at 0.12 atm, $R_k\ T^3 \approx 11 \times 10^2$ cm^{2}K^4/W.

2 Fermi Liquid Theory and Conservation Laws

2.1 Distribution Function and Fermi Liquid Theory in Nonequilibrium System

In this chapter Fermi liquid theory is explained briefly.[4,5] It becomes to be needed in Chapter 3. To treat Fermi liquid as a nonequilibrium and heterogeneous system, a distribution function $\mathbf{n}_{ps}$ ($\mathbf{rt}$) is needed for quasiparticles and it depends on a position r and the time t. In this treatment a momentum p and the position r are considered simultaneously. So the uncertainty principle must be argueed. The problem is a localization of the quasiparticle in the canonical space (**r p**). Conditions to use the distribution function $\mathbf{n}_{ps}$ (**r t**) must be the following points.

(a) A range of localization for the quasiparticles is as same as a heterogeneous range of the system. This range is expressed as λ . Of course, the heterogeneous range is macroscopic (not to be small as the sizes of molecules or atoms).

(b) The distribution function seems to be changing in a range $\Delta p = k_B T\ /\ v_f$ at a momentum space. From this reason, the quasiparticle cannot be localized in the momentum space less than Δp.

Heisenberg uncertainty relation for the above (a) and (b) is expressed as follows:

$$\lambda \gg (h/2\pi)\ v_f / k_B T\ . \qquad (1)$$

If this relation is satisfied , there are no problems to use the distribution function. For example, the normal liquid of ^{3}He has a value

$$(h/2\pi)\ v_f / k_B T = 1.25 \times 10^{-15}\ [\text{ÅK}]/T[\text{K}].$$

If a temperature is 1 mK, the value is sufficiently small macroscopically.

To use Wigner distribution function N (r_1 σ_1, $r_2\sigma_2$;t) is the most general method for the distribution of quasiparticles. It means probability amplitude for a system to return to the initial state when a quasipaticle is removed from a position r_1 with a spin σ_1 and another quasiparticle is put in a position r_2 with a spin σ_2 simultaneously at a time t. Creation and annihilation operators are expressed with $a^{+}_{p\sigma}$ and $a_{p\sigma}$. The Wigner distribution function is defined as follows:

$$N(\mathbf{r}_1\sigma_1, \mathbf{r}_2\sigma_2 ; t) = \int \frac{d^3p_1}{h^3} e^{i2\pi p_1 \cdot r_1/h} \int \frac{d^3p_2}{h^3} e^{-i2\pi p_2 \cdot r_2/h} \langle a^{+}_{p_2\sigma_2} a_{p_1\sigma_1} \rangle , \tag{2}$$

where < > means an average at a time t. The Fourier transform of Eq.(2) is an extension of distribution function $(n_p)_{\sigma\sigma'}$ to a heterogeneous system. It is a 2 x 2 matrix in the spin space. Hence, the definition of the distribution function becomes

$$[n_p(\mathbf{r}, t)]_{\sigma\sigma'} = \int d^3r' \, e^{-i2\pi p \cdot r/h}$$

$$N(\mathbf{r}+\frac{\mathbf{r}'}{2}, \sigma ; \mathbf{r}-\frac{\mathbf{r}'}{2}, \sigma' ; t) \tag{3}$$

Baym and Pethick considered the Fermi liquid with this distribution function quantum-mechanically as a nonequilibrium and heterogeneous system.[4] Classical mechanic distribution function was considered by Chapman-Enskog[6] in nonequilibrium system. Number density n (r,t) at a location r can be expressed as follows:

$$n(\mathbf{r}, t) = \sum_{\sigma} N(\mathbf{r}\sigma, \mathbf{r}\sigma ; t) = \sum_{\sigma} \int \frac{d^3p}{h^3} [n_p(\mathbf{r},t)]_{\sigma\sigma} \tag{4}$$

Number density of quasiparticles with momentum is defined as follows:

$$n_p(\mathbf{r}, t) = \sum_{\sigma} \int d^3r [n_p(\mathbf{r}, t)]_{\sigma\sigma}$$
$$= \sum_{\sigma} \int d^3r_1 \int d^3r_2 e^{-i2\pi p \cdot (r_1 - r_2)/h} N(\mathbf{r}_1\sigma, \mathbf{r}_2\sigma ; t) \tag{5}$$

We are considering a heterogeneous system. External forces may be exerted, and the system must not be in an equilibrium state. Energy $\mathbf{E}$(t) of the whole system contains interactions with external fields. Energy of a quasiparticle at location r should be expressed by $\varepsilon_{p\sigma}$ (rt) . It is defined as a variation of the distribution function $n_{p\sigma}$(rt) :[7]

$$\delta\mathbf{E}(t) \equiv \int d^3r E(\mathbf{r}, t)$$
$$= \sum_{\sigma} \int \frac{d^3p}{h^3} \int d^3r \, \varepsilon_{p\sigma}(\mathbf{r}t) \, \delta n_{p\sigma}(\mathbf{r}t) \tag{6}$$

where $\delta n_{p\sigma}$(rt) is a variation of the distribution function. And it could be arbitrary before the variation, i.e. , $\delta n_{p\sigma}$(rt) could be a variation from interactions of the particles or an external perturbation. An effective interaction of the quasiparticles is defined with a variation of $\varepsilon_{p\sigma}$(rt) :

$$\delta\varepsilon_{p\sigma}(\mathbf{r}t) = \sum_{\sigma} \int \frac{d^3p'}{h^3} \int d^3r' f_{p\sigma, p'\sigma'}(\mathbf{r}, \mathbf{r}', t) \, \delta n_{p'\sigma'}(\mathbf{r}'t), \tag{7}$$

where $\delta f_{p\sigma,p'\sigma'}$ (r, r′, t) is an Landau parameter in a heterogeneous system. A range of interactions between quasiparticles in a neutral Fermi liquid is appropriately $h/2\pi p_f$. If a change of distribution function $n_{p'\sigma'}$ (r′, t) is small in this range, $\delta n_{p'\sigma'}$ (r′ , t) could be moved out of the integral with respect to d^3r' in Eq.(7). In this case,

$$f_{p\sigma, p'\sigma'}(\mathbf{r}, t) \equiv \int d^3r' f_{p\sigma, p'\sigma'}(\mathbf{r}, \mathbf{r}', t) \tag{8}$$

is a local Landau parameter. Normal liquid ^{3}He has a short range interaction. Hence, the Landau parameter is defined usually as in Eq.(8).[8]

2.2 Parameters and Hartree-Fock Approximation in Fermi Liquid Theory

Parameters of Fermi liquid theory are given in Hartree-Fock approximation from an interaction Hamiltonian.[10,11] The Hamiltonian can be define as follows:

$$\hat{H} = \sum_{k\sigma} \varepsilon^{0}_{k\sigma} c^{+}_{k\sigma} c_{k\sigma}$$
$$+ \frac{1}{2\Omega} \sum_{\substack{k\sigma, k'\sigma' \\ q}} \nu(\mathbf{q}) \, c^{+}_{k+q\sigma} c^{+}_{k'-q\sigma'} c_{k'\sigma'} c_{k\sigma}, \tag{1}$$

where ν (q) is the Fourier transform of interactions between particles, $\varepsilon^{0}_{k\sigma} = (h/2\pi)^2k^2/2m$ is kinetic energy of a particle. Energy is expressed with Hartree-Fock approximation as follows:

$$E^{\mathrm{H.F.}}=\sum_{k\sigma}\varepsilon^0_{k\sigma}n^0_{k\sigma}+\frac{\nu(0)}{2\Omega}\left(\sum_{k\sigma}n^0_{k\sigma}\right)^2 -\frac{1}{2\Omega}\sum_{k\sigma,k'\sigma'}\nu(\mathbf{k}-\mathbf{k}')\,\delta_{\sigma\sigma'}n^0_{k\sigma}n^0_{k'\sigma'} \qquad (2)$$

where

$$n^0_{k\sigma}=\langle c^+_{k\sigma}\ c_{k\sigma}\rangle_0 \qquad (3)$$

is the Fermi distribution function for the ground state. Making a variation

$$n^0_{k\sigma}\rightarrow n^0_{k\sigma}+\delta n_{k\sigma} \qquad (4)$$

with a notice of the spherical symmetry of the interaction, (i.e. $\nu\ (k-k')=\nu\ (k'-k)$), we get the following Hartree-Fock approximation:

$$\delta E^{\mathrm{H.F.}}=\sum_{k\sigma}\varepsilon^{\mathrm{H.F.}}_{k\sigma}\ \delta n_{k\sigma}\ , \qquad (5)$$

$$\varepsilon^{\mathrm{H.F.}}_{k\sigma}=\varepsilon^0_{k\sigma}+\frac{1}{\Omega}\sum_{k'\sigma'}f^{\mathrm{H.F.}}_{k\sigma,k'\sigma'}n^0_{k'\sigma'}, \qquad (6)$$

$$f^{\mathrm{H.F.}}_{k\sigma,k'\sigma'}=\nu(0)-\nu(\mathbf{k}-\mathbf{k}')\,\delta_{\sigma,\sigma'}. \qquad (7)$$

$f^{\mathrm{H.F.}}_{k\sigma,k'\sigma'}$ is a Landau parameter in Hartree-Fock approximation. From this parameter, symmetric and antisymmetric parameters are defined as follows:

$$f^s_{k,k'}=\frac{1}{2}\left\{f^{\mathrm{H.F.}}_{k\uparrow,k'\uparrow}+f^{\mathrm{H.F.}}_{k\uparrow,k'\downarrow}\right\} =\nu(0)-\frac{1}{2}\ \nu(k-k')\ , \qquad (8)$$

$$f^a_{k,k'}=\frac{1}{2}\left\{f^{\mathrm{H.F.}}_{k\uparrow,k'\uparrow}-f^{\mathrm{H.F.}}_{k\uparrow,k'\downarrow}\right\}=-\frac{1}{2}\ \nu(k-k')\ . \qquad (9)$$

If the interaction can be assumed as δ-function, $\nu\ (q)=\nu=$const., then the parameters become

$$f^{\mathrm{H.F.}}_{k\sigma,k'\sigma'}=\nu\ (1-\delta_{\sigma,\sigma'})\ , \qquad (10)$$

$$f^s_{k,k'}=-f^a_{k,k'}=\frac{\nu}{2}. \qquad (11)$$

These equations show that f-parameters do not depend on wave numbers and $f^{\mathrm{H.F.}}_{k\sigma,k'\sigma'}$ becomes zero for two particles with the same spins. Expansions with spherical functions for $f^s_{k,k'}$ and $f^a_{k,k'}$ on the Fermi surface become

$$f^s_l=-f^a_l=\frac{2l+1}{2}\int_{-1}^{1}\frac{\nu}{2}P_l(\zeta)\,d\zeta =\frac{\nu}{2}\delta_{l,0} \qquad (12)$$

where ζ is a cosine of an angle between wave vectors of the two particles. Because of $f^S_l=0$, a ratio of the effective mass to the particle mass become

$$\frac{m^*}{m}=1+\frac{1}{3}\ F^S_l=1\ . \qquad (13)$$

In this expression, F^S_l is a product of f^S_l and the state density on the Fermi surface. The self energy of the Hartree-Fock approximation becomes

$$\Sigma_{k\sigma}=\frac{1}{\Omega}\sum_{k'\sigma'}f^{\mathrm{H.F.}}_{k\sigma,k'\sigma'}n^0_{k'\sigma'}.$$

Table 2 Fermi Liquid parameters(Experiments)[Ref.4]

³He

pressure	0 antm	27 atm
m^*/m	3.01	5.63
F^s_0	10.07	74.38
F^s_1	6.04	13.90
F^a_0	-0.67	-0.74
F^a_1	-0.67	-0.53
p_f ($\times 10^{-20}$gcm/sec)	8.28	9.235
v_f (cm/sec)	5.48×10^3	3.27×10^3
c_1 (cm/sec)	1.829×10^4	3.893×10^4

³He in ⁴He

	1.3%		5.0%	
	Theor.	Exper.	Theor.	Exper.
$F^s_1/3$	0.0174		0.055	
m^*/m_3	2.38	2.38±0.04	2.47	2.46±0.04
F^a_0	0.09	0.09±0.03	0.04	0.08±0.03
F^s_0	-0.20		-0.42	

This energy also does not depend on wave vectors. Because of these facts, the effective mass m* is equivalent to the particle mass m.

Table 2 shows Landau parameters for the liquid ^{3}He and the dilute solution of ^{3}He in the superfluid ^{4}He.[4]

2.3 Conservation Laws of Field and Stress Tensor

Stress tensor of the liquid ^{3}He can be derived from the theory of field. In this section, we derive it from Hamiltonian of the system assumed as follows:

$$H=\int\left\{\frac{(h/2\pi)^2}{2m}\nabla\psi^+(\mathbf{x})\cdot\nabla\psi(\mathbf{x})+V_0(\mathbf{x})\rho(\mathbf{x})\right\}d^3x$$
$$+\frac{1}{2}\iint\phi(\mathbf{x}-\mathbf{x}')\,\psi^+(\mathbf{x})\rho(\mathbf{x}')\,\psi(\mathbf{x})\,d^3x'd^3x,$$

where V_0 (x) is an external potential, ϕ $(x-x')$ is an interaction potential between particles. $\psi(\mathbf{x})$, $\psi^+(\mathbf{x})$ are the quantities of the field. $\rho(x)$ is number of density of the particles. It is expressed with creation and annihilation operators a_k and a_k^+ as follows:

$$\psi(\mathbf{x})=\frac{1}{\sqrt{\Omega}}\sum_k a_k e^{ik\cdot x},$$
$$\psi^+(\mathbf{x})=\frac{1}{\sqrt{\Omega}}\sum_k a_k^+ e^{ik\cdot x},$$
$$\rho(\mathbf{x})=\psi^+(\mathbf{x})\,\psi(\mathbf{x})=\frac{1}{\Omega}\sum_{kq}a_k^+\,a_{k+q}\,e^{iq\cdot x}. \qquad (1)$$

The conservation for number of particles is given from the equation of motion for $\rho(\mathbf{x})$. The commutation relation $[\rho(\mathbf{x}),\ H]$ gives the following formula :

$$\frac{\partial\rho(\mathbf{x})}{\partial t}+\nabla\cdot\mathbf{j}(\mathbf{x})=0,$$
$$\mathbf{j}(\mathbf{x})=\frac{h/2\pi}{2mi}\{\psi^+(\mathbf{x})\nabla\psi(\mathbf{x})-\nabla\psi^+(\mathbf{x})\,\psi(\mathbf{x})\}, \qquad (2)$$

where $\mathbf{j}(\mathbf{x})$ is flux of particles.

Momentum density can be defined with mass of the particle m ; and $\mathbf{j}(\mathbf{x})$ is defined as $\mathbf{g}(\mathbf{x})=m\mathbf{j}(\mathbf{x})$. By the same method as in deriving Eq.(2), the conservation of momentum becomes

$$\frac{\partial\mathbf{g}(\mathbf{x})}{\partial t}+\nabla\cdot\Pi^{gas}(\mathbf{x})+\mathbf{B}(\mathbf{x})=-\rho(\mathbf{x})\nabla V_0(\mathbf{x}), \qquad (3)$$

$$\Pi^{gas}_{\beta\alpha}(\mathbf{x})=\frac{(h/2\pi)^2}{2m}\left\{\frac{\partial\psi^+(\mathbf{x})}{\partial x_\beta}\frac{\partial\psi(\mathbf{x})}{\partial x_\alpha}+\frac{\partial\psi^+(\mathbf{x})}{\partial x_\alpha}\frac{\partial\psi(\mathbf{x})}{\partial x_\beta}-\frac{1}{2}\frac{\partial^2\rho(\mathbf{x})}{\partial x_\beta\partial x_\alpha}\right\},$$
$$\mathbf{B}(\mathbf{x})=\psi^+(\mathbf{x})\int\nabla\phi(\mathbf{x}-\mathbf{x}')\,\rho(\mathbf{x}')\,d^3x'\ \ \psi(\mathbf{x}). \qquad (3b)$$

The right hand side of Eq.(3) is a term from the external force. Π^{gas} $(\mathbf{x})$ is a stress tensor that is derived from the kinetic energy of the Hamiltonian. $\mathbf{B}(\mathbf{x})$ cannot be expressed as a divergence of a tensor if we only calculate it from the commutation relation $[\mathbf{g}(\mathbf{x}),\ H]$. However, it is possible if a effective range is short for the particle interactions.

In the following discussion, it is not needed to get the stress tensor explicitly, but we need the integral with a product of a function and the tensor. Let any function be expressed by $\mathbf{A}(\mathbf{x})$. The function is assumed not to change rapidly within the range of the effective distance of the particle interactions. An integral of the product $\mathbf{A}(\mathbf{x})$ and $\mathbf{B}(\mathbf{x})$ becomes as follows:

$$\sum_\alpha\int A_\alpha(\mathbf{x})\,B_\alpha(\mathbf{x})\,d^3x$$
$$=\sum_\alpha\int d^3x\int A_\alpha(\mathbf{x})\frac{\partial\phi(\mathbf{x}-\mathbf{x}')}{\partial x_\alpha}F(\mathbf{x},\mathbf{x}')\,d^3x', \qquad (4)$$

where $F(\mathbf{x},\ \mathbf{x}')\equiv\psi^+(\mathbf{x})\,\rho(\mathbf{x}')\,\psi(\mathbf{x})=F(\mathbf{x}',\ \mathbf{x})$. The interaction ϕ $(\mathbf{x}-\mathbf{x}')$ has the spherical symmetry. Hence, the following equation is valid.

$$\frac{\partial\phi(\mathbf{x}\text{-}\mathbf{x}')}{\partial x_\alpha}=-\frac{\partial\phi(\mathbf{x}\text{-}\mathbf{x}')}{\partial x'_\alpha}=-\frac{\partial\phi(\mathbf{x}'\text{-}\mathbf{x})}{\partial x'_\alpha}.$$

With this feature of the interaction, Eq.(4) becomes

$$-\sum_\alpha\int d^3x\int A_\alpha(\mathbf{x}')\frac{\partial\phi(\mathbf{x}-\mathbf{x}')}{\partial x_\alpha}F(\mathbf{x},\mathbf{x}')\,d^3x'. \qquad (5)$$

This is equivalent to Eq.(4). Therefore, a half of an addition of these two expressions is also equivalent to Eqs.(4) and (5). This expression becomes

$$\sum_\alpha\int d^3x\int\frac{1}{2}\{A_\alpha(\mathbf{x})-A_\alpha(\mathbf{x}')\}$$
$$\frac{\partial\phi(\mathbf{x}-\mathbf{x}')}{\partial x_\alpha}F(\mathbf{x},\mathbf{x}')\,d^3x'. \qquad (6)$$

The change of $\mathbf{A}(\mathbf{x})$ is assumed to be slow within the effective range of the interactions. The difference $A_\alpha(\mathbf{x})$ and $A_\alpha(\mathbf{x}')$ in Eq.(6) can be written as

$$A_\alpha(\mathbf{x})-A_\alpha(\mathbf{x}')\approx\sum_\beta(x_\beta-x'_\beta)\frac{\partial A_\alpha(\mathbf{x})}{\partial x_\beta},$$

Making use of the above equation and the partial integral with respect to $\mathbf{x}$, Eq.(6) becomes

$$\sum_\alpha\int d^3x\, A_\alpha(\mathbf{x})\sum_\beta\frac{\partial}{\partial x_\beta}\int\frac{-1}{2}(x_\beta-x'_\beta)\frac{\partial\phi(\mathbf{x}-\mathbf{x}')}{\partial x_\alpha} F(\mathbf{x},\mathbf{x}')d^3x'. \quad (7)$$

With comparison of Eq.(4) and Eq.(7), we get an expression for $B_\alpha(\mathbf{x})$, i.e.

$$B_\alpha(\mathbf{x})\approx\sum_\beta\frac{\partial}{\partial x_\beta}\frac{-1}{2}\int(x_\beta-x'_\beta)\frac{\partial\phi(\mathbf{x}-\mathbf{x}')}{\partial x_\alpha} F(\mathbf{x},\mathbf{x}')d^3x'. \quad (8)$$

Therefore, a stress tensor for the interactions between the particles can be expressed as follows :

$$\Pi^{\text{int}}_{\beta\alpha}(\mathbf{x})=\frac{-1}{2}\int(x_\beta-x'_\beta)(x_\alpha-x'_\alpha)\frac{1}{r}\frac{\partial\phi(\mathbf{r})}{\partial r} F(\mathbf{x},\mathbf{x}')d^3x', \quad (9)$$

where $r=|\mathbf{x}-\mathbf{x}'|$.

Assuming δ-type function for the interactions of the particles, we can derive $\Pi^{\text{int}}(\mathbf{x})$. All elements of the stress tensor are not needed hereafter. Hence, only $\Pi^{\text{int}}_{zz}(\mathbf{x})$ will be considered. Eq.(9) can be expressed as follows:

$$\Pi^{\text{int}}_{zz}(\mathbf{x})=-\frac{1}{2}\psi^+(\mathbf{x})I(\mathbf{x})\psi(\mathbf{x}), \quad (10)$$

$$I(\mathbf{x})\equiv\int(z-z')^2\frac{1}{r}\frac{\partial\phi(r)}{\partial r}\rho(\mathbf{x}')d^3x'. \quad (11)$$

$\rho(\mathbf{x}')$ can be expanded with plane waves, i.e.

$$\rho(\mathbf{x}')=\frac{1}{\Omega}\sum_q\rho_q e^{iq\cdot x'}.$$

Substituting this expression into Eq.(11) we get

$$I(\mathbf{x})=\frac{1}{\Omega}\sum_q e^{iq\cdot x}\rho_q\int\cos^2\theta\, r\frac{\partial\phi(r)}{\partial r}e^{iq\cdot r}d^3r, \quad (12)$$

where θ is an angle between $\mathbf{r}$ and the z-axis. Configuration of $\mathbf{q}$, $\mathbf{r}$, z-axis is shown in Fig.2. We use the Rayleigh formula for a plane wave and the formula of spherical functions; i.e.

$$e^{iq\cdot r}=\sum_{l=0}^{\infty}(2l+1)i^l j_l(qr)P_l(\cos\theta),$$

$$\cos^2\theta=\frac{1}{3}+\frac{2}{3}P_2(\cos\theta),$$

where j_l is the spherical Bessel function and P_l is the Legendre polynomial. Making use of these formulae, Eq.(12) becomes

$$I(\mathbf{x})=\frac{1}{\Omega}\sum_q e^{iq\cdot X}\rho_q\sum_{l=0}^{\infty}(2l+1)i^l J_l(q)K_l(\theta_q), \quad (13)$$

where

$$J_l(q)\equiv\int_0^\infty r^3 j_l(qr)\frac{d\phi(r)}{dr}dr, \quad (14)$$

$$K_l(\theta_q)\equiv\int\left\{\frac{1}{3}+\frac{2}{3}P_2(\cos\theta)\right\}P_l(\cos\theta)d\omega$$
$$=\frac{4\pi}{3}\delta_{l,0}+\frac{8\pi}{15}\delta_{l,2}P_2(\cos\theta_q). \quad (15)$$

Terms of Eq.(13) for $l=0$ and 2 must be considered because of Eq.(15). Therefore,

$$I(\mathbf{x})=\frac{1}{\Omega}\sum_q e^{iq\cdot x}\rho_q\left\{\frac{4\pi}{3}J_0(q)-\frac{8\pi}{3}J_2(q)P_2(\cos\theta_q)\right\}. \quad (16)$$

We assume δ-type function for the interaction of particles, i.e.

$$\phi(r)=\nu\delta(r)=\nu\lim_{s\to\infty}\left(\frac{s}{\pi}\right)^{3/2}e^{-sr^2}.$$

The integrals of Eq.(14) for $l=0$ and 2 become

$$J_0(q)=-\frac{\nu}{4\pi}\lim_{s\to\infty}\left(3-\frac{q^2}{2s}\right)e^{-q^2/4s}=-\frac{3\nu}{4\pi},$$

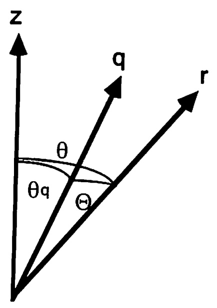

Fig.2

$$J_2(q) = -\frac{3\nu}{4\pi} \lim_{s\to\infty} e^{-q^2/4s} - J_0(q) = 0 \quad (17)$$

Making use of Eq.(16) and Eq.(17); Eq.(10) is reduced to

$$\Pi^{\mathrm{int}}_{zz}(\mathbf{x}) = \frac{\nu}{2}\, \psi^{+}(\mathbf{x})\, \rho(\mathbf{x})\, \psi(\mathbf{x}) \quad (18)$$

The total stress tensor can be got with an addition of Eq.(18) that is interaction part and Eq.(3) that is the kinetic energy part; as follows :

$$\begin{aligned}\Pi_{zz}(\mathbf{x}) &\equiv \Pi^{\mathrm{gas}}_{zz}(\mathbf{x}) + \Pi^{\mathrm{int}}_{zz}(\mathbf{x}) \\ &= \frac{(h/2\pi)^2}{4m}\left\{ 2\frac{\partial\psi^{+}(\mathbf{x})}{\partial z}\frac{\partial\psi(\mathbf{x})}{\partial z} - \frac{\partial^2\psi^{+}(\mathbf{x})}{\partial^2 z}\psi(\mathbf{x}) \right. \\ &\left. - \psi^{+}(\mathbf{x})\frac{\partial^2\psi(\mathbf{x})}{\partial^2 z}\right\} + \frac{\nu}{2}\psi^{+}(\mathbf{x})\,\psi^{+}(\mathbf{x})\psi(\mathbf{x})\psi(\mathbf{x}).\end{aligned} \quad (19)$$

3 Thermal Resistance between ^{3}He and a Solid System in Hamiltonian Formalism

3.1 Hamiltonian of ^{3}He and a Solid System

Heat flux between a solid and the liquid ^{3}He is caused by an interaction at a boundary. From this heat flux we can get Kapitza resistance. In this section the interaction between a solid and ^{3}He, H'_{SL}, is derived from stress tensor of the liquid and displacement of the surface of the solid.[12,13] However, we neglect the magnetic feature of the solid, so that a magnetic interaction between the spins of ^{3}He and the solid is also neglected.

With this assumption, the Hamiltonian of the interaction becomes

$$H'_{SL} = \int \psi^{+}(\mathbf{x}) \sum_n V_A(\mathbf{R}_n + \mathbf{u}_n - \mathbf{x})\, \psi(\mathbf{x})\, d^3x\,, \quad (1)$$

where V_A is the potential energy between the solid atoms and the liquid particles, $\mathbf{R}_n$ is the a lattice point of the solid, $\mathbf{u}_n$ is a displacement at the lattice point. In low temperatures energy of phonons are rather small. Hence, the displacement $\mathbf{u}_n$ is small enough to be considered in the first order of V_A expansion with respect to $\mathbf{u}_n$. We can also assume that V_A is a short range potential. Also in the solid, $\psi(\mathbf{x})$ decreases rapidly. Therefore, the dominant contribution in Eq.(1) is from the surface of the solid. With these assumptions, the summation becomes

$$\begin{aligned}&\sum_n V_A(\mathbf{R}_n + \mathbf{u}_n - \mathbf{x}) \\ &\quad \approx \sum_n V_A(\mathbf{R}_n - \mathbf{x}) - \mathbf{u}_n(\mathbf{x}, \mathbf{y}, 0)\cdot \nabla \sum_n V_A(\mathbf{R}_n - \mathbf{x}).\end{aligned}$$

In this expression, the interface between the solid and the liquid is assumed to be at z=0. Because of the symmetry of the system, the potential energy depends only on z. Hence, we can express

$$\sum_n V_A(\mathbf{R}_n - \mathbf{x}) = V_0(z).$$

From the above discussion, Eq.(1) can be expressed as follows:

$$\begin{aligned}H'_{SL} &= \int \rho(\mathbf{x})\, V_0(z)\, d^3x \\ &\quad - \int u_z(x, y, 0)\, \rho(\mathbf{x}) \frac{dV_0(z)}{dz} d^3x.\end{aligned} \quad (2)$$

The first term is static and expresses a contribution from the solid to the liquid ^{3}He. This can be included in non-perturbed part of the total Hamiltonian. The second term contributes to the heat flow. We express it as H_{SL} in later sections.

The conservation of momentum is got in section 2.3 as

$$\begin{aligned}&\frac{\partial g_z(\mathbf{x})}{\partial t} + \frac{\partial \Pi_{zx}(\mathbf{x})}{\partial x} + \frac{\partial \Pi_{zy}(\mathbf{x})}{\partial y} + \frac{\partial \Pi_{zz}(\mathbf{x})}{\partial z} \\ &= -\rho(x)\frac{\partial V_0(z)}{\partial z}.\end{aligned} \quad (3)$$

$V_0(z)$ can be assumed to have a rapid change in the range $z = +\varepsilon$ to $-\varepsilon$. Assuming that $\rho(\mathbf{x})$, $\Pi(\mathbf{x})$, and $\mathbf{g}(\mathbf{x})$ vanish rapidly in the solid; we can get an integral of Eq.(3) as follows:

$$\begin{aligned}&-\int_{-\varepsilon}^{\varepsilon} \rho(x) \frac{dV_0(z)}{dz} dz \\ &= \int_{-\varepsilon}^{\varepsilon} \left\{ \frac{\partial g_z}{\partial t} + \frac{\partial \Pi_{zx}}{\partial x} + \frac{\partial \Pi_{zy}}{\partial y} + \frac{\partial \Pi_{zz}}{\partial z} \right\} dz \\ &= \left\{ \frac{\partial g_z}{\partial t} + \frac{\partial \Pi_{zx}}{\partial x} + \frac{\partial \Pi_{zy}}{\partial y} \right\}_{z=\varepsilon-0} \int_{\xi}^{\varepsilon} dz \\ &+ \{\Pi_{zz}\}_{z=+\varepsilon} - \{\Pi_{zz}\}_{z=-\varepsilon}.\end{aligned}$$

In the solid $\{\Pi_{zz}\}_{z=-\varepsilon}$ vanishes. Assuming $\partial g_z/\partial t$ and Π have monotone increasing feature, we have made use of the second mean value theorem for the first term of the last line in the above equation, and $-\varepsilon < \xi < \varepsilon$. If ε is small enough, we can neglect this term. The dominant term should be $\{\Pi_{zz}\}_{z=+\varepsilon}$. We then get Hamiltonian for the interactions between the solid and the liquid ^{3}He as follows:

$$H_{SL}=\int u_z(x,y,0)\,\Pi_{zz}(x,y,0)\,dxdy. \qquad (4)$$

The Hamiltonian for the liquid ^{3}He can be expressed with the above discussion as follows:

$$\begin{aligned}H_L=&\int\frac{(h/2\pi)^2}{2m}\nabla\psi^+(\mathbf{x})\cdot\nabla\psi(\mathbf{x})d^3x\\&+\int\rho(\mathbf{x})v_0(\mathbf{x})d^3x\\&+\frac{1}{2}\int d^3x\int\phi(\mathbf{x}-\mathbf{x}')\,\psi^+(\mathbf{x})\,\rho(\mathbf{x}')\,\psi(\mathbf{x})\,d^3x'.\end{aligned} \qquad (5)$$

The first and second terms are non-perturbed Hamiltonian. The third term expresses interactions in ^{3}He particles. We will refer it by H_{int} in later sections.

3.2 Heat Conduction:Fermi Gas

A stress tensor in the liquid ^{3}He without interactions are given from Eq.(3a) in the section 2.3 as follows:

$$\Pi^{gas}(\mathbf{x})=\sum_{k,p}M^0_{k,p}c^+_{k+p}c_k e^{-ip\cdot x}, \qquad (1)$$

$$M^0_{k,p}\equiv\frac{(h/2\pi)^2}{m\Omega}(k_z+\frac{1}{2}p_z)^2. \qquad (2)$$

c^+_{k+p} and c_k are creation and annihilation operators for ^{3}He particles. The displacement of the liquid normalized in the half infinite space is

$$\mathbf{u}(x,y,z)=2\sum_q\left(\frac{h/2\pi}{2\rho_s\Omega_s\omega_q}\right)^{1/2}$$

$$\mathbf{e}_q(a_q+a^+_{-q})e^{iq_{//}\cdot x}\cos q_z z.$$

At the interface of the solid , it becomes

$$\mathbf{u}(x,y,0)=\sum_q\left(\frac{2h/2\pi}{\rho_s\Omega_s\omega_q}\right)^{1/2}\mathbf{e}_q(a_q+a^+_{-q})e^{i(q_x x+q_y y)}; \qquad (3)$$

where a_q and a^+_{-q} are the creation and annihilation operators for phonons in the solid, $\mathbf{e}_q$ is a polarization vector with wave vector $\mathbf{q}$, ω_q is an angular frequency, ρ_s is mass density of the solid and Ω_s is volume of the solid. Substituting these results to Eq.(4) in the section 3.1, we can get a Hamiltonian for the interactions between the solid and liquid:

$$H_{SL}=\sum_{k,p,q}T^q_{k,p}(a_q+a^+_{-q})c^+_{k+p}c_k, \qquad (4)$$

$$T^q_{k,p}\equiv A\delta_{q_{//},p_{//}}e_{qz}\left(\frac{2h/2\pi}{\rho_s\Omega_s\omega_q}\right)^{1/2}M^0_{k,p}=T^{-q}_{k,-p}, \qquad (5)$$

where A is an area of the solid surface, e_{qz} is a z-component of $\mathbf{e}_q$, $\mathbf{q}_{//}$, $\mathbf{p}_{//}$ are wave vectors parallel to the solid surface.

Heat flux $\dot{Q}$ from the solid to the liquid ^{3}He can be expressed with the phonon distribution function. Considering H_{SL}, we get:

$$\begin{aligned}\dot{Q}=&-\sum_q(h/2\pi)\,\omega_q\dot{n}_q\\=&-\sum_q(h/2\pi)\,\omega_q\sum_{k,p}\frac{2\pi}{h/2\pi}|T^q_{k,p}|^2\\&\{f^0_{k+p}(1-f^0_k)(1+n_q)-f^0_k(1-f^0_{k+p})n_q\}\\&\delta(\varepsilon_{k+p}-\varepsilon_k-(h/2\pi)\,\omega_q),\end{aligned}$$

where ε_k is an energy of a ^{3}He particle with the equilibrium distribution f^0_k at a temperature T. Phonons in the solid are assumed to be in the equilibrium distribution n_q at temperature $T+\Delta T$. Assuming $\Delta T/T \ll 1$, we can expand n_q to the first order of ΔT. We express the distribution function of phonons at temperature T as n^0_q. The distribution functions n^0_q and f^0_k at the same temperature have vanishing collision integrals because of the energy conservation law. Hence, the heat flux becomes

$$\begin{aligned}\dot{Q}=&-\sum_{k,p,q}(h/2\pi)\,\omega_q\frac{2\pi}{(h/2\pi)}|T^q_{k,p}|^2\\&\Delta T\frac{\partial n^0_q}{\partial T}\left(f^0_{k+p}-f^0_k\right)\delta(\varepsilon_{k+p}-\varepsilon_k-(h/2\pi)\,\omega_q).\end{aligned}$$

The heat flux per area is expressed as follows:

$$J_{FG} \equiv Q/A = \varDelta T \frac{2(h/2\pi)^2}{\rho_s \Omega_s} \sum_q e_{qz}^2 \frac{\partial n_q^0}{\partial T} R_{FG}(\omega_q), \tag{6}$$

$$R_{FG}(\omega_q) \equiv A \frac{2\pi}{h/2\pi} \sum_{k,p} \delta_{q//,p//} |M_{k,p}^0|^2 \left(f_k^0 - f_{k+p}^0 \right) \delta(\varepsilon_{k+p} - \varepsilon_k - (h/2\pi)\omega_q) . \tag{7}$$

Temperatures we are considering now are in an order of 0.1K. The Fermi temperature becomes $\varepsilon_F/k_B \approx 5K$, then

$$\frac{q}{k_F} \ll \frac{v_F}{2c_S} \ll 1 , \tag{8}$$

where k_F is the Fermi wave number, q is the wave number of phonons , v_F is the Fermi velocity ($\approx$ 100m/s), and c_s is the velocity of phonons in the solid ($\approx$ 3000m/s). Considering the conservation laws of energy and momentum, we can get the following inequality:

$$\frac{1}{2k_F}(2k_z\ p_z + p_z^2) + \frac{k_{//}}{k_F} .\ q_{//} + \frac{q_{//}^2}{2k_F}$$

$$= \frac{c_S}{v_F}\ q \gg q .$$

Terms including $k_{//}$ and / or $q_{//}$ could be neglected because of $k_{//}k_F < 1$ and $q_{//} < q \ll k_F$. Therefore,

$$p_z = -2k_z - \frac{c_S q k_F}{v_F\ k_Z} \approx -2k_z ,$$

and

$$p_z = \frac{c_S q k_F}{v_F\ k_Z} = \frac{m\omega_q}{(h/2\pi)\ k_Z} .$$

Because $M_{k,p}^0 \approx 0$ when $p_z \approx -2k_z$;we should consider only the case of $p_z = m\omega_q/(h/2\pi)\ k_z$. Eq.(7) then becomes

$$R_{FG}\ (\omega_q) = \frac{\omega_q\ k_F^4}{4\pi^2} . \tag{9}$$

Substituting this result into Eq.(6), we get the thermal resistance as follows:

$$R_K^{-1} = J_{FG}/\Delta T = \frac{3\pi^2\ \rho_H\ v_F}{45\rho_S\ c_S^3}\ k_B \left(\frac{k_B\ T}{h/2\pi}\right)^3 . \tag{10}$$

We should notice here that this value and Eq.(23) in chapter 1 from the zero sound have the same orders.

3.3 Heat Conduction:Fermi Liquid

The component of the stress tensor given by Eq.(18) in the section 2.3 can be expressed with spins as follows:

$$\Pi_{zz}^{int}(\mathbf{x}) = \frac{\nu}{2\Omega^2} \sum_{\substack{k_1,k_2 \\ k_3,k_4 \\ \sigma,\sigma'}} c_{k_4\sigma}^{+} c_{k_3\sigma'}^{+} c_{k_2\sigma'} c_{k_1\sigma} e^{i(k_1+k_2-k_3-k_4)\cdot x} . \tag{1}$$

Making use of Hartree-Fock approximation, we can get the following expression instead of Eq.(1).

$$\overline{\Pi}_{zz}^{int}(\mathbf{x}) = \frac{n_3 \nu}{2\Omega} \sum_{k,p,\sigma} c_{k+p,\sigma}^{+} c_{k,\sigma} e^{-ip\cdot x} - \frac{\nu}{4} n_3^2, \tag{2}$$

where n_3 is number density of ^{3}He particles. Adding $\overline{\Pi}_{zz}^{int}$ and Π_{zz}^{gas} makes the component of the total stress tensor as follows:

$$\overline{\Pi}_{zz}(\mathbf{x}) = \sum_{k,p,\sigma} \frac{(h/2\pi)^2}{m\Omega} \left\{ \left(k_z + \frac{1}{2} p_z \right)^2 + \frac{1}{3} k_F^2 F_0 \right\} c_{k+p,\sigma}^{+} c_{k,\sigma} e^{-ip\cdot x} - \frac{1}{3} F_0 n_3 \varepsilon_F , \tag{3}$$

where

$$F_0 = N(\varepsilon_F)\ \frac{\nu}{2},\ \ \varepsilon_F = \frac{m k_F}{(\pi\ h/2\pi)^2}. \tag{4}$$

If we assume δ-function for interactions of particles, the effective mass of ^{3}He is the same as its original mass m. N (ε_F) is the state density on the Fermi surface. The average of Eq.(3) becomes

$$\left\langle \overline{\Pi}_{zz} \right\rangle = \frac{2}{5}\ n_3\ \varepsilon_F\ \left(1 + \frac{5}{6}\ F_0 \right), \tag{5}$$

where $2n_3\ \varepsilon_F/5$ is the pressure of the ideal Fermi gas at temperature T=0.

Comparing Eq.(1) in the section 3.2 and Eq.(3) in this section, we can get the following expression instead of Eq.(2) in the section 3.2:

$$M_k(\mathbf{p}) = \frac{(h/2\pi)^2}{m\Omega} \left\{ \left(k_z + \frac{1}{2} p_z \right)^2 + \frac{1}{3} k_F^2 F_0 \right\}. \tag{6}$$

Toombs et al.[12] got a similar expression from the Fermi liquid theory. Their results can be got if we put p=0 in Eq.(6). They assumed $p_z \ll k_z$. Here we also use this assumption.

Scattering amplitude for renormalized quasiparticles becomes

$$A_{k',k}\,(\mathbf{p},\ \omega)=\frac{F_0}{N(\varepsilon_F)\,[1+F_0\,g(\lambda)]}, \qquad (7)$$

where $\lambda=\omega/v_F\,q$ and

$$g(\lambda)=\frac{1}{N(\varepsilon_F)}\frac{2}{\Omega}\sum_{k}\chi^0{}_k(\mathbf{q},\omega)$$
$$=\frac{1}{N(\varepsilon_F)}\left[1-\frac{\lambda}{2}\log\frac{\lambda+1}{\lambda-1}\right], \qquad (8)$$

$$\chi^0{}_k\,(\mathbf{q},\ \omega)=\frac{f^0_k-f^0_{k+p}}{\varepsilon_{k+p}-\varepsilon_k-(h/2\pi)\,\omega}. \qquad (9)$$

A diagram of dynamical screening for the interactions between the solid and the liquid is shown in Fig.3. The expression for the diagram is

$$\bar{M}_k(\mathbf{p},\omega+i\delta)$$
$$=M_k-\frac{1}{\Omega}\sum_{k'}M_{k'}\chi^0{}_{k'}(\mathbf{p},\omega+i\delta)\,A_{k',k}(\mathbf{p},\omega+i\delta). \qquad (10)$$

We should remind the assumption $|\,p_{/\!/}\,|<<|\,p_z\,|$ and note $A_{k',k}(\mathbf{p},\ \omega+\mathbf{i}\delta)$ does not depend on $\mathbf{k}'$. The result of calculation of Eq.(10) becomes

$$\bar{M}_k(\mathbf{p},\omega)=\frac{(h/2\pi)^2k^0_F}{m\Omega}\ \frac{\xi^2+(\xi^2-\lambda^2)F_0\,g(\lambda)}{1+F_0\,g(\lambda)}, \qquad (11)$$

where $\lambda=\omega/pv_F\approx\omega/p_zv_F$, $\xi=\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}$, and $\hat{\mathbf{p}}$ is almost parallel to the z axis. With this expression, we considered the many-body correlation. Therefore, we can use the same equation for the Fermi gas, i.e., we can use Eq. (7) in section 3.2 with substituting $\bar{M}_k$ for M_k. Making use of the following formula:

$$Im\ \{\chi^0_k(\mathbf{p},\ \omega_q\ +\ i\delta)\}$$
$$=(f^0_k-f^0_{k+p})\,\pi\delta(\varepsilon_{k+p}-\varepsilon_k-(h/2\pi)\,\omega_q), \qquad (12)$$

and reminding the assumptions $\mathbf{q}_{/\!/}\approx0$, $\mathbf{p}_{/\!/}\approx0$; we can get a thermal resistance for the Fermi liquid as follows:

$$R_{FL}(\omega_q)=\frac{2A}{h/2\pi}\sum_{k,p}\delta_{q/\!/,p/\!/}\,|\bar{M}_k(\mathbf{p},\omega_q+i\delta)|^2$$
$$Im\{\chi^0_k(\mathbf{p},\omega_q+i\delta)\}$$
$$=\frac{2A}{h/2\pi}Im\sum_{k,p_z}|\bar{M}_k(\mathbf{p},\omega_q+i\delta)|^2\frac{\delta(\varepsilon_k-\varepsilon_F)\xi}{\xi-(\lambda+i\eta)}, \qquad (13)$$

where $\xi=\hat{k}\cdot\hat{p}$, $\lambda=\omega/p_zv_F$, $\eta=\delta/p_zv_F\rightarrow0$. With substitutions

$$\sum_{k}\delta(\varepsilon_k-\varepsilon_F)\rightarrow\Omega\,N(\varepsilon_F)\frac{1}{2}\int_{-1}^{1}d\xi,$$

$$\sum_{p_z}\rightarrow\frac{\Omega}{A\pi}\int_0^\infty dp_z=\frac{\Omega}{A\pi}\int_0^\infty\frac{\omega_q d\lambda}{v_F\,\lambda^2},$$

we get the thermal resistance for the Fermi liquid as follows:

$$R_{FL}(\omega_q)=\frac{\Omega^2N(\varepsilon_F)}{(h/2\pi)v_F}\omega_q\int_{-1}^{1}d\xi\int_0^\infty\frac{d\lambda}{\lambda^2}\frac{1}{\pi}$$
$$Im\left[|\bar{M}_k(\xi,\lambda+i\eta)|^2\frac{\xi}{\xi-(\lambda+i\eta)}\right]. \qquad (14)$$

In the rest of this section, we show that the R_{FL} consists of terms for the individual excitations and the zero sound excitations.

3.3.1 Thermal Resistance from Individual Excitation

If we put $\lambda=\xi$ in the matrix element of Eq. (14), it represents a term of the individual excitations, i.e.

$$\bar{\Pi}_k(\xi,\ \lambda+i\eta)]_{\lambda=\xi}$$
$$=\frac{(h/2\pi)^2\,k_F{}^2}{m\Omega}\ \frac{\xi^2}{1+F_0\,g(\xi+i\eta)}. \qquad (15)$$

In this expression $p_z=\omega/(h/2\pi)k_F$. This fact corresponds to the change of momentum in the Fermi gas. If we restrict the integral with respect to positive value of ξ, we get the thermal resistance R_{SP} for single particle:

$$R_{SP}(\omega_q)=\frac{\Omega^2N(\varepsilon_F)}{(h/2\pi)v_F}\,\omega_q\int_0^1 d\xi\int_0^\infty$$
$$|\bar{M}_k(\xi,\lambda+i\eta)|^2{}_{\lambda=\xi}\frac{\eta/\pi}{(\xi-\lambda)^2+\eta^2}\,\xi\frac{d\lambda}{\lambda^2}. \qquad (16)$$

Noting $\frac{\eta/\pi}{(\xi-\lambda)^2+\eta^2}\rightarrow\delta(\xi-\lambda)$, we can notice $\bar{M}_k$ is of the Fermi gas. Therefore,

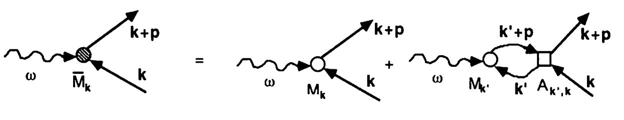

Fig.3

$$R_{SP}(\omega_q)=\frac{\Omega^2 N(\varepsilon_F)}{(h/2\pi)v_F}\omega_q\left(\frac{(h/2\pi)^2 k_F{}^2}{m\Omega}\right)^2$$
$$\int_0^1\frac{d\xi}{\xi}\left|\frac{\xi^2}{1+F_0 g(\xi+i\eta)}\right|^2$$
$$=\frac{k_F^4\,\omega_q}{4\pi^2}\int_0^1\frac{4\xi^3}{|1+F_0 g(\xi+i\eta)|^2}d\xi. \tag{17}$$

Ratio to $R_{FG}(\omega_q)$ becomes

$$R_{SP}(\omega_q)/R_{FG}(\omega_q)$$
$$=\int_0^1\frac{4\xi^3}{|1+F_0 g(\xi+i\eta)|^2}d\xi\equiv\gamma(s), \tag{18}$$

where s is a quantity which decide the dispersion relation of zero sound, and it is got from F_0. The denominator of $\gamma(s)$ can be expressed as follows:

$$|1+F_0\,g(\xi+i\eta)|^2$$
$$=|1+F_0\,g_1(\xi)|^2+|F_0\,g_2(\xi)|^2, \tag{19}$$

where

$$g_1(\xi)=1-\frac{\xi}{2}\log\frac{1+\xi}{1-\xi},$$

$$g_2(\xi)=\frac{\pi}{2}\xi.$$

They are the real and imaginary parts of $g(\xi+i\eta)$, respectively.

3.3.2 Thermal Resistance from Zero Sound Excitation

If we put $\lambda=s$ in the matrix element of Eq. (14), it represents a term of zero sound, i.e.

$$\bar{M}_k(\xi,\ \lambda+i\eta)]_{\lambda\to s}$$
$$\to\frac{(h/2\pi)^2\,k_F{}^2}{m\Omega}\,\frac{-s^2\,g(s)}{1+F_0\,g(\lambda+i\eta)}. \tag{20}$$

Because this denominator vanishes, we expand it around $\lambda=s$. Hence,

$$\bar{M}_k(\xi,\ \lambda+i\eta)$$
$$\to\frac{(h/2\pi)^2\,k_F{}^2}{m\Omega}\,\frac{-s^2\,g(s)}{(\lambda+i\eta-s)\,g'(s)}. \tag{21}$$

where

$$1+F_0\,g(s)\equiv 0\qquad(s>1),$$

$$g'(s)\equiv\frac{dg(s)}{ds}=\frac{s}{s^2-1}-\frac{1}{2}\log\frac{s+1}{s-1}. \tag{22}$$

Making use of these results, we get the thermal resistance $R_{zs}(\omega_q)$ from the zero sound as follows:

$$R_{ZS}(\omega_q)=\frac{\Omega^2 N(\varepsilon_F)}{(h/2\pi)v_F}\omega_q\int_{-1}^1 d\xi$$
$$\int_{(\lambda\approx s)}\frac{d\lambda}{\lambda^2}\left(\frac{(h/2\pi)^2 k_F^2}{m\Omega}\right)^2\left|\frac{s^2 g(\lambda)}{(\lambda-s+i\eta)g'(s)}\right|^2\frac{\eta\xi/\pi}{(\xi-\lambda)^2+\eta^2}$$
$$=\frac{\omega_q k_F^4}{\pi^2}\left(\frac{s\,g(s)}{g'(s)}\right)^2\int_{-1}^1\frac{\xi^2}{(\xi-s)^2}d\xi. \tag{23}$$

Ratio to $R_{FG}(\omega_q)$ becomes

$$R_{ZS}(\omega_q)/R_{FG}(\omega_q)=\frac{4}{3}\,\phi(s), \tag{24}$$

$$\phi(s)\equiv\frac{2s^2\,g^2(s)}{g'(s)}. \tag{25}$$

4 Conclusion

We study the thermal resistance between the solid and the liquid ^{3}He from the point of view with a microscopic theory. For this purpose, we have to get Hamiltonian H_L (Eq. (5) in the section 3.1) which represents interactions between the particles of the liquid ^{3}He and Hamiltonian H_{SL} in the section 3.1 which represents interactions between the solid surface and the liquid ^{3}He. With these Hamiltonians the thermal resistance is derived as in the section 3.3.

To get these results, we should express the stress tensor Π_{zz} in the second quantization. The tensor is derived in Chapter 3. If we assume the short range interaction between the particles, $\nu\delta(\mathbf{x}-\mathbf{x}')$;the tensor coincides with the results which was derived by Toombs et al. 12) using the Fermi liquid theory. Also we can thoroughly make it clear what kind of approximations they have employed. With the diagram method, we can consider the collective excitations and the individual excitations simultaneously in the formula that we use to get the thermal conduction. As is mentioned above, we have shown the thermal conduction between the solid and the liquid ^{3}He microscopically. The method we have discussed is a fundamental theory for explaining a mechanism of the thermal conduction for any Fermi particle system from the first princiles.

Appendix

A1 Derivation of Formulae for Heat Flux

A formula for heat flux in the paper of Legett, Vourio[14] is derived in this section. Two systems 1 and 2 are assumed in equilibrium with $\beta_1=1/k_BT_1$ and $\beta_2=1/k_BT_2$, respectively. Let Hamiltonians of the systems be H_1 and H_2, their energies $E_\mu^{(1)}$, $E_\nu^{(2)}$, i.e.,

$$H_1 \mid \mu \rangle = E_\mu^{(1)} \mid \mu \rangle ,$$
$$H_2 \mid \nu \rangle = E_\nu^{(2)} \mid \nu \rangle . \qquad (1)$$

The subscript μ is used only for the system 1, and ν for the system 2. Probability for the two systems to have a total energy $E_{\mu\nu}=E_\mu^{(1)}+E_\nu^{(2)}$ is

$$P_{\mu\nu}=e^{\beta_1(\Omega_1-E_u^{(1)})}\ e^{\beta_2(\Omega_2-E_\nu^{(2)})} \qquad (2)$$

When an interaction is exerted between these two systems with Hamiltonian

$$H_{int}=g\sum_k A_k B_k=H_{int}^+ , \qquad (3)$$

we can derive a heat flux. In Eq.(3), A_k is an operator for the system 1, and B_k is for the system 2. The transition probability per time for the system 1 and 2 from a state $\mid \mu\nu \rangle$ to $\mid \mu'\nu' \rangle$ is

$$R_{\mu\nu->\mu'\nu'}=\frac{2\pi}{h/2\pi}$$
$$\mid \langle \mu'\nu' \mid H_{int} \mid \mu\nu\rangle \mid^2 \delta(E_{\mu\nu}-E_{\mu'\nu'}) \qquad (4)$$

The heat flux by this transition become

$$\dot{Q}_{1->2}=\sum_{\mu\nu}\sum_{\mu'\nu'} P_{\mu\nu}(E_\mu^{(1)}-E_{\mu'}^{(1)})\, R_{\mu\nu->\mu'\nu'} . \qquad (5)$$

With reminding $R_{\mu'\nu'->\mu\nu}=R_{\mu\nu->\mu'\nu'}$, Eq.(5) can be expressed as follows:

$$\dot{Q}_{1->2}=-\sum_{\mu\nu}\sum_{\mu'\nu'} P_{\mu'\nu'}(E_\mu^{(1)}-E_{\mu'}^{(1)})\, R_{\mu\nu->\mu'\nu'} . \qquad (6)$$

Half of the addition of Eq.(5) and Eq.(6) gives

$$\dot{Q}_{1->2}=\frac{1}{2}\sum_{\mu\nu}\sum_{\mu'\nu'}(P_{\mu\nu}-P_{\mu'\nu'})(E_\mu^{(1)}-E_{\mu'}^{(1)})$$
$$R_{\mu\nu->\mu'\nu'} . \qquad (7)$$

Making use of the energy conservation which is expressed by the δ function in the right hand side of Eq.(4), we can express the subtraction of the probabilities in Eq.(7) as follows:

$$P_{\mu\nu}-P_{\mu'\nu'}=e^{\beta_1(\Omega_1-E_u^{(1)})}\ e^{\beta_2(\Omega_2-E_\nu^{(2)})}$$
$$[1-e^{(\beta_2-\beta_1)(E_u^{(1)}-E_u^{(1)})}]. \qquad (8)$$

Assuming $\beta_1=\beta$, $\beta_2=\beta+\Delta\beta$, and $\Delta\beta/\beta \ll 1$;we can get an expression of Eq.(8) till the first order of $\Delta\beta$ in the expansion as follows:

$$P_{\mu\nu}-P_{\mu'\nu'}=e^{\beta_1(\Omega_1-E_u^{(1)})}\ e^{\beta_2(\Omega_2-E_\nu^{(2)})}$$
$$\Delta\beta(E_\mu^{(1)}-E_\mu^{(1)}. \qquad (9)$$

Substituting this expression into Eq.(7), we can get the heat flux as follows:

$$\dot{Q}_{1->2}=\frac{\Delta\beta}{2}\sum_{\mu\nu}\sum_{\mu'\nu'} e^{\beta(\Omega_1-E_\mu^{(1)})}e^{\beta(\Omega_2-E_\nu^{(2)})}$$
$$(E_\mu^{(1)}-E_{\mu'}^{(1)})^2 R_{\mu\nu->\mu'\nu'} . \qquad (10)$$

This expression is general and not depends on the expressions of the interaction H_{int}. We have just use the energy conservation in the transition probability $R_{\mu\nu->\mu'\nu'}$ and $\Delta\beta/\beta \ll 1$.

Substituting Eq.(3) into Eq.(4), we can get the transition probability expressed as follows:

$$R_{\mu\nu-\mu'\nu'}=\frac{2\pi}{h/2\pi}g^2\sum_{kl}\langle\mu|A_k^+|\mu'\rangle\langle\mu'|A_l|\mu\rangle$$
$$\langle\nu|B_k^+|\nu'\rangle\langle\nu'|B_l|\nu\rangle\,\delta(E_{\mu\nu}-E_{\mu'\nu'}). \qquad (11)$$

Using an identity

$$\delta(E_{\mu\nu}-E_{\mu'\nu'})=\int_{-\infty}^{\infty}(h/2\pi)\,d\omega\,\delta(E_\mu^{(1)}-E_{\mu'}^{(1)}$$
$$+(h/2\pi)\,\omega)\,\delta(E_\nu^{(2)}-E_{\nu'}^{(2)}-(h/2\pi)\,\omega), \qquad (12)$$

and substituting Eq.(11) into Eq.(10), we can get

$$\dot{Q}_{1->2}=\pi\Delta\beta\sum_{kl}\int_{-\infty}^{\infty}d\omega\, g^2((h/2\pi)\,\omega)^2$$
$$\Phi_{A_k^+A_l}(\omega)\,\Phi_{B_k^+B_l}(-\omega) , \qquad (13)$$

where $\Phi_{XY}(\omega)$ is a spectrum of the correlation function defined by

$$\Phi_{XY}(\omega)\equiv\sum_{\mu\mu'}e^{\beta(\Omega-E_\mu)}\langle\mu|X|\mu'\rangle\langle\mu'|Y|\mu\rangle$$
$$\delta(E_\mu-E_{\mu'}+(h/2\pi)\,\omega)$$
$$=\frac{1}{2\pi(h/2\pi)}\int_{-\infty}^{\infty}\langle X(t)\,Y(0)\rangle e^{i\omega t}\,dt . \qquad (14)$$

Because we can inverse the sign of ω in Eq.(12) and H_{int} is a Hermitian matrix, the product of the correlation function can be expressed as follows:

$$\Phi_{A_k \cdot A_l}(-\omega)\ \Phi_{B_k \cdot B_l}(\omega),$$
$$\text{or } \Phi_{A_k A_l^\cdot}(\pm\omega)\ \Phi_{B_k B_l^\cdot}(+\omega).$$

In the second expression double signs must be took in the same order. We define a spectrum density function as follows:

$$\chi_{XY}(\omega) \equiv \Phi_{XY}(\omega) - \Phi_{YX}(-\omega) = (1-e^{-\beta(h/2\pi)\omega})\Phi_{XY}(\omega). \tag{15}$$

Hence, Eq.(13) becomes

$$\dot{Q}_{1->2} = \pi\varDelta\beta\sum_{kl}\int_{-\infty}^{\infty} d\omega g^2((h/2\pi)\omega^2 \frac{\chi_{A_k^+A_l}(\omega)\chi_{B_k^+B_l}(-\omega)}{(1-e^{-\beta(h/2\pi)\omega})(1-e^{\beta(h/2\pi)\omega})}. \tag{16}$$

If the correlation function vanishes when k is not equal to l in the integrand of Eq.(16), and also if A_k and B_k are Hermitian, the heat flux can be expressed as follows:

$$\dot{Q}_{1->2} = 2\pi\varDelta\beta\sum_{k}\int_{0}^{\infty} d\omega g^2((h/2\pi)\omega)^2 \varPhi_{A_k}(\omega)\varPhi_{B_k}(-\omega)$$
$$= 2\pi\varDelta\beta\sum_{k}\int_{0}^{\infty} d\omega g^2((h/2\pi)\omega)^2 \frac{\chi_{A_k}(\omega)\chi_{B_k}(\omega)}{(1-e^{-\beta(h/2\pi)\omega})(e^{\beta(h/2\pi)\omega}-1)} \tag{17}$$

Because the operators are Hermitian and their correlations become self correlations, the suffixes of Φ and χ are reduced to just one suffix .

A2 Heat Flux between a Solid and a Fermi Gas

We show how to get the heat flux between the solid and the Fermi gas from the results discussed in section A1. We assume interactions are given by H_{SL} defined as follows:

$$H_{SL} = \sum_{k,p,q} T^q_{k,p}(a_q + a^+_{-q})c^+_{k+p}c_k, \tag{1}$$

$$T^q_{k,p} \equiv A\delta_{q//,p//}e_{qz}\left(\frac{2(h/2\pi)}{\rho_s\Omega_s\omega_q}\right)^{1/2}M^0_{k,p} = T^{-q}_{k+p,-p}. \tag{2}$$

H_{SL} can be expressed with a product of two operators that include creation and annihilation operators of phonons in the solid and of ^{3}He particles as follows:

$$H_{SL} = \sum_q A_q B_q\ , \tag{3}$$

$$A_q = (a_q + a^+_{-q}) = A^+_{-q}, \tag{4}$$

$$B_q = \sum_{k,p} T^q_{k,p}c^+_{k+p}c_k = B^+_{-q}\ . \tag{5}$$

The correlation function of A_q become

$$\Phi_{A_q^+,A_q}(\omega) = (1+n^0_q)\delta((h/2\pi)\omega_q - (h/2\pi)\omega) + n^0_q\delta((h/2\pi)\omega_q + (h/2\pi)\omega), \tag{6}$$

$$\chi_{A_q^+,A_q}(\omega) = \delta((h/2\pi)\omega_q - (h/2\pi)\omega) - \delta((h/2\pi)\omega_q + (h/2\pi)\omega), \tag{7}$$

where n^0_q is the distribution function of phonons with an angular frequency ω_q. The other correlation functions with combinations of phonon operators become zero. The correlation function of B_q becomes

$$\varPhi_{B_q^+,B_q}(\omega) = \left(\sum_k T^q_{k,0}f^0_k\right)^2\delta((h/2\pi)\omega) + \sum_{k,p}|T^q_{k,p}|^2 f^0_k(1-f^0_{k+p}) \delta(\varepsilon_k - \varepsilon_{k+p} + (h/2\pi)\omega), \tag{8}$$

$$\chi_{B_q^+,B_q}(\omega) = \sum_{k,p}|T^q_{k,p}|^2(f^0_k - f^0_{k+p}) \delta(\varepsilon_{k+p} - \varepsilon_k - (h/2\pi)\omega). \tag{9}$$

where f^0_k is the distribution function of ^{3}He particles. Substituting Eq.(7) and Eq.(9) into Eq.(16) of the section A1, and making use of a feature of $T^q_{k,p}$ in Eq.(2) , we can get the heat flux as follows:

$$\dot{Q} = \frac{2\pi\varDelta\beta}{h/2\pi} \sum_{k,p,q}\frac{((h/2\pi)\omega_q)^2|T^q_{k,p}|^2(f^0_k - f^0_{k+p})}{(1-e^{-\beta(h/2\pi)\omega_q})(1-e^{\beta(h/2\pi)\omega_q})} \delta(\varepsilon_{k+p} - \varepsilon_k - (h/2\pi)\omega_q). \tag{10}$$

Noting the following relation

$$\frac{((h/2\pi)\omega_q)^2\ \Delta\beta}{(1-e^{-\beta(h/2\pi)\omega_q})(1-e^{\beta(h/2\pi)\omega_q})} = (h/2\pi)\ \omega_q\ \Delta T\frac{\partial n^0_q}{\partial T}, \tag{11}$$

we can reform Eq.(10) as follows:

$$\dot{Q}=\frac{2\pi}{h/2\pi}\sum_{k,p,q}|T^{q}_{k,p}|^{2}(h/2\pi)\omega_{q}\Delta T\frac{\partial n^{0}_{q}}{\partial T}$$
$$(f^{0}_{k}-f^{0}_{k+p})\delta(\varepsilon_{k+p}-\varepsilon_{k}-(h/2\pi)\omega_{q})\ . \qquad (12)$$

This heat flux coincide with the expression we got in the section 3.2 , that is derived from the Boltzmann equation of phonons.

References

1) G. D. Mahan, Chap.10, *Many-Particle Physics*, Plenum Press (1981)

2) I. M. Khalatnikov, *An Introduction to the Theory of Superfluidity*,
Chap.23, p138, Benjamin (1965)

3) J. Gavoret, Phys. Rev. **137**, A721 (1965)

4) Baym and Pethick, *physic. of Liq. Sol. Helium Part II*,
Chap.1 section 1.1; section 1.2, Chap.2 p140, ed. by Bennemann and Ketterson, John Wiley & Sons (1978)

5) Landau, JETP **3**, 920 (1957); JETP **35**, 70 (1959)
D. N. Zubarev, *Neravnovesnaya Statisticheskaya Termodinamika*, section 14. 2; section 21.1 , Nauka (1971).

6) D. D. Fitts, *Nonequilibrium Thermodynamics*, McGraw-Hill (1962)

7) H. Goldstein, *Classical Mechanics 2nd Ed.*, Addison-Wesley (1950)

8) P. Nozieres, *Theory of Interacting Fermi Systems*, Chap1, section 3, p18, Benjamin (1964)

9) Jones and March, *Theor. Sol. State Phys. vol. 1*, p154, John Wiley & Sons (1973)

10) F. W. Sheard, R. M. Bowley, and G. A. Toombs, Phys. Rev. **A8** 3135 (1973)

11) D. N. Zubarev, *Neravnovesnaya Statisticheskaya Termodinamika*, Chap.IV, Nauka (1971)

12) G. A. Toombs, F. W. Sheard and M. J. Rice, J. Low Temp. Phys. **39**,273 (1980)

13) L. J. Challis, J. Phys. **C7**,481 (1974)
J. P. Harrison, J. Low Temp. Phys. **37**,467 (1979)

14) A. L. Leggett, M. Vuorio, J. Low Temp. Phys. **3**, 359 (1970)

15) E. Evans, D. L. Mills, Phys. Rev. **B5**, 4126 (1972)

16) F. W. Sheard, G. A. Toombs, J.Phys. **c5**, L 166 (1972)
F. W. Sheard, G. A. Toombs and L.J.Challis, Phys. Rev. Lett. **27**, 1117 (1971)
I. L. Bekarevich and I. M. Khalatnikov, JETP **19**, 1187 (1961)
M. J. Rice, Phys. Rev. Lett. **27**, 92 (1971)

17) E. P. Bashkin and A. E. Meyerovich, Journal de Physique, **c7**, 61 (1980)

18) T. McMuller, J. Low Temp. Phys. **51**, 33 (1983)