A Note on Conditional Expectations

Toshitada SHINTANI*

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Abstract. If f is any real-valued function in $L^1(\Omega, a, P)$ and B is any sub-σ-field of a then $E(f/B)=f$ a. e. Here, in general, the exceptional set $\notin$ B. By using this it is shown that the paths of Brownian Motion are a. e. differentiable.

Let (Ω, a, P) be a probability space and $E(f/B)$ the conditional expectation of f with respect to the sub-σ-field B of a. Let $B \neq \{\phi, \Omega\}$.

Theorem 1. Let $A \in a$ and let B be any sub-σ-field of a then $E(\chi_A/B) = \chi_A$ a. e.
Here the exceptional set e, in general, $e \notin B$.

Proof. Let $A \in a$, $\forall B \subset a$ and $A \notin B$. (If $A \in B$ then the Theorem is well-known.)

$$\begin{aligned}\int_\Omega E(\chi_A/B)(\omega)\,dP &= \int_\Omega \chi_A(\omega)\,dP \\ &= \int_A \chi_A(\omega)\,dP + \int_{A^c} \chi_A(\omega)\,dP \\ &= 1\cdot P(A) + 0\cdot P(A^c) \\ &= P(A) \quad \text{(It may be supposed that } 0<P(A)<1.)\end{aligned}$$

On the other hand, since $A \in a$ and $B \subset a$,

$$\int_\Omega E(\chi_A/B)(\omega)\,dP = \int_A E(\chi_A/B)(\omega)\,dP + \int_{A^c} E(\chi_A/B)(\omega)\,dP$$

Let $Z(\omega)$ be any a-measurable random variable. Then, by the mapping when B is given

$$F : Z(\omega) \longmapsto E(Z/B)(\omega) \quad (\forall \omega \in \Omega),$$

$E(\chi_A/B)(\omega) =$ a. e. constant a on A
and $E(\chi_A/B)(\omega) =$ a. e. constant b on A^c.
Notice that the exceptional sets are a-measurable so that these union $\notin$ B since $A \notin B$, in general.
Then

$$(*) \qquad P(A) = \int_\Omega E(\chi_A/B)(\omega)\,dP = a\cdot P(A) + b\cdot P(A^c)$$
$$= a\cdot P(A) + b\cdot(1-P(A)). \quad (1-P(A) \neq 0.)$$

By $(*)$ $\quad b=0 \Longrightarrow a=1 \quad$ and $\quad a=1 \Longrightarrow b=0$

So $\quad a=1 \Longleftrightarrow b=0.$

By contraposition of the above statement,

$$a \neq 1 \Longleftrightarrow b \neq 0.$$

We shall show that if we suppose that $a \neq 1$ and $b \neq 0$ then we have a contradiction. For instance, suppose $a=1/k$ and $b=1/k$ $(k>1)$ (so suppose $a=b$). Then, by $(*)$,

$$1 > P(A) = 1/k\cdot P(A) + 1/k\cdot(1-P(A)) = 1/k.$$

So take k such that $k > \frac{1}{P(A)}$ then $P(A) < P(A)$.

This is a contradiction. So it is not $a \neq 1$ thus it is not $b \neq 0$. That is, $a=1$ and $b=0$.
Therefore $E(\chi_A/B)(\omega) = \chi_A(\omega)$ a. e. and the exceptional set e, in genral, $e \notin B$. (q. e. d.)

* 一般教科助教授，数学

Remark. If a=b then $E(\chi_A/B)(\omega)$ = a. e. constant on Ω.
As $f\in L^1(\Omega, a, P)$ is a limit of sequence of simple functions, $E(f/B)(\omega)$ = a. e. constant on Ω. This can not happen since $E(f/B)(\omega)$ is the function of ω.
Therefore $E(\chi_A/B) \fallingdotseq$ a. e. constant on Ω.
Theorem 2. Let $f\in L^1(\Omega, a, P)$ and $\forall B\subset a$.
Then $E(f/B)=f$ a. e. (The exceptional set e, in general, $e\notin B$.)
Proof. Since $f\in L^1 \Longleftrightarrow |f|=f^+ + f^- \in L^1$ so that $f^+, f^-\in L^1$, f^+ and f^- are a-measurable.
For f^+ and f^- there are sequences of a-measurable simple functions $\{g_n\}_{n\geqslant 1}$ and $\{h_n\}_{n\geqslant 1}$ such that
$g_n(\omega)\geqq 0$, $g_n(\omega)\uparrow f^+(\omega)$; $h_n(\omega)\geqq 0$, $h_n(\omega)\uparrow h^-(\omega)$ $(\forall \omega\in\Omega)$.
Set $f_n=g_n-h_n$ $(n=1, 2, \ldots)$.
Then, for a-measurable function f, $\{f_n\}_{n\geqslant 1}$ is the sequence of a-measurable simple functions such that

$$|f_n(\omega)|\leqq|f(\omega)|\in L^1 \text{ and } \lim_{n\to\infty} f_n(\omega)=f(\omega) \quad (\forall \omega\in\Omega).$$

Here, for f^+ g_n is defined as follows :
for $i=0, 1, \ldots, n2^n-1$ and $n=1, 2, \ldots$

$$\text{set} \quad A_{ni}=\left\{\frac{i}{2^n} \leqq f^+ < \frac{i+1}{2^n}\right\},\ A_{n\ n}2^n=\{f^+\geqq n\}$$

and define g_n by

$$g_n=\sum_{i=1}^{n2^n} \frac{i}{2^n}\cdot \chi_{A_{ni}}$$

Then $A_{ni}\in a$ and $\{g_n\}_{n\geqslant 1}$ satisfies above property.
Define similarly h_n for f^-.
Next, by $E(\chi_A/B)=\chi_A$ a. e.,

$$E(g_n/B)=g_n \text{ a. e. and } E(h_n/B)=h_n \text{ a. e.}$$

so that $E(f_n/B)=f_n$ on $\Omega\setminus e_n$, $P(e_n)=0$, so on $\Omega\setminus \bigcup_n e_n$.
By the Lebesgue's convergence theorem

$$\lim_{n\to\infty} E(f_n/B) \underset{\text{a. e.}}{=\!=} E(\lim_{n\to\infty} f_n/B)=E(f/B)$$

and the most left side $\underset{\text{a. e.}}{=\!=} \lim_{n\to\infty} f_n = f$.
So $E(f/B)=f$ a. e. for $\forall B\subset a$. (q. e. d.)
Theorem 3. Let $f=(f_t)_{t\geqslant 0}$ be any real-valued martingale on the probability space $(\Omega, a, \{a_t\}, P)$.

$$\text{Then} \quad P\left(\lim_{t\to s}\frac{f_t(\omega)-f_s(\omega)}{t-s}=0\right)=1.$$

Proof. As f is a martingale it may be supposed that the paths are continuous.
Since $f_t \underset{\text{a. e.}}{=\!=} E(f_t/a_s)=f_s$, $f_t(\omega)=f_s(\omega)$ a. e..
So take any $s\geqslant 0$ and fix this
and let

$$f_t(\omega)=f_s(\omega) \text{ on } \Omega\setminus e_t,\ P(e_t)=0,$$

and

$$f_{t'}(\omega)=f_s(\omega) \text{ on } \Omega\setminus e_{t'},\ P(e_{t'})=0,$$

$$\text{So} \quad \frac{f_t(\omega)-f_s(\omega)}{t-s}=0 \text{ on } \Omega\setminus e_t\ (t\fallingdotseq s)\ (\text{i, e., for } \forall\omega\in\Omega\setminus e_t.)$$

and this holds also for any t′ instead of t.
Take any $\omega\in\Omega\setminus e_t\cup e_{t'}$. Then

$$\lim_{t\to t'}\frac{f_t(\omega)-f_s(\omega)}{t-s}=0.$$

In fact, when $t\to t'$, $f_t(\omega)$ becomes $f_{t'}(\omega)$ since paths are continuous.

$$\lim_{t\to t'}\frac{f_t(\omega)-f_s(\omega)}{t-s}=\frac{f_{t'}(\omega)-f_s(\omega)}{t'-s}=0$$

on $\Omega \setminus e_{t'}$ (thus, $=0$ on $\Omega \setminus e_t \cup e_{t'}$).
So, for $\forall\ t' \geqq 0$, on $\Omega \setminus e_t \cup e_{t'}$

$$(*) \quad \frac{f_t(\omega)-f_s(\omega)}{t-s}=0=\lim_{t \to t'} \frac{f_t(\omega)-f_s(\omega)}{t-s}.$$

(Notice that $t \to t'$ is, in general, $t \neq t'$ and $t \to t'$.)
Now, $f_s(\omega)=f_s(\omega)$ on Ω so that $e_s=\phi$.
Let $t'=s$ in $(*)$ then $e_{t'}=e_s=\phi$ and

$$0=\frac{f_t(\omega)-f_s(\omega)}{t-s}=\lim_{t \to s} \frac{f_t(\omega)-f_s(\omega)}{t-s} \text{ on } \Omega \setminus e_t,\ P(e_t)=0.$$

Therefore, $\lim_{t \to s} \frac{f_t(\omega)-f_s(\omega)}{t-s}=0$ a. e.,

that is,

$$P\left(\lim_{t \to s} \frac{f_t(\omega)-f_s(\omega)}{t-s}=0\right)=1. \quad \text{(q. e. d.)}$$

Corollary. Let $B=(B_t)_{t \geqq 0}$ be any real-valued Brownian motion then

$$P\left(\lim_{t \to s} \frac{B_t(\omega)-B_s(\omega)}{t-s}=0\right)=1.$$

Remark. Since $B_t(\omega)=B_s(\omega)$ a. e., the proofs of Paley-Wiener-Zygmund theorem fail.

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Department of Mathematics, Tomakomai National College of Technology, Tomakomai, Hokkaido 059-12, Japan