# On $L^P$-boundedness of martingales in Banach space with Radon-Nikodým property

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(Received November 28,1997)

**Abstract.** Let $f = (f_n)$ be a martingale with values in Banach space X having Radon-Nikodým property. $\|\cdot\|$ denotes a norm on X and E denotes Lebesgue integration over a probability space $(\Omega, a, P)$. Then f is $L^1_X$-bounded if and only if f is $L^p_X$-bounded $(1<p<\infty)$. Moreover if $f = (f_n)$ is $L^1_X$-bounded martingale then $\{\|f_n\|\}$ is uniformly integrable. Thus, $L^1$-bounded positive submartingale $f = (f_n)$ is uniformly integrable. In fact, f converges in $L^1$.

Let $\|f\|_1 = \sup_n E\|f_n\|$ and let $M^1 = \{\text{martingale } f = (f_n) : \|f\|_1 < \infty\}$.
Here notice that $\|f(\omega)\|$ is a positive function over $\Omega$.

**Theorem I.** f is $L^1_X$-bounded $\Leftrightarrow$ f is $L^p_X$-bounded $(1<p<\infty)$.

**Proof.** $f \in M^1 \Rightarrow f_n \to {}^{\exists} f_\infty$ a. e. as $n \to \infty$.

$$E\|f_\infty\| \leqslant \underline{\lim}_n E\|f_n\| \quad (\text{by Fatou's lemma})$$
$$\leqslant \overline{\lim}_n E\|f_n\|$$
$$\leqslant \sup_n E\|f_n\|$$
$$< \infty$$

And $E\|f_n\| \leqslant \sup_n E\|f_n\| < \infty$ for all n.

So $E\|f_n\| \leqslant {}^{\exists}G < \infty \ (n = 1, 2, \ldots, \infty)$
so that $\|f_n\| \leqslant {}^{\exists}K = K(\omega) < \infty$ a. e. $(n = 1, 2, \ldots, \infty)$,
i. e., $\|f_n\| \leqslant {}^{\exists}K < \infty$ on $\Omega \backslash e_n$, $P(e_n) = 0$, so on $\Omega \backslash \underset{n}{U} e_n$ by Zorn's lemma.

Thus, $\sup_n \|f_n(\omega)\| \leqslant K < \infty$ a. e., on $\Omega \backslash \underset{n}{U} e_n$.

From $\sup_n \|f_n\| \geqslant \|f_n\| \ (\geqslant 0)$

$K^p \underset{a.e.}{\geqslant} (\sup_n \|f_n\|)^p \geqslant \|f_n\|^p$ for all n $(1 \leqslant p < \infty)$.

So $\sup_n \|f_n(\omega)\|^p \leqslant K^p < \infty$ a. e., i. e., on $\Omega \backslash \underset{n}{U} e_n$.

On the other hand,

$$\| \|f_n\|^p \|_\infty = \underset{\omega}{\text{ess.sup}} | \|f_n(\omega)\|^p |$$
$$= \underset{\omega}{\text{ess.sup}} \|f_n(\omega)\|^p$$
$$= \inf_{e \in N} \sup_{\substack{\Omega\backslash e \\ P(e)=o}} \|f_n(\omega)\|^p$$

( Here $N$ is the family of all null set e. )

$$\leqslant \inf_{e \in N} \sup_{\substack{\Omega\backslash e \\ P(e)=o}} \left\{ \sup_n \|f_n(\omega)\|^p \right\}$$
$$\leqslant \inf_{e \in N} \sup_{\substack{\Omega\backslash e \\ P(e)=o}} K^p \leqslant \sup_{\substack{\Omega\backslash e \\ P(e)=o}} K^p < \infty \text{ for all n. (Here let } e = \underset{n}{U} e_n)$$

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And $\| 1 \|_1 < \infty$. That is , $\|f_n\|^P \in L_\infty$ ,$1 \in L^1$.
Thus, by Hölder's inequality, for all n,

$$\int_\Omega \|f_n\|^P \, dP = \int_\Omega \|f_n\|^P \cdot 1 \, dP \leqq \| \|f_n\|^P \|_\infty \cdot \| 1 \|_1$$

$$\leqq \sup_n \| \|f_n\|^P \|_\infty \cdot \| 1 \|_1 \leqq \sup_{\substack{\Omega \setminus e \\ P(e)=o}} K^P \cdot \| 1 \|_1 < \infty.$$

so that $\sup_n \| \|f_n\| \|_P < \infty$ .

Therefore f is $L^1_X$-bounded $\Rightarrow$ f is $L^p_X$-bounded ( $1 < p < \infty$ ).
Conversely, f is $L^p_X$-bounded ( $1 < p < \infty$ )$\Rightarrow$
Since $\|f_n\| \in L^P$ and $1 \in L^q$ ( $1/P + 1/q = 1$ ),

$$\int_\Omega \|f_n\| \, dP = \int_\Omega \|f_n\| \cdot 1 \, dP$$

$$\leqq \| \|f_n\| \|_P \cdot \| 1 \|_q \text{ ( by Hölder's inequality )}$$

$$= \left\{ \int_\Omega \|f_n\|^P \, dP \right\}^{1/P} \cdot \| 1 \|_q \quad \text{( Here notice } 1 \in L^q. \text{ )}$$

$$\leqq \sup_n \| \|f_n\| \|_P \cdot \| 1 \|_q < \infty.$$

That is, f is $L^p_X$-bounded $\Rightarrow$ f is $L^1_X$-bounded. ( q. e. d. )

**Theorem 2** If $f = ( f_n ) \in M^1$ then $\{ \| f_n \| \}$ is uniformly integrable.

**Proof.** Let $f \in M^1$ then $f_n \xrightarrow{\exists} f_\infty \in L^1$ a. e. ( $n \to \infty$ ).

$$\lim_{n\to\infty} | \|f_n\| - \|f_\infty\| | = | \lim_{n\to\infty} ( \|f_n\| - \|f_\infty\| ) |$$

$$= | \lim_{n\to\infty} \|f_n\| - \|f_\infty\| |$$

$$= | \| \lim_{n\to\infty} f_n\| - \|f_\infty\| | \text{ ( by the continuity of norm )}$$

$$= | \|f_\infty\| - \| f_\infty\| |$$

$$= 0 \text{ a. e.}$$

On the other hand, since ( $f_n, a_n$ ) is a martingale

$$\| f_n \| = \| E( f_{n+1} / a_n ) \|$$

$$\leqq E( \| f_{n+1} \| / a_n ) \text{(( } \| f_n \| \text{ ) is a real submartingale. )}$$

$$\overline{\overline{\text{a. e.}}} \; \| f_{n+1} \| \quad \text{( See Remark. )}$$

$$= \| E( f_{n+2} / a_{n+1} ) \|$$

$$\leqq E( \| f_{n+2} \| / a_{n+1} )$$

$$\overline{\overline{\text{a. e.}}} \; \| f_{n+2} \| \uparrow \lim_{j\to\infty} \| f_{n+j} \| = \| f_\infty \| .$$

And, by Fatou's lemma,

$$E \| f_\infty \| \leqq \underline{\lim}_{n\to\infty} E \|f_n\| \leqq \overline{\lim}_{n\to\infty} E \|f_n\| \leqq \sup_n E \|f_n\| < \infty.$$

That is, $\|f_n\| \leqq \|f_\infty\|$ a. e. and $\|f_\infty\| \in L^1$.
So $| \|f_n\| - \|f_\infty\| | \leqq | \|f_n\| | + | \|f_\infty\| | = \|f_n\| + \|f_\infty\| \leqq 2 \cdot \|f_\infty\| \in L^1$.
By Lebesgue's convergence theorem

$$\lim_{n\to\infty} E\ \big|\|f_n\| - \|f_\infty\|\big| = E\big(\lim_{n\to\infty} \big|\|f_n\| - \|f_\infty\|\big|\big) = E(0) = 0.$$

That is, $\{\|f_n\|\}$ is uniformly integrable. ( q. e. d. )

Thus, $L^1$ – bounded positive submartingale $f = (f_n)$ is uniformly integrable. In fact, f converges in $L^1$.

**Remark**. In the proof suppose that $a_n$ is not finitely generated. If some $a_n$ ($\neq\{\phi, \Omega\}$) is finitely generated then every $a_m$, $m>n$, is finitely generated since $a_m \supset a_n$ so that there is $g\in L^1$ such that $\|f_n\| \leq g$ ($n=1, 2, \cdots$) since $\|f_m\| = E(\|f_m\|/a_m)$ = simple function ($m>n$).
Thus, then $\{\|f_n\|\}$ is uniformly integrable.

## References.


1. D.L.Burkholder : Martingale transforms. Ann. Math. Statist., 37(1966) ,1494-1504.
2. ―――――― : A geometric characterization of Banach spaces in which martingale difference sequences are unconditional. Ann. Probability, 9(1981), 997-1011.
3. S.D.Chatterji : Martingale convergence and the Radon-Nikodým theorem in Banach spaces. Math. Scand.,22(1968),21-41.
4. T.Shintani : Martingale transforms in a Banach space.(1997), (preprint).


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