Guide to Applied Mathematics for Foreign Students II

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Abstract

This article is a self-study note for foreign students.
The content is a part of lectures, particularly in Fourier series and its application to Partial differential equations.

1 Fourier series

1.1 Definitions of Fourier series and its coefficients

Let $\vec{e}_n$ be a basis vector, which is characterized as

$$<\vec{e}_m, \vec{e}_n> = \delta_{mn} \tag{1}$$

where the notation $<,>$ denotes scalar (or inner) product. The equation (1) is said to be orthonormal and $\vec{e}$ is also called orthonormal vector. Let V^N be N-dimensional vector space, $\vec{X}$ any element of V^N. $\vec{X}$ may be expanded in terms of the set of basis vectors $\{\vec{e}\}_{n=1,2,\cdots,N}$

$$\vec{X} = \sum_{n=1}^{N} c_n \vec{e}_n \tag{2}$$

On multiplying by $\vec{e}_n$ in both members of the equation (2) and using the equation (1), the expanded coefficients c_n are given.

$$c_n = <\vec{X}, \vec{e}_n> \tag{3}$$

The magnitude of vector $\vec{X}$ is expressed

$$|\vec{X}| = \sqrt{|\vec{X}|^2} = \sqrt{(\vec{X}, \vec{X})} = \sqrt{\sum_{n=1}^{N} c_n^2} \tag{4}$$

or

$$|\vec{X}|^2 = \sum_{n=1}^{N} c_n^2 \tag{5}$$

which is the extention to N-dimension of Phthagoras' theorem.

By analogy with the above manner, consider the expansion of a function $f(x)$, which is at first assumed to be periodic, that is, $f(x) = f(x+T)$. T denotes the period, for example, $T=2\pi$. Let $f(x)$ define in the range $[-\pi, \pi]$. Hereafter we assume that the function is not totally arbitrary, but should be piecewise continuous, and the square of its absolute value should have a finite integral over all space.

Let $f_n(x)$ be a basis function, which is characterized as

$$<f_m(x), f_n(x)> = \delta_{mn} \tag{6}$$

where the nortation $<,>$ denotes scalar (or inner) product, which is defined

$$<f_m(x), f_n(x)> \stackrel{\text{def}}{=} \int_{-\pi}^{\pi} f_m(x) f_n(x)\, dx \tag{7}$$

or in the case that the functions are complex

$$\stackrel{\text{def}}{=} \int_{-\pi}^{\pi} \overline{f_m(x)} f_n(x)\, dx \tag{8}$$

$f_n(x)$ is also called an orthonormal function. $f(x)$ may be expanded in terms of the set of basis functions $\{f_n(x)\}_{n=1,2\cdots,\infty}$

$$f(x) = \sum_{n=1}^{\infty} c_n f_n(x) \tag{9}$$

where

$$c_n = <f(x), f_n(x)> = \int_{-\pi}^{\pi} f(x) f_n(x)\, dx \tag{10}$$

provided that
$<f(x), f_n(x)> = 0 \Rightarrow f(x) = 0$

The magnitude of function $f(x)$ is written

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$$|f(x)| = \sqrt{< f(x), f(x)>}$$
$$= \sqrt{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_m c_n < f_m(x), f_n(x) >}$$
$$= \sqrt{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_m c_m \delta_{mn}} = \sqrt{\sum_{n=1}^{\infty} c_n^2} \tag{11}$$

Instead of the notation $|f(x)|$, it is custmary to use the notation $\|f(x)\|$, which is called "norm" in the functional space:

$$\|f(x)\|^2 = \sum_{n=1}^{\infty} c_n^2 \tag{12}$$

which is called Parseval's equality.

As first example for basis sets, we take the following trigonometrical functions;

$$\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos(nx), \frac{1}{\sqrt{\pi}}\sin(nx)\}_{n=1,2,\cdots,\infty} \tag{13}$$

where

$$<\frac{1}{\sqrt{\pi}}\cos mx, \frac{1}{\sqrt{\pi}}\cos nx> = \frac{1}{\pi} <\cos mx, \cos nx>$$
$$= \delta_{mn} = \begin{cases} 1\ (m=n=0) \\ 0\ (m=n=0) \end{cases} \tag{14}$$
$$<\frac{1}{\sqrt{\pi}}\sin mx, \frac{1}{\sqrt{\pi}}\sin nx> = \frac{1}{\pi} <\sin mx, \sin nx>$$
$$= \delta_{mn} \tag{15}$$
$$<\frac{1}{\sqrt{\pi}}\cos mx, \frac{1}{\sqrt{\pi}}\sin nx> = \frac{1}{\pi} <\cos mx, \cos nx>$$
$$= 0 \tag{16}$$

Using the basis set (13), we rewrite the expansion series(9)

$$f(x) = \sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{17}$$

or

$$f(x) = \frac{a_0}{\sqrt{2\pi}} + \sum_{n=1}^{\infty} (a_n \frac{1}{\sqrt{\pi}} \cos nx + b_n \frac{1}{\sqrt{\pi}} \sin nx) \tag{18}$$

On multiplying the equation (18) by basis functions (by $\frac{1}{\sqrt{2\pi}}$ or by $\frac{1}{\sqrt{\pi}}$ $\cos nx$ or by $\frac{1}{\sqrt{\pi}}$ $\sin nx$) and integrating term-by-term, the expanded coefficients are given

$$a_0 = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} f(x)\cdot 1 dx$$
$$a_n = \frac{1}{\sqrt{\pi}}\int_{-\pi}^{\pi} f(x)\cos nx dx$$
$$b_n = \frac{1}{\sqrt{\pi}}\int_{-\pi}^{\pi} f(x)\sin nx dx$$

Such a trigonometrical series uniformly convergent in some range is called Fourier series and its coefficients are Fourier ones. Substitution of these coefficients in the formula (18) gives

$$f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) dx$$
$$+ \sum_{n=1}^{\infty}\left\{\left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(\xi)\cos n\xi d\xi\right)\cos nx\right.$$
$$\left. + \left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(\xi)\sin n\xi d\xi\right)\sin nx\right\}$$
$$= \frac{a'_0}{2} + \sum_{n=1}^{\infty} (a'_n \cos nx + b'_n \sin nx) \tag{19}$$

where

$$a'_n = \frac{a_n}{\sqrt{\pi}} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx \tag{20}$$
$$b'_n = \frac{b_n}{\sqrt{\pi}} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx \tag{21}$$

It is convenient to describe the trigonometrical series (19) and its coefficients (20) and (21) as Fourier series and Fourier coefficients associated with $f(x)$.

As for the expansion series, when $f(x)$ is even, then it follows that

$$(x) = \frac{a'_n}{2} + \sum_{n=1}^{\infty} a' \cos nx \tag{22}$$

$$f(x) = \sum_{n=1}^{\infty} a'_n \sin nx \tag{23}$$

and when $f(x)$ is odd, then

Each are called Fourier cosine and sine series respectively.

Example 1. Find Fourier series for each of following periodic functions($T = 2\pi$).

(1) $f(x)=x(-\pi \leq x \leq \pi)$

(2) $f(x)=x(0 \leq x \leq 2\pi)$

(3) $f(x)=x^2(-\pi \leq x \leq \pi)$

(4) $f(x)=x^2(0 \leq x \leq 2\pi)$

Solution.

(1) $a' = 0$ since $f(x)$ is odd.

$$b'_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx dx = (-1)^{n-1}\frac{2}{n}$$

Therefore, the required series is

$$x = 2\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}\sin nx$$
$$= 2 \quad \sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{2} - \frac{\sin 4x}{2} - \ldots\ldots$$

Here, let $x = \frac{\pi}{2}$ in the above expansion series,2 we have

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

(2) $f(x)=x$ is neither even nor odd in the range $[0, 2\pi]$.

$$a'_0 = \frac{1}{2\pi}\int_0^{2\pi} x \cdot 1 dx = \pi$$
$$a'_n = \frac{1}{\pi}\int_0^{2\pi} x\cos nx dx = 0$$
$$b'_n = \frac{1}{\pi}\int_0^{2\pi} x\sin nx dx = -\frac{2}{n}$$
$$x = \pi - \sum_{n=1}^{\infty}\frac{2}{n}\sin nx$$
$$= \pi - 2\left[\sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \frac{\sin 4x}{4} + \cdots\right]$$

(3) $b_n = 0$ since $f(x)$ is even.

$$a'_0 = \frac{2}{2\pi}\int_0^{\pi} x^2 \cdot 1dx = \frac{\pi^2}{3}$$
$$a'_n = \frac{2}{\pi}\int_0^{\pi} x^2\cos nx dx = (-1)^n\frac{4}{n^2}$$
$$b_n = 0$$
$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^2}\cos nx$$
$$= \frac{\pi^2}{3}\ 4\left[\cos x - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} - \cdots\right]$$

Here, let $x = \pi$ in the above expansion series

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

let $x = 0$, then

$$\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots \qquad (24)$$

(4) $f(x) = x^2$ is neither even nor odd in the range $0, 2\pi$.

$$a'_0 = \frac{1}{2\pi}\int_0^{2\pi} x^2 \cdot 1dx = \frac{4\pi^2}{3}$$
$$a'_n = \frac{1}{\pi}\int_0^{2\pi} x^2\cos nx dx = \frac{4}{n^2}$$
$$b'_n = \frac{1}{\pi}\int_0^{2\pi} x^2\sin nx dx = -\frac{4\pi}{n}$$
$$x^2 = \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty}\ \frac{1}{n^2}\cos nx - \frac{\pi}{n}\sin nx \qquad (25)$$

1.2 Meaning of Fourier seires and its coefficient

A linear combination of trigonomatrical functions $A\cos kx + B\sin kx$, where A, B, k are the arbitrary constants, is the general solution of the differential equation

$$D^2y(x) + k^2y(x) = \frac{d^2y(x)}{dx^2} + k^2y(x) = 0$$

On taking into account the boundary conditions on the above equation,

$$y(\pi) = y(-\pi) \text{ and } \frac{dy(\pi)}{dx} = \frac{dy(-\pi)}{dx}$$

the constant k in the solution has to be

$k = n =$ integer

where the integer n is called eigen value.

$$-\frac{d^2y_n(x)}{dx^2} = n^2y_n(x)$$

The corresponding solution for eigen equation with respact to eigen value n is called eigen function, which are denoted by cos nx or sin nx. These eigen functions, which can always be normalized, are orthogonal;

$$\langle\cos mx,\ \sin nx\rangle = 0$$

Then, by the use of the set of orthonormalized eigen functions,

$$\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos nx, \frac{1}{\sqrt{\pi}}\sin nx\}_{n=1,2,\cdots,\infty}$$

any function in the range $-\pi$, π can be expanded

$$f(x)=\sum_{n=0}^{\infty} c_n y_n(x)$$

$$\left[y_n(x)\in\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos nx \frac{1}{\sqrt{\pi}}\sin nx\}_{n=1,2,\cdots,\infty}\right]$$

where

$$c_n=< f(x), y_n(x) >$$

That is to say, Fourier series is the expansion by means of all the eigen functions.

Next, consider a meaning of Fourier coefficients c_n. Let $\sum_{i=0}^{n} d_i y_i(x)$ be a finite approximate formula of $f(x)$. The coefficient d_i is determined due to the minimum condition of the error of mean square,

$$\lim_{n\to\infty}\int_{-\pi}^{\pi}\{f(x)-\sum_{i=0}^{n} d_i y_i(x)\}^2=0,$$

$$\begin{aligned}
(\mathrm{I}) &= \int_{-\pi}^{\pi}\{f(x)-\sum_{i=0}^{n} d_i y_i(x)\}^2 dx \\
&= \int_{-\pi}^{\pi}\{f(x)\}^2 dx \\
&\quad - 2\sum_{i=0}^{n} d_i \int_{-\pi}^{\pi} f(x) y_i(x) dx + \sum_{i=0}^{n} d_i^2 \\
&= \int_{-\pi}^{\pi}\{f(x)\}^2 dx - 2\sum_{i=0}^{n} c_i d_i + \sum_{i=0}^{n} d_i^2 \\
&= \int_{-\pi}^{\pi}\{f(x)\}^2 dx + \sum_{i=0}^{n}(d_i-c_i)^2 - \sum_{i=0}^{n} c_i^2
\end{aligned}$$

Obviously, since the quantity (I) above is positive, it follows that

$$\int_{-\pi}^{\pi}\{f(x)\}^2 dx \geq \sum_{i=0}^{n} c_i^2 \tag{26}$$

which is called Bessel's inequality. When $d_i = c_i$, it follows that

$$\int_{-\pi}^{\pi}\{f(x)\}^2 dx = \sum_{i=0}^{n} c_i^2 \tag{27}$$

which is called Parseval's equality. Therefore, Fourier series is said to be a best fit approximation. Since $\int_{-\pi}^{\pi}\{f(x)\}^2 dx$ is assumed to have a finite value, the coefficient c_n has to be

$$c_n \to 0 \ (n\to\infty) \tag{28}$$

which is called Cauchy−Riemann's theorem.

1.3 Differentiation and Integration of Fourier series

Consider differentiation and integration of Fourier series. As an example, we shall begin the discussion on Fourier series of the functions x and x^2 given in Example 1.

$$x = 2\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}\sin nx \tag{29}$$

$$x^2=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^2}\cos nx$$

By differentiating in both members of x^2

$$\begin{aligned}
2x &= 4\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}(-1)\sin nx \\
&= 4\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}\sin nx
\end{aligned}$$

Consequently, we obtain

$$x = 2\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}\sin nx$$

In general, when $\frac{df(x)}{dx}$, Fourier series of g(x) may be given by termwise differentiation of that of $f(x)$

$$\frac{df(x)}{dx}=g(x)=\sum_{n=1}^{\infty}(-na'_n\sin nx+nb'_n\cos nx)$$

$$=\sum_{n=1}^{\infty} n(b'_n\cos nx - a'_n\sin nx) \tag{29}$$

By integrating in both members of x of the equation(1.1)

$$(\text{L.S.}) = \int_0^x t dt = \frac{1}{2} x^2$$

$$(\text{L.S.}) = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{1}{n} \int_0^x \sin nt dt$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1}\frac{1}{n}\left\{-\frac{1}{n}(\cos nx - 1)\right\}$$

$$= 2\sum_{n=1}^{\infty} (-1)^{n}\frac{1}{n^2}\cos nx + 2\sum_{n=1}^{\infty} (-1)^{n-1}\frac{1}{n^2}$$

Consequently, we obtain

$$x^2 = 4\sum_{n=1}^{\infty} (-1)^{n-1}\frac{1}{n^2} + 4\sum_{n=1}^{\infty} (-1)^{n}\frac{1}{n^2}\cos nx$$

$$= \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} (-1)^{n}\frac{1}{n^2}\cos nx$$

where

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^2} \overset{(24)}{=} \frac{\pi^2}{12}$$

In general, when $\int_0^x f(x)dt = g(x)$, Fourier series of $g(x)$ may be given by termwise integration of that of $f(x)$

$$\int_0^x f(t)dt = g(x)$$

$$= \frac{a_0'}{2}\int_0^x dt + \left\{\int_0^x (a_n'\cos nt + b_n'\sin nt)dt\right\}$$

1.4 Conplex form of Fourier series

The functions cos nx and sin nx are expressed in terms of exponential functions

$$\cos nx = \frac{1}{2}(e^{inx} + e^{-inx}) \qquad (30)$$

$$\sin nx = \frac{1}{2i}(e^{inx} + e^{-inx}) \qquad (31)$$

A set of basis functions, $\left\{\frac{1}{\sqrt{2\pi}} e^{inx}\right\}_{n=0,\pm1,\pm2,\cdots,\pm\infty,}$ is called an orthonormal system:

$$<\frac{1}{\sqrt{2\pi}} e^{inx}, \frac{1}{\sqrt{2\pi}} e^{inx}> = \delta_{mn} \qquad (32)$$

By substituting these relations into the ex pansion series (9), $f(x)$ is reduced to

$$f(x) = \frac{a_0'}{2} + \sum_{n=1}^{\infty} (a_n'\cos nx + b_n'\sin nx)$$

$$= \frac{a_0'}{2} + \sum_{n=1}^{\infty}\left\{a_n'\frac{1}{2}(e^{inx} + e^{inx}) + b_n'\frac{1}{2i}(e^{inx} - e^{-inx})\right\}$$

$$= c_0' + \sum_{n=1}^{\infty}\left\{\frac{a_n' - ib_n'}{2} e^{inx} + \frac{a_n' + ib_n'}{2} e^{-inx}\right\}$$

$$= c_0' + \sum_{n=1}^{\infty}(c_n' e^{inx} + c_{-n}' e^{-inx})$$

$$= \sum_{n=1}^{\infty} c_n' e^{inx} = \sum_{n=1}^{\infty} c_n \frac{1}{\sqrt{2\pi}} e^{inx} \qquad (33)$$

which is called complex Fourier series. Its coefficients are

$$c_n' = \frac{c_n}{\sqrt{2\pi}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx} dx, \ c_0' = \frac{a_0'}{2} \qquad (34)$$

The relations of coefficients between c_n' and (a_n', b_n') are given

$$c_n' = \frac{a_n' - ib_n'}{2} \text{ and } \overline{c_n'} = \frac{a_n' - ib_n'}{2} = c_{-n}' \qquad (35)$$

Example 2. Find complex Fourier series in the range $|-\pi, \pi|$ for each of the functions.

(1) $f(x) = x$ (2) $f(x) = x^2$

(3) $f(x) = \cos x$ (4) $f(x) = \sin x$

Solution.

(1)

$$C' = \frac{1}{2\pi}\int_{-\pi}^{\pi} xe^{-inx} dx = (-1)^{n+1}\frac{1}{in}$$

$$x = \sum_{n=-\infty}^{\infty} (-1)^{n+1}\frac{1}{in} e^{-nx}$$

$$= \frac{1}{i}\left(e^{ix} - \frac{e^{i2x}}{2} + \frac{e^{i3x}}{3} - \cdots\cdots\right.$$

$$\left. - e^{-ix} + \frac{e^{-i2x}}{2} - \frac{e^{-i3x}}{3} + \cdots\right)$$

$$= \frac{1}{i}\left(2i\sin x - \frac{2i\sin 2x}{2} + \frac{3i\sin 3x}{3} - \cdots\right)$$

$$= 2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right)$$

$$= 2\sum_{n=1}^{\infty} (-1)^{n+1}\frac{\sin nx}{n} \qquad (36)$$

(2)

$$c'_0=\frac{1}{2\pi}\int_{-\infty}^{\infty}x^2dx=\frac{\pi^2}{3}$$

$$c'_n=\frac{1}{2\pi}\int_{-\infty}^{\infty}x^2e^{-inx}dx=\frac{1}{2\pi}(-1)^n\frac{4\pi}{n^2}$$

$$=(-1)^n\frac{2}{n}$$

$$x^2=\frac{\pi^2}{3}-\sum_{n=-\infty}^{\infty}{}'(-1)^n\frac{2}{n^2}e^{inx}$$

$$=\frac{\pi^2}{3}-4\sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{n^2}$$

where the prime dentoes the term for which $n=0$ is omitted.

(3)

$$c'_n=\frac{1}{2\pi}\int_{-\infty}^{\infty}\cos xe^{-inx}dx$$

$$=\frac{1}{2\pi}\frac{1}{2}\int_{-\infty}^{\infty}(e^{ix}+e^{-ix})e^{-inx}dx$$

$$=\frac{1}{2\pi}\frac{1}{2}\int_{-\infty}^{\infty}(e^{-(n-1)x}+e^{-i(n+1)x})dx$$

$$=\frac{1}{2}(\delta_{n1}+\delta_{n-1})$$

Consequently, remaining coefficients are $c'_{\pm1}=\frac{1}{2}$. Then,

$$\cos x=\frac{1}{2}e^{-ix}+\frac{1}{2}e^{ix}=\frac{1}{2}(e^{ix}+e^{-ix})$$

(4)

$$c'_n=\frac{1}{2\pi}\int_{-\infty}^{\infty}\sin xe^{-inx}dx$$

$$=\frac{1}{2\pi}\frac{1}{2i}\int_{-\infty}^{\infty}(e^{ix}-e^{-ix})e^{-inx}dx$$

$$=\frac{1}{2\pi}\frac{1}{2i}\int_{-\infty}^{\infty}(e^{-(n-1)x}-e^{-i(n+1)x})dx$$

$$=\frac{1}{2i}(\delta_{n.1}-\delta_{n.-1})$$

Consequently, remaining coefficients are $c'_1=\frac{1}{2i}$, $c'_{-1}=-\frac{1}{2i}$. Then,

$$\sin x=\frac{1}{2i}e^{ix}-\frac{1}{2i}e^{-ix}=\frac{1}{2i}(e^{ix}-e^{-ix})$$

1.5 Fourier series for arbitrary interval

As the period T, in taking $2p$ instead of 2π, where p is arbitrary real number, the expansion series and its coefficients may be written

$$f(x)=\sum_{n=-\infty}^{\infty}c'_n e^{i\frac{n\pi\infty}{p}}$$

$$=c'_0+\sum_{n=1}^{\infty}(c'_n e^{i\frac{n\pi\infty}{p}}+c_{-n}e^{-i\frac{n\pi\infty}{p}}) \quad (37)$$

where

$$c'_n=\frac{1}{2p}\int_{-p}^{p}f(x)e^{-i\frac{n\pi\infty}{p}}dx,\quad \overline{c'_n}=c'_{-n} \quad (38)$$

Corresponding real-type Fourier series may be written

$$f(x)=\frac{a'_0}{2}+\sum_{n=1}^{\infty}\left\{a'_n\cos\frac{n\pi x}{p}+b'_n\sin\frac{n\pi x}{p}\right\} \quad (39)$$

where

$$a'_n=\frac{1}{p}\int_{-p}^{p}f(x)\cos\frac{n\pi x}{p}dx \quad (40)$$

$$b'_n=\frac{1}{p}\int_{-p}^{p}f(x)\sin\frac{n\pi x}{p}dx \quad (41)$$

Example 3. Find Fourier series for each of the following periodic functions ($T=2p$).

(1) $f(x)=x(-p\le x\le p)$ (2) $f(x)=x(0\le x\le 2p)$
(3) $f(x)=x^2(-p\le x\le p)$ (4) $f(x)=x^2(0\le x\le 2p)$

Solution.

(1) $a'_n=0$ since $f(x)$ is odd.

$$b'_n=\frac{1}{p}\int_{-p}^{p}x\sin\frac{n\pi x}{p}dx=(-1)^{n-1}\frac{2p}{n\pi} \quad (42)$$

Therefore, the required series is

$$x=2\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2p}{n\pi}\sin\frac{n\pi x}{p} \quad (43)$$

(2) $f(x)=x$ is neither even nor odd in the range $[0,2p]$.

$$a_0'=\frac{1}{2p}\int_0^{2p}x\cdot 1dx=p$$

$$a_n'=\frac{1}{p}\int_0^{2p}x\cos\frac{n\pi x}{p}dx=0$$

$$b_n'=\frac{1}{p}\int_0^{2p}x\sin\frac{n\pi x}{p}dx=-\frac{2p}{n\pi}$$

$$x=p-\sum_{n=1}^{\infty}\frac{2p}{n\pi}\sin\frac{n\pi x}{p} \qquad (44)$$

(3) $b_n=0$ since $f(x)$ is even.

$$a_0'=\frac{1}{2p}\int_0^{2p}x\cdot 1dx=\frac{p^2}{3}$$

$$a_n'=\frac{1}{p}\int_0^{2p}x\cos\frac{n\pi x}{p}dx=(-1)^n\frac{4p}{n^2\pi}$$

$$b_n=0$$

$$x^2=\frac{p^2}{3}+4\sum_{n=1}^{\infty}(-1)^n\frac{p}{n^2\pi}\cos\frac{n\pi x}{p} \qquad (45)$$

(4) $f(x)=x^2$ is neither even nor odd in the range $[0,2p]$.

$$a_0'=\frac{1}{2p}\int_0^{2p}x^2\cdot 1dx=\frac{4p^2}{3}$$

$$a_n'=\frac{1}{p}\int_0^{2p}x\cos\frac{n\pi x}{p}dx=\frac{4p^2}{(n\pi)^2}$$

$$b_n'=\frac{1}{p}\int_0^{2p}x^2\sin\frac{n\pi x}{p}dx=-\frac{4p}{n\pi}$$

$$x^2=\frac{4p^2}{3}$$

$$+4\sum_{n=1}^{\infty}\left(\frac{p^2}{(n\pi)^2}\cos\frac{n\pi x}{p}-\frac{p}{n\pi}\sin\frac{n\pi x}{p}\right)$$

[Exercise] Expand the following periodic functions to Fourier series :

(1) $f(x)=|x|\ (-\pi\leq x\leq\pi)$

(2) $f(x)=\begin{cases}-1 & (-1\leq x<0)\\ 1 & (0\leq x\leq 1)\end{cases}$

(3) $f(x)=\begin{cases}-1-x & (-1\leq x<-\frac{1}{2})\\ x & (-\frac{1}{2}\leq x<\frac{1}{2})\\ 1-x & (\frac{1}{2}\leq x\leq 1)\end{cases}$

(4) $f(x)=\begin{cases}x(1+x) & (-1\leq x<0)\\ x(1-x) & (0\leq x\leq 1)\end{cases}$

Solution.

(1) The function $f(x)$ is even and has a period of $2l=2\pi$, $l=\pi$.

$$\sigma_0=\frac{2}{\pi}\int_0^{\pi}xdx=\pi$$

$$\sigma_n=\frac{2}{\pi}\int_0^{\pi}x\cos nxdx=\frac{2}{\pi}\times(I)$$

$$(I)=-\frac{1}{n}[x\sin nx]_0^{\pi}-\frac{1}{n}\int_0^{\pi}\sin nxdx$$

$$=\frac{1}{n^2}[\cos nx]_0^{\pi}=\frac{1}{n^2}(\cos n\pi-1)$$

$$=\frac{1}{n^2}((-1)^n-1)=\begin{cases}\frac{-2}{n^2} & (n=odd)\\ 0 & (n=even)\end{cases}$$

$$\sigma_0=\frac{2}{\pi}\frac{-2}{n^2}=\frac{-4}{\pi n^2}$$

$$f(x)=|x|=\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos(2n-1)^x$$

(2) The function $f(x)$ is odd and and has a period of $2l=2$, $l=1$.

$$b_n=\frac{2}{1}\int_0^1\sin nxdx$$

$$=-\frac{1}{n\pi}[\cos n\pi x]_0^1$$

$$=-\frac{1}{n\pi}(\cos n\pi x-1)=\frac{1}{n\pi}(1-(-1)^n)$$

$$=\begin{cases}\frac{2}{n\pi} & (n=odd)\\ 0 & (n=even)\end{cases}$$

$$f(x)=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin(2n-1)^{\pi x}$$

(3) The function $f(x)$ is odd and and has a period of $2l=2$, $l=1$.

$$b_n=\frac{2}{1}\left[\int_0^{\frac{1}{2}}x\sin n\pi xdx+\int_{\frac{1}{2}}^1 x\sin x\pi xdx\right]$$

$$=2\{(I)+(II)\}$$

$$(I)=-\frac{1}{n\pi}[x\cos n\pi x]_0^{\frac{1}{2}}+\frac{1}{n\pi}\int_{\frac{1}{2}}^1\cos n\pi xdx$$

$$=\frac{1}{(n\pi)^2}\sin\frac{n\pi}{2}$$

$$(II) = -\frac{1}{n\pi}[(1-x)\cos n\pi x]_0^{\frac{1}{2}} - \frac{1}{n\pi}\int_{\frac{1}{2}}^{1}\cos n\pi x dx$$

$$= \frac{1}{(n\pi)^2}\sin\frac{n\pi}{2}$$

$$2\{(I)+(II)\} = \frac{4}{(n\pi)^2}\sin\frac{n\pi}{2}$$

$$f(x) = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin n\pi x$$

(4) The function $f(x)$ is odd and and has a period of $2l=2$, $l=1$.

$$b_n = \int_{-1}^{1} f(x)\sin n\pi x dx = 2\int_0^1 f(x)\sin n\pi x dx$$

$$= 2\left(\int_0^1 x\sin n\pi x dx - \int_0^1 x^2\sin n\pi x dx\right)$$

$$= 2\{(I)-(II)\}$$

$$(I) = \frac{-1}{n\pi}[x\cos n\pi x]_0^1 + \frac{1}{n\pi}\int_0^1\cos n\pi x dx$$

$$= \frac{-1}{n\pi}(\cos n\pi) + \left(\frac{1}{n\pi}\right)^2[\sin n\pi x]_0^1$$

$$= \frac{-1}{n\pi}(-1)^n$$

$$(II) = \frac{-1}{n\pi}[x^2\sin n\pi x]_0^1 + \frac{2}{n\pi}\int_0^1 x\cos n\pi x dx$$

$$= \frac{-1}{n\pi}(-1)^n + \frac{2}{(n\pi)^3}\{(-1)^n-1\}$$

$$b_n = \frac{4}{(n\pi)^3}\{1-(-1)^n\} = \begin{cases}\frac{8}{(n\pi)^3} & (n=odd)\\ 0 & (n=even)\end{cases}$$

$$f(x) = \frac{4}{(\pi)^3}\sum_{n=1}^{\infty}\frac{1}{n^3}\{1-(-1)^n\}\sin n\pi x$$

$$= \frac{8}{(\pi)^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\sin(2n-1)\pi x$$

1.6 Application to Partial Differential Equations

Laplace equation

The Laplace equation or Potential equation on two dimension, say x and y, is

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0 \qquad (46)$$

We will investigate a solution for $u(x, y)$ under the following boundary conditions which mean that boundary values lie on the boundary of rectangle;

$$u(0, y) = u(a, y) = 0 \qquad (47)$$

$$u(x, 0) = g(x), \quad u(x, b) = 0 \qquad (48)$$

where a function $g(x)$ may be expanded into Fourier Series.

First of all, we assume that a form of solution has

$$u(x, y) = X(x)Y(y) \qquad (49)$$

which is called the solution of separated variables. Substitutions (49) into eq. (46) gives

$$\frac{X''(x)}{X(x)} = \frac{Y''(y)}{Y(y)}$$

The expression on the left involves functions depending only x while the expression on the left involves functions depending only y. Hence both expressions must be equal to a constant denoted $-\lambda$, where the sign of ninus $-$ is arbitrary. The process of the separation is straightforward and simple, and results in

$$X''(x) + \lambda X(x) = 0 \qquad (50)$$

$$Y''(y) - \lambda Y(y) = 0 \qquad (51)$$

The equation, $X''(x)+kX(x)=0$ (k=*constant*), can be solved at once; its general solution may be written

$$X(x) = c_1 e^{\sqrt{kx}} + c_2 e^{-\sqrt{kx}} \qquad (k>0)$$

$$= c_1 + c_2 x \qquad (k=0)$$

$$= c_1\cos\sqrt{kx} + c_2\sin\sqrt{kx} \qquad (k<0)$$

where c_1 and c_2 are constants.

With the help of the above solutions, we can write solutions of eqs. (50) and (51):

$$X(x) = A \text{ xos } \sqrt{\lambda x} + B\sin\sqrt{\lambda x}$$

Here the boundary conditions, $u(0, y) = u(a, y) = 0$, require that $A=0$ and $\sqrt{\lambda} = \frac{n\pi}{a}$. Thus, we rewrite

$$X_n(x) = B_n\sin\frac{n\pi}{a}x \quad (n=1,2,\cdots) \qquad (52)$$

In like manner, the solution $Y(y)$ is

$$Y(y) = Ce^{\frac{n\pi}{a}y} + De^{-\frac{n\pi}{a}y}$$

Considering the boundary condition $u(x, b)=X(x)Y(b)=0$, we have

$$Y_n(y)=C_n\sinh\frac{n\pi}{a}(y-b) \tag{53}$$

Finally we have to find a solution for $u(x, y)$ in the form

$$u(x, y) = \sum_{n=0}^{\infty} X_n(x)Y_n(y)$$
$$= \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{a}x\sinh\frac{n\pi}{a}(y-b) \tag{54}$$

where renewed coefficient b_n can be determined in terms of the boundary condition $u(x, 0)=g(x)$,

$$u(x, 0)=g(x) = \sum_{n=1}^{\infty} b_n\sinh\left(-\frac{n\pi}{a}b\right)\sin\frac{n\pi}{a}x$$
$$= \sum_{n=1}^{\infty} b_n' \sin\frac{n\pi}{a}x$$

The above expression is the Fourier sine expansion of the function $g(x)$, hence we obtain

$$b_n'=\frac{2}{a}\int_0^a g(x)\sin\frac{n\pi}{a}xdx$$

The solving procedure of partial differential equation, which is called the method of the separation of variabls, may schematically be devided into three parts;

STEP I. A solution of separated variables is assumed:

$$u(x, y)=X(x)Y(y)$$

STEP II. Solve two ordinary differetial equations separated of $X(x)$ and $Y(y)$ in terms of the boundary condition.

STEP Ⅲ. Set a general form of solution by a linear conbination of each separated solutions, in other words, principle of superposition of each separated solutions and determine expansion coefficients in terms of another boundary condition.

[Example Ⅰ] Solve the Laplace equation (46) subject to the following boundary conditions,

(1) $u(0, y)=0$
$u(x, 0)=\sin\pi x$, $u(x, 1)=0$

(2) $u_x(0, y)=u_x(1, y)=0$
$u_y(0, y)=\cos\pi x$, $u_y(x, 1)=0$

where $u_x(0, y)=\left.\frac{\partial u(x, y)}{\delta x}\right|_{x=0}$, etc..

Solution.

(1) STEP Ⅰ.

$$u(x, y)=X(x)Y(y)$$

$$X''(x)+\lambda X(x)=0$$
$$Y''(y)+\lambda Y(y)=0$$

STEP Ⅱ.

$$X_n(x)=B_n\sin n\pi x(n=1,2,\ldots)$$
$$Y_n(y)=C_n\sin hn\pi(y-1)$$

STEP Ⅲ.

$$u(x, y) = \sum_{n=0}^{\infty} X_n(x)Y_n(y)$$
$$= \sum_{n=1}^{\infty} b_n\sin n\pi x\sinh n\pi(y-1)$$

where renewed coefficient b_n can be determined in terms of the boundary condition $u(x, 0)=\sin\pi x$,

$$u(x, 0)=\sin\pi x = \sum_{n=1}^{\infty} b_n\sinh(-n\pi)\sin n\pi x$$
$$= \sum_{n=1}^{\infty} b_n' \sin n\pi x$$

A remaining term is $n=1$ only. Then,

$$b_1'=1 \text{ or } b_1=\frac{1}{\sinh(-\pi)}=\frac{-1}{\sinh(\pi)}$$

where $\sinh(-\pi)=-\sinh(\pi)$. A desired solution is

$$u(x, y)=\frac{-1}{\sinh(\pi)}\sin\pi x\sinh\pi(y-1)$$

(2) STEP Ⅰ.

$$u(x, y)=X(x)Y(y)$$

$$X''(x)+\lambda X(x)=0$$
$$Y''(y)+\lambda Y(y)=0$$

STEP Ⅱ.

$$X_n(x) = A_n\cos n\pi x\ (n=1,2,\ldots)$$
$$Y_n(y) = C_n\cosh n\pi(y-1)$$

STEP III.

$$u(x, y) = \sum_{n=0}^{\infty} X_n(x) Y_n(y)$$

$$= \sum_{n=0}^{\infty} a_n \cos n\pi x \cosh n\pi(y-1)$$

where renewed coefficient a_n can be determined in terms of the boundary condition $u_y(x, 0) = \cos \pi x$,

$$u_y(x, 0) = \cos \pi x = \sum_{n=1}^{\infty} a_n n\pi \cosh(-n\pi) \cos n\pi x$$

$$= \sum_{n=1}^{\infty} a_n' \text{co } n\pi x$$

A remaining term is $n = 1$. Then,

$$a_1' = 1 \quad \text{or} \quad a_1 = \frac{1}{\pi \cosh(-\pi)} = \frac{1}{\pi \cosh(\pi)}$$

where $\cosh(-\pi) = \cosh(\pi)$.

A desired solution is

$$u(x, y) = \frac{1}{\pi \cosh(\pi)} \cos n\pi x \cosh \pi(y-1)$$

[**Example 2**] Laplace equation with polar coordinates, γ and θ ($x = \gamma \cos\theta$, $y = \gamma \sin\theta$), is

$$\frac{\partial^2 u(\gamma, \theta)}{\partial \gamma^2} + \frac{1}{\gamma} \frac{\partial u(\gamma, \theta)}{\partial \gamma} + \frac{1}{\gamma^2} \frac{\partial^2 u(\gamma, \theta)}{\partial \theta^2} = 0 \quad (55)$$

Solve the equation under the following boundary conditions which mean that boundary values lie on circle (a is radius);

$$u(a, \theta) = g(\theta), \quad g(-\pi) = g(\pi) \quad (56)$$

where a function $g(\theta)$ may be expanded into Fourier series.

Solution. Putting $u(\gamma, \theta) = R(\gamma)\Theta(\theta)$ into eq. (55), we obtain two ordinary differential equations separated:

$$\gamma^2 R''(\gamma) + \gamma R'(\gamma) + \lambda R(\gamma) = 0 \quad (57)$$

$$\Theta''(\theta) + \lambda \Theta(\theta) = 0 \quad (58)$$

The solution of $\Theta''(\theta) + \lambda\Theta(\theta) = 0$ is

$$\Theta(\theta) = A \cos\sqrt{\lambda\theta} + B \sin\sqrt{\lambda\theta}$$

where a constant λ has to be n (integer) subject to the boundary codition; $\lambda = n$

Let $R(\gamma) = \gamma^u$ as a solution of the equation, $\gamma^2 R''(\gamma) + \gamma R'(\gamma) + \lambda R(\gamma) = 0$. The exponent μ is determined by

$$u(u-1)u - n^2 = 0 \quad \text{or} \quad u = \pm n$$

However, the solution of γ^{-n} is excluded due to irregularity at $\gamma = 0$. Hence we write a solution

$$u(\gamma, \theta) = \sum_{n=0}^{\infty} \gamma^n (A_n \cos n\theta + B_n \sin n\theta)$$

where renewed coefficients A_n and B_n are constants to be determined in terms in terms of the periodic boundary condition:

$$u(a, \theta) = \sum_{n=0}^{\infty} a^n (A_n \cos n\theta + B_n \sin n\theta)$$

$$= g(\theta)$$

The expression of Fourier expansion of $g(\theta)$ has

$$g(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) \quad (59)$$

where

$$a_n = \frac{1}{\pi} \int_{-\pi}^{n} g(t) \cos nt dt$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(t) \sin nt dt$$

In comparision between eqs. (59) and (59), the coefficients A_n and B_n are

$$A_0 = \frac{a_0}{2}, \quad B_0 = 0, \quad a^n A_n = a^n B_n = b_n$$

or

$$A_n = \frac{1}{a^n \pi} \int_{-\pi}^{\pi} g(t) \cos nt dt$$

$$B_n = \frac{1}{a^n \pi} \int_{-\pi}^{\pi} g(t) \sin nt dt$$

Then we have

$$u(\gamma, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) dt + \frac{1}{\pi} \sum_{n=1}^{\infty} \left(\frac{\gamma}{a}\right)^n$$

$$\times \int_{-\pi}^{\pi} g(t) (\cos nt \cos n\theta + \sin nt \sin n\theta) dt$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) dt + \frac{1}{\pi} \sum_{n=1}^{\infty} \left(\frac{\gamma}{a}\right)^n$$

$$\times \int_{-\pi}^{\pi} g(t) \cos n(t-\theta) dt$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) \left[1 + 2 \sum_{n=1}^{\infty} \left(\frac{\gamma}{a}\right)^n \cos n(t-\theta)\right] dt$$

where the bracket [⋯] on the right-hand side the last equality reduces to

$$[\cdots] = 1+\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\left(e^{in(t-\theta)}+e^{-in(t-\theta)}\right)$$

$$= 1+\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\left[\frac{e^{i(t-\theta)}}{1-e^{i(t-\theta)}}+\frac{e^{-i(t-\theta)}}{1-e^{-i(t-\theta)}}\right]$$

$$= 1+\frac{2ar\cos(t-\theta)+2r^2}{a^2-2ar\cos(t-\theta)+r^2}$$

$$= \frac{a^2-r^2}{a^2-2ar\cos(t-\theta)+r^2}$$

Hence we rewrite

$$u(r,\theta)=\frac{1}{2\pi}\int_{-\pi}^{\pi}g(t)\frac{a^2-r^2}{a^2-2ar\cos(t-\theta)+r^2}dt \quad (60)$$

Wave equation

The standard form of Wave equation in the two variables, say x and t, is

$$c^2\frac{\partial^2 u(x,t)}{\partial x^2}=\frac{\partial^2 u(x,t)}{\partial t^2} \quad (61)$$

where c is constant. We will investigate a solution for $u(x,t)$ under the following boundary and initial conditions;

$$u(0,t)=u(a,t)=0 \quad (62)$$

$$u(x,0)=g(x),\ u_t(x,0)=G(x) \quad (63)$$

where functions $g(x)$ and $G(x)$ may be expanded into Fourier series. The equation (61) can be solved by the separation of variables in the previous section. First. let

$$u(x,t)=X(x)T(t) \quad (64)$$

Substitution (64) into eq. (61) gives

$$\frac{X''(x)}{X(x)}=\frac{1}{c^2}\frac{T''(t)}{T(t)}=-\lambda$$

Then, it follows that

$$X''(x)+\lambda X(x)=0 \quad (65)$$

$$T''(t)(c^2\lambda)T(t)=0 \quad (66)$$

We can write solutions of eqs. (65) and (66):

$$X_n(x)=B_n\sin\frac{n\pi}{a}x \quad (n=1,2,\cdots)$$

$$T_n(t)=C_n\cos\frac{n\pi}{a}t+D_n\sin\frac{n\pi}{a}t$$

Finally we look for a solution for $u(x,y)$ in the form

$$u(x,t)=\sum_{n=0}^{\infty}X_n(x)T_n(t)$$

$$=\sum_{n=1}^{\infty}\sin\frac{n\pi}{a}x\left(a_n\cos\frac{n\pi}{a}ct+b_n\sin\frac{n\pi}{a}ct\right) \quad (67)$$

where renewed coefficients a_n *and* b_n can be determined in terms of the initial conditions, $u(x,0)=g(x)$ and $u_t(x,0)=G(x)$, respectively,

$$u(x,0)=g(x)=\sum_{n=1}^{\infty}a_n\sin\frac{n\pi}{a}x \quad (68)$$

and then

$$a_n=\frac{2}{a}\int_0^a g(t)\sin\frac{n\pi}{a}t\,dt$$

On the other hand,

$$u_t(x,0)=G(x)=\sum_{n=1}^{\infty}b_n\frac{n\pi}{a}\sin\frac{n\pi}{a}x \quad (69)$$

and then

$$\frac{n\pi c}{a}b_n=\frac{2}{a}\int_0^a G(t)\sin\frac{n\pi}{a}t\,dt$$

To demonstrate the propagation of wave, we can rewrite the expression (67), obtaining in terms of the factor formulas in trigonometric fuction,

$$a_n\sin\frac{n\pi}{a}x\cos\frac{n\pi}{a}ct$$

$$=\frac{a_n}{2}\left\{\sin\frac{n\pi}{a}(x+ct)+\sin\frac{n\pi}{a}(x-ct)\right\} \quad (70)$$

and

$$b_n\sin\frac{n\pi}{a}x\sin\frac{n\pi}{a}ct$$

$$=\frac{-b_n}{2}\left\{\cos\frac{n\pi}{a}(x+ct)+\cos\frac{n\pi}{a}(x-ct)\right\} \quad (71)$$

From these, we may write as a general form of solution

$$u(x,t)=af_1(x+ct)+bf_2(x-ct) \quad (72)$$

which is called d'Alebert's solution of the equation (61).

The wave equation (61) can be changed to the equations $\frac{\partial^2 u(\eta,t)}{\partial\eta\partial\xi}=0$ by the linear transformation of $\eta=x+ct$ and $t=x-ct$. The repeated integral with respect to η and t for $\frac{\partial^2 u(\eta,t)}{\partial\eta\partial\xi}=0$ gives the solution (72).

[Example I] Solve the wave equation subject to the following boundary and initial conditions,

(1) $u(0,t)=u(1,t)=0$
$u(x,0)=\sin\pi x,\ u_t(x,0)=0$

(2) $u_x(0,t)=u_x(1,t)=0$
$u(x,0)=0,\ u_t(x,0)=\cos\pi x$

Solution.

(1) STEP I.

$$u(x,t)=X(x)T(t)$$

$$\frac{X''(x)}{X(x)}=\frac{1}{c^2}\frac{T''(t)}{T(t)}=-\lambda$$

STEP II.

$$X''(x)+\lambda X(x)=0$$
$$T''(t)+(c^2\lambda)T(t)=0$$

$$X_n(x)=B_n\sin n\pi x\quad(n=1,2,\cdots)$$
$$T_n(t)=C_n\cos n\pi ct+D_n\sin n\pi ct$$

STEP III.

$$u(x,t)=\sum_{n=0}^{\infty}X_n(x)T_n(t)$$
$$=\sum_{n=0}^{\infty}\sin n\pi x(a_n\cos n\pi ct+b_n\sin n\pi ct)$$

$$u(x,0)=\sin\pi x=\sum_{n=1}^{\infty}a_n\sin n\pi x$$

$$n=1\ \Rightarrow\ a_1=1$$

On the other hand,

$$u_t(x,0)=0\ \Rightarrow\ b_n=0$$

Hence,

$$u(x,t)=\sin\pi x\cos\pi ct$$

(2) STEP II.

$$X''(x)+\lambda X(x)=0$$
$$T''(t)+(c^2\lambda)T(t)=0$$

$$X_n(x)=A_n\cos n\pi x\quad(n=1,2,\cdots)$$
$$T_n(t)=C_n\cos n\pi ct+D_n\sin n\pi ct$$

STEP III.

$$u(x,t)=\sum_{n=0}^{\infty}X_n(x)T_n(t)$$
$$=\sum_{n=0}^{\infty}\cos n\pi x(a_n\cos n\pi ct+b_n\sin n\pi ct)$$

$$u(x,0)=0\ \Rightarrow\ a_n=0$$

On the other hand,

$$u_t(x,0)=\cos\pi x=\sum_{n=1}^{\infty}b_n(n\pi c)\cos n\pi x)$$

$$\Rightarrow\ n=1,\ b_1=\frac{1}{\pi c}$$

Hence,

$$u(x,t)=\frac{1}{\pi c}\cos\pi x\cos\pi ct$$

Heat equation

The standard form of (one dimensional) Heat equation in the two variables, say x and t, is

$$c^2\frac{\partial^2 u(x,t)}{\partial x^2}=\frac{\partial u(x,t)}{\partial t}\tag{73}$$

where c is a constant. We will investigate a solution for $u(x,t)$ under the following boundary and initial conditions;

$$u(0,t)=u(a,t)=0\tag{74}$$
$$u(x,0)=g(x)\tag{75}$$

where a function $g(x)$ may be expanded into Fourier series.

First, let

$$u(x,t)=X(x)T(t)\tag{76}$$

Substitution (76) into eq. (73) gives

$$\frac{X''(x)}{X(x)}=\frac{1}{c^2}\frac{T'(t)}{T(t)}=-\lambda$$

Then, it follows that

$$X''(x) + \lambda X(x) = 0 \quad (77)$$

$$T'(t) + (c^2 \lambda) T(t) = 0 \quad (78)$$

We can write solutions of eqs. (77) and (78):

$$X_n(x) = B_n \sin \frac{n\pi}{a} x \quad (n = 1, 2, \cdots)$$

$$T_n(t) = C_n e^{-(\frac{n\pi c}{a})^2 t}$$

Finally we look for a solution for $u(x, y)$ in the form

$$u(x, t) = \sum_{n=0}^{\infty} X_n(x) T_n(t)$$

$$= \sum_{n=1}^{\infty} b_n e^{-(\frac{n\pi c}{a})^2 t} \sin \frac{n\pi}{a} x \quad (79)$$

where renewed coefficients b_n can be determined in terms of the initial condition, $u(x, 0) = g(x)$,

$$u(x, 0) = g(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{a} x \quad (80)$$

and then

$$b_n = \frac{2}{a} \int_0^a g(t) \sin \frac{n\pi}{a} t dt$$

[Example Ⅰ] Solve the Heat equation subject to the following boundary and initial conditions,

(1) $u(0, t) = u(2, t) = 0$

$u(x, 0) = \sin \pi x$

(2) $u(0, t) = u(1, t) = 0$

$$u(x, 0) \begin{cases} x(1+x) & (-1 \le x < 0) \\ x(1-x) & (0 \le x < 1) \end{cases}$$

Solution

(1) STEP Ⅰ.

$$u(x, t) = X(x) T(t)$$

$$X''(x) + \lambda X(x) = 0$$

$$T''(t) - \lambda T(t) = 0$$

STEP Ⅱ.

$$X(x) = A \cos \sqrt{\lambda x} + B \sin \sqrt{\lambda x}$$

$$X(0) = A = 0, \quad X(2) = B \sin 2\sqrt{\lambda} = 0$$

$$\sqrt{\lambda} = \frac{n\pi}{2} \text{ or } \lambda = \left(\frac{n\pi}{2}\right)^2 \quad (n = 1, 2, \cdots)$$

$$T(t) = Ce^{-\lambda t} = Ce^{-(\frac{n\pi}{2})^2 t}$$

STEP Ⅲ.

$$u(x, t) = \sum_{n=0}^{\infty} X_n(x) T_n(t)$$

$$= \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{2} e^{-(\frac{n\pi}{2})^2 t}$$

$$u(x, 0) = \sin \pi x = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{2} x$$

$$b_2 = 1$$

$$u(x, y) = \sin \pi x e^{-\pi^2 t}$$

(2) STEP Ⅱ.

$$X(x) = A \cos \sqrt{\lambda x} + B \sin \sqrt{\lambda x}$$

$$X(0) = A = 0, \quad X(1) = B \sin \sqrt{\lambda} = 0$$

$$\sqrt{\lambda} = n\pi \text{ or } \lambda = (n\pi)^2 \quad (n = 12, \cdots)$$

$$T(t) = Ce^{-\lambda t} = Ce^{-(n\pi)^2 t}$$

STEP Ⅲ.

$$u(x, 2t) = \sum_{n=0}^{\infty} X_n(x) T_n(t)$$

$$= \sum_{n=1}^{\infty} b_n \sin(n\pi x) e^{-(n\pi)^2 t}$$

$$u(x, 0) = \begin{cases} x(1+x) \\ x(1-x) \end{cases} = \sum_{n=1}^{\infty} b_n \sin n\pi x$$

where

$$b_n = \frac{4}{(n\pi)^3} \{1 - (-1)^n\} = \begin{cases} \frac{8}{(n\pi)^3} & (n = odd) \\ 0 & (n = even) \end{cases}$$

by **Exercise** (4) in the previous section. The solution is written

$$u(x, t) = \frac{4}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^3} \{1 - (-1)^n\} \sin(n\pi x) e^{-(n\pi)^2 t}$$