Stochastic integral for L^1

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(Received 30 November, 1998)

Abstract. Consider on a probability space (Ω, a, P). Let $v=(v_t)$ $(v_t \in L^1)$ be a stochastic process and $f=(f_t)$ a semi-martingale then a Stieltjes type stochastic integral $\int_0^1 v_t \, df_t$ converges a. e.

Theorem 1. ([3]) Semi-martingale $f=(f_t)$ is of a. e. bounded variation.

Theorem 2. ([1]) For a stochastic process $v=(v_t)$ and for an increasing process $A=(A_t)$ there exists an a. e. continuous process $v^{(n)}=(v_t^{(n)})$ such that

$$\lim_{n\to\infty} \int_0^1 \left| v_t - v_t^{(n)} \right| dA_t = 0 \quad \text{a. e.} \quad (\text{Here } v_t^{(\infty)} = v_t \text{ a. e.}).$$

By Theorem 1 f is a. e. represented as the difference of two increasing processes $\tilde{f}$ and $\tilde{g}$.

Theorem 3. (**Main Theorem**). Let $v=(v_t)$ $(v_t \in L^1)$ and let $f=(f_t)$ a semi-martingale. Then the stochastic integral $\int_0^1 v_t \, df_t$ exists a. e. and it is a. e. limit of Stieltjes type integral $\int_0^1 v_t^{(n)} \, df_t$.

Proof. Let $A_t = \tilde{f}_t$. For almost all ω, $\int_0^1 v_t^{(n)}(\omega) \, d\tilde{f}_t(\omega)$ exsists as a Stieltjes integral.

$$\begin{aligned} &\left| \int_0^1 v_t^{(m)}(\omega) \, d\tilde{f}_t(\omega) - \int_0^1 v_t^{(n)}(\omega) \, d\tilde{f}_t(\omega) \right| \\ &\leqslant \int_0^1 \left| v_t^{(m)}(\omega) - v_t^{(n)}(\omega) \right| d\tilde{f}_t(\omega) \ \text{(This is a Stieltjes integral)} \\ &\leqslant \int_0^1 \left| v_t^{(m)}(\omega) - v_t(\omega) \right| d\tilde{f}_t(\omega) + \int_0^1 \left| v_t(\omega) - v_t^{(n)}(\omega) \right| d\tilde{f}_t(\omega) \\ &\longrightarrow 0 \quad \text{as } m, n \longrightarrow \infty \quad \text{by Theorem 2.} \end{aligned}$$

So $\left\{ \int_0^1 v_t^{(n)}(\omega) \, d\tilde{f}_t(\omega) \right\}_{n \geqslant 1}$ is a Cauchy sequence in R or C.

Since the state space R or C is complete then $\int_0^1 v_t^{(\infty)} \, d\tilde{f}_t \left(= \int_0^1 v_t \, d\tilde{f}_t \text{ a. e.}\right)$ converges in the sense of a. e. convergence.

Similarly $\int_0^1 v_t \, d\tilde{g}_t$ exists in the sense of a. e. convergence.

Therefore the Stieltjes type stochastic integral $\int_0^1 v_t \, df_t$ converges a.e. (q. e. d.)

Remark. This shows that the stochastic calculus is obtained by the similar way for ordinary differential calculus.

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