Guide to Applied Mathematics for Foreign Students III

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Abstract

This article is self-study for foreign students. The content is a part of lectures, particularly in Fourier transformation and its application to differential equations.

1 Fourier Transformation

1.1 Definition of Fourier Transformation

We shall define Fourier transformation as the extension of Fourier series. Complex Fourier series is written

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi x}{p}}$$

where p is the period and the coefficient c_n is given

$$c_n = \frac{1}{2p}\int_{-p}^{p} f(y)e^{-i\frac{n\pi y}{p}}dy$$

Now, let $\dfrac{n\pi}{p} = t_n$ and then,

$$t_{n+1} - t_n = \frac{\pi}{p} = \Delta t$$

Then, it follows that

$$f(x) = \sum_{n=-\infty}^{\infty} \left\{ \frac{\Delta t}{2\pi}\int_{-p}^{p} f(y)e^{-it_n y}dy \right\} e^{it_n x}$$

By taking p sufficiently large, Δt goes to zero as n is fixed. Changing the discrete variable t_n to the continuous t as $n \to \pm\infty$, that is, on replacing summation to integral in the above equation, then $f(x)$ becomes

$$f(x) = \int_{-\infty}^{\infty} \left\{ \frac{1}{2\pi}\int_{-\infty}^{\infty} f(y)e^{-ity}dy \right\}$$
$$\times\ e^{itx}dt \stackrel{\text{def}}{=} \frac{1}{2\pi}\int_{-\infty}^{\infty} F(t)e^{itx}dt \qquad (1)$$

where

$$F(t) = \mathcal{F}(f(x)) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} f(x)e^{-itx}dx \qquad (2)$$

which is called Fourier transform of $f(x)$. The equation (1) is thought to be Foureir series with the period of the infinity. The notation $\mathcal{F}$ denotes Fourier's linear integral operator, which means that

$$\mathcal{F}(af(x) + bg(x)) = a\mathcal{F}(f(x)) + b\mathcal{F}(g(x))$$
$$= aF(t) + bG(t) \qquad (3)$$

The equation (1) is called inverse Fourier transform or inversion formula:

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} \left\{ \frac{1}{2\pi}\int_{-\infty}^{\infty} f(y)e^{-ity}dy \right\} e^{itx}dt \\ &= \int_{-\infty}^{\infty} f(y) \left\{ \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{it(x-y)}dt \right\} dy \\ &= \int_{-\infty}^{\infty} f(y)\delta(x-y)dy \stackrel{\text{def}}{=} f(x) \qquad (4) \end{aligned}$$

where we have used the definition of δ function without verfication entirely. This is the origin of the name of inversion formula. Here δ function is expressed

$$\begin{aligned} \delta(x-y) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{it(x-y)}dt \\ &= \frac{1}{2\pi} < e^{ity}, e^{itx} > \qquad (5) \end{aligned}$$

The equation (5) is thought to be the orthonormal condition for continuous variables.

To show the inversion formula, it is available to describe symbolically as follows:

$$f(x) = \mathcal{F}^{-1}(F(t)) = \mathcal{F}^{-1}(\mathcal{F}(f(x)))$$
$$= \mathcal{F}^{-1}\mathcal{F}(f(x)) = f(x) \qquad (6)$$

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provided that

$$\mathcal{F}^{-1}\mathcal{F} = \mathcal{F}\mathcal{F}^{-1} = I \tag{7}$$

where the notation $\mathcal{F}^{-1}$ denotes Fourier's inverse linear integral operator and the notation I unit one.

The Fourier transformation between $f(x)$ and $F(t)$ is expressed symbolically

$$f(x) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} F(t) \quad \text{or} \quad f(x) \Longleftrightarrow F(t) \tag{8}$$

Hereafter we often call it the pair of Fourier transformation or the pair for simplicity.

In the definition of δ function (5), for simplicity, substitution of $x - y = x$ gives

$$\delta(x) \overset{\text{def}}{=} \frac{1}{2\pi}\int_{-\infty}^{\infty} 1 \cdot e^{itx}dt = \mathcal{F}^{-1}(1)$$

$$\mathcal{F}(\delta(x)) = \int_{-\infty}^{\infty} \delta(x)e^{-itx}dx = 1$$

These definitions mean that $\delta(x)$ is inverse Fourier transform of $F(t) = 1$ and $F(t) = 1$ is Fourier transform of $\delta(x)$, and then symbolically

$$\delta(x) \Longleftrightarrow 1 \tag{9}$$

Example 1. Find Fourier transforms for the following functions by use of the formula (5):

(a) $\cos ax$ (b) $\sin ax$

Solution.

$$(a)\ \mathcal{F}(\cos ax) = \mathcal{F}\left(\frac{1}{2}(e^{iax} + e^{-iax})\right)$$

$$= \frac{1}{2}\int_{-\infty}^{\infty}\left(e^{i(a-t)x} + e^{-i(a+t)x}\right)dx$$

$$= \frac{1}{2}\{\delta(a-t) + \delta(a+t)\}$$

In a similar manner,

$$(b)\ \mathcal{F}(\sin ax) = \mathcal{F}\left(\frac{1}{2i}(e^{iax} - e^{-iax})\right)$$

$$= \frac{1}{2i}\int_{-\infty}^{\infty}\left(e^{i(a-t)x} - e^{-i(a+t)x}\right)dx$$

$$= \frac{1}{2i}\{\delta(a-t) - \delta(a+t)\}$$

$$\cos ax \Longleftrightarrow \frac{1}{2}\{\delta(t+a) + \delta(t-a)\} \tag{10}$$

$$\sin ax \Longleftrightarrow \frac{i}{2}\{\delta(t+a) - \delta(t-a)\} \tag{11}$$

Here we give similar formulas for the functions, $\cosh ax$ and $\sinh ax$:

$$\cosh ax \Longleftrightarrow \frac{1}{2}\{\delta(t+ia) + \delta(t-ia)\} \tag{12}$$

$$\sinh ax \Longleftrightarrow \frac{1}{2}\{\delta(t+ia) - \delta(t-ia)\} \tag{13}$$

Different forms of Fourier transformation may be defined by the devision of the numerical factor $\frac{1}{2\pi}$

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} f(y)e^{-ity}dy\right\} \\ &\times\ e^{itx}dt = \int_{-\infty}^{\infty} \widehat{F'(t)}e^{itx}dx \end{aligned} \tag{14}$$

or

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(y)e^{-ity}dy\right\} \\ &\times\ e^{itx}dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \widehat{F''(t)}e^{itx}dx \end{aligned} \tag{15}$$

or

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} f(y)e^{-i2\pi ty}dy\right\} \\ &\times\ e^{i2\pi tx}dt = \int_{-\infty}^{\infty} \widehat{F'''(t)}e^{i2\pi tx}dx \end{aligned} \tag{16}$$

Now, construct the forms of real-type Fourier transformation. We return to the equation (1)

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y)e^{it(x-y)}dydt$$

The real part of exponential function in the above eqaution is

$$\begin{aligned} \Re e^{it(x-y)} &= \cos t(x-y) \\ &= \cos tx \cos ty + \sin tx \sin ty \end{aligned}$$

which is even with respect to t. Then, we rewrite the above equation

$$\begin{aligned} f(x) &= \frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(y)\cos t(x-y)dydt \\ &= \frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(y)(\cos ty \cos tx \\ &+\ \sin ty \sin tx)dydt \\ &= \frac{1}{\pi}\int_{0}^{\infty}\left\{\left(\int_{-\infty}^{\infty} f(y)\cos ty dy\right)\cos tx\right. \end{aligned}$$

$$+ \left(\int_{-\infty}^{\infty} f(y)\sin ty\, dy\right)\sin tx\Big\}\, dt$$
$$= \frac{1}{\pi}\int_0^{\infty}\{A(t)\cos tx + B(t)\sin tx\}\, dt \quad (17)$$

where

$$A(t) = \int_{-\infty}^{\infty} f(x)\cos tx dx$$
$$= 2\int_0^{\infty} f(x)\cos tx dx$$
$$B(t) = \int_{-\infty}^{\infty} f(x)\sin tx dx$$

The equation (17) is called Fourier's double integral formula.

In particular, when $f(x)$ is even or odd, Fourier transformation may be written

$$f(x) = \frac{1}{\pi}\int_0^{\infty} A(t)\cos tx dt \quad (18)$$

and

$$f(x) = \frac{1}{\pi}\int_0^{\infty} B(t)\sin tx dt \quad (19)$$

which is called inverse Fourier cosine or sine transform.

Integrating with respect to t in the equation (17) on the assumption that the order of integration may be reversed, we have

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(y)\cos t(x-y)dydt$$
$$= \lim_{T\to\infty}\frac{1}{\pi}\int_0^{\infty} f(y)\left\{\int_{-T}^{T}\cos t(x-y)dt\right\}dy$$
$$= \lim_{T\to\infty}\frac{1}{\pi}\int_0^{\infty} f(y)\frac{\sin T(x-y)}{x-y}dy$$

which is called Fourier's single integral formula.

We may modify the definition in the equation (5) of δ function as follows:

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{itx}dt$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}(\cos tx + i\sin tx)dt$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\cos tx dt = \frac{1}{\pi}\int_0^{\infty}\cos tx dt \quad (20)$$

$\delta(x)$ is said to be inverse Fourier cosine transform of $A(t) = 1$.

Finally consider Fourier transform of Heaviside function $H(x)$ (or Unit step function $U(x)$);

$$\mathcal{F}(H(x)) = \int_{-\infty}^{\infty} H(x)e^{-itx}dx$$
$$= \int_0^{\infty} e^{-itx}dx = \lim_{a\to 0}\int_0^{\infty} e^{-ax}e^{-itx}dx$$
$$= \lim_{a\to 0}\int_0^{\infty} e^{-(a+it)x}dx = \lim_{a\to 0}\frac{1}{it+a} \quad (21)$$

From this, it is wrong because of the neglect of discontinuous effect at $t = 0$ that Fourier transform of $H(x)$ becomes $\frac{1}{it}$. Then, we may rewrite the expression on the right in the last equation (21)

$$\lim_{a\to 0}\frac{1}{it+a} = \lim_{a\to 0}\frac{1}{i}\left(\frac{t}{t^2+a^2} + i\frac{a}{t^2+a^2}\right)$$

where

$$\frac{t}{t^2+a^2} = \begin{cases} \frac{1}{t} & (t \neq 0, a \to 0) \\ 0 & (t = 0, a \to 0) \end{cases}$$

and

$$\frac{a}{t^2+a^2} = \begin{cases} 0 & (t \neq 0, a \to 0) \\ \infty & (t = 0, a \to 0) \end{cases}$$

Then, the expression of $\lim_{a\to 0}\frac{1}{it+a}$ is given

$$\lim_{a\to 0}\frac{1}{it+a} = \frac{1}{it} + \pi\delta(0)$$
$$= -ip.v.\left(\frac{1}{t}\right) + \pi\delta(t) \quad (22)$$

where the notation $p.v.$ denotes Cauchy's principal value and δ function is expressed as another expression

$$\lim_{a\to 0}\frac{1}{\pi}\frac{a}{t^2+a^2} = \delta(t)$$

The following is the required expression:

$$H(x) \Longleftrightarrow \frac{1}{it} + \pi\delta(t)$$

Example 2. Find Fourier transforms for the functions,

(a) $H(x)\cos ax$ (b) $H(x)\sin ax$

Solution

(a) $\mathcal{F}(H(x)\cos ax) = \frac{1}{2}\mathcal{F}(H(x)(e^{iax} + e^{-iax}))$

$$= \frac{1}{2}\{\mathcal{F}(e^{iax}H(x) + e^{-iax}H(x))\}$$

$$
\begin{aligned}
&= \left\{ \frac{1}{i(t-a)} + \pi\delta(t-a) + \frac{1}{i(t+a)} \right. \\
&\qquad \left. + \pi\delta(t+a) \right\} \\
&= \frac{\pi}{2}\{\delta(t-a)+\delta(t+a)\} \\
&\qquad + \frac{1}{2i}\left\{ \frac{1}{t-a} + \frac{1}{t+a} \right\} \\
&= \frac{\pi}{2}\{\delta(t-a)+\delta(t+a)\} + \frac{1}{i}\frac{t}{t^2-a^2}
\end{aligned}
$$

(b) $\mathcal{F}(H(x)\sin ax) = \frac{i}{2}\mathcal{F}(H(x)(e^{iax}-e^{-iax}))$

$$
\begin{aligned}
&= \frac{i}{2}\{ \frac{1}{i(t-a)} + \pi\delta(t-a) - \frac{1}{i(t-a)} \\
&\qquad - \pi\delta(t-a)\} \\
&= \frac{i\pi}{2}\{\delta(t-a)-\delta(t+a)\} + \frac{a}{t^2-a^2}
\end{aligned}
$$

1.2 Examples of Fourier transformations

We shall find Fourier transforms for several functions of importance in the following sections and from them show inversion formula or Fourier's integral formula directly. Moreover we shall show that the results obtained may be used to evaluate intergals.

(1) Consider Fourier transform for the function defined by the equation

$$
f(x) = \begin{cases} 1 & (|x|<a) \\ 0 & (|x|>a) \end{cases}
$$

which is expressed by means of Heaviside function as

$$
f(x) = H(x+a) - H(x-a) = H(a-|x|)
$$

This function is a kind of rectangular functions or window functions and is symbolically denoted $\mathrm{rect}(ax)$.

From the definition (1) of Fourier transform, it follows that

$$
\begin{aligned}
F(t) &= \int_{-\infty}^{\infty} f(x)e^{-itx}dx = \int_{-a}^{a} 1\cdot e^{-itx}dx \\
&= -\frac{1}{it}(e^{-ita}-e^{ita}) = \frac{2\sin at}{t} \qquad (23)
\end{aligned}
$$

The function $\frac{\sin ax}{ax}$ is symbolically denoted $\mathrm{sinc}\, ax$;

$$
\frac{2\sin at}{t} \equiv 2a\ \mathrm{sinc}\, at
$$

From the definition (1), it follows that

$$
\begin{aligned}
f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(t)e^{itx}dt \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\sin at}{t}e^{itx}dt \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin at}{t}(\cos tx + i\sin tx)dt \\
&= \frac{2}{\pi}\int_{0}^{\infty} \frac{1}{t}\sin at\cos txdt \qquad (24) \\
&= \frac{1}{\pi}\int_{0}^{\infty} \frac{1}{t}\{\sin(a+x)+\sin(a-x)\}dt
\end{aligned}
$$

Using the following formula

$$
\int_0^\infty \frac{1}{t}\sin atdt = \frac{\pi}{2} sign\ a
$$

where a is an arbitrary number and the definition of the *sign* function is

$$
sign\ x = \begin{cases} -1 & (x<0) \\ 1 & (x>0) \end{cases} \qquad (25)
$$

or using by the complex form

$$
\int_{-\infty}^{\infty} \frac{1}{t}e^{iax}dt = i\pi sign\ a
$$

Then, we get

$$
f(x) = \begin{cases} \frac{1}{\pi}\left(\frac{\pi}{2}+\frac{\pi}{2}\right) = 1 & |x|<a \\ \frac{1}{\pi}\left(\frac{\pi}{2}-\frac{\pi}{2}\right) = 0 & |x|>a \end{cases}
$$

This is the required expression:

$$
H(a-|x|) \Longleftrightarrow \frac{2\sin at}{t}
$$

or

$$
\mathrm{rect}(ax) \Longleftrightarrow 2a\ \mathrm{sinc}(at)
$$

Using Fourier's integral formula, we may write from the equation (24)

$$
f(x) = \frac{2}{\pi}\int_0^\infty \frac{1}{t}\sin at\cos txdt = \begin{cases} 1\ (|x|<a) \\ 0\ (|x|>a) \end{cases}
$$

Then, we have

$$
\int_0^\infty \frac{1}{t}\sin at\cos txdt = \begin{cases} \frac{\pi}{2} & (|x|<a) \\ 0 & (|x|>a) \end{cases}
$$

which is called Dirichlet's discontinuous factor. Putting $a=1$ and $x=0$ in the expression of the integral, we obtain a well-known formula

$$\int_0^\infty \frac{\sin t}{t} dt = \frac{\pi}{2}$$

(2) Consider Fourier transform for the *sign* function;

$$sign\ x = \begin{cases} -1 & (x < 0) \\ 1 & (x > 0) \end{cases}$$

$$\begin{aligned} F(t) &= -\int_{-\infty}^0 1 \cdot e^{-itx} dx + \int_0^\infty 1 \cdot e^{-itx} dx \\ &= \frac{2}{it} - \frac{1}{it} \lim_{N\to\infty} (e^{-itN} - e^{itN}) \end{aligned}$$

where the $\lim_{N\to\infty}(\cdots)$ does not exist. Then, we introduce the damping factor (or the convergence factor) multiplying e^{-itx} as follows.

$$\begin{aligned} F(t) &= \lim_{a\to 0} \left(-\int_{-\infty}^0 e^{ax} e^{-itx} dx \right. \\ &+ \left. \int_0^\infty e^{-ax} e^{-itx} dx \right) \\ &= \lim_{a\to 0} \left(-\frac{1}{a - it} + \frac{1}{a + it} \right) \\ &= \lim_{a\to 0} \left(\frac{1}{it - a} + \frac{1}{it + a} \right) \\ &= \lim_{a\to 0} \left(\frac{1}{t} - i\pi\delta(t) + \frac{1}{t} + i\pi\delta(t) \right) \\ &= \frac{2}{it} \end{aligned}$$

Verify inversion formula directly

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^\infty \frac{2}{it} e^{itx} dt = \frac{1}{i\pi} \int_{-\infty}^\infty \frac{1}{t} e^{itx} dt$$

$$= \frac{1}{i\pi} i\pi sign\ x = sign\ x$$

This is the required expression:

$$sign\ x \Longleftrightarrow \frac{2}{it}$$

(3) Consider Fourier transform for the function defined by the equation

$$f(x) = \begin{cases} 1 - \frac{|x|}{a} & (|x| < a) \\ 0 & (|x| > a) \end{cases}$$

which is symbolically denoted $\Lambda\left(\frac{x}{a}\right)$

$$\begin{aligned} F(t) &= \int_{-\infty}^\infty \left(1 - \frac{|x|}{a}\right) e^{-itx} dx \\ &= \int_{-a}^0 \left(1 + \frac{x}{a}\right) e^{-itx} dx \\ &+ \int_0^a \left(1 - \frac{x}{a}\right) e^{-itx} dx \\ &= \frac{1}{at^2}(2 - 2\cos at) = \frac{1}{a} \left(\frac{\sin \frac{at}{2}}{\frac{t}{2}} \right)^2 \\ &= a \left(\frac{\sin \frac{at}{2}}{\frac{at}{2}} \right)^2 \end{aligned} \tag{26}$$

Verify inversion formula:

$$\begin{aligned} f(x) &= \frac{1}{2\pi} \int_{-\infty}^\infty a \left(\frac{\sin \frac{at}{2}}{\frac{at}{2}} \right)^2 e^{itx} dt \\ &= \frac{1}{2\pi} \frac{2}{a} \int_{-\infty}^\infty \frac{1}{t^2} (1 - \cos at) e^{itx} dt \\ &= \frac{1}{a\pi} \left\{ \int_{-\infty}^\infty \frac{1}{t^2} e^{itx} dt \right. \\ &- \left. \int_{-\infty}^\infty \frac{1}{t^2} \cos at e^{itx} dt \right\} \\ &= \frac{1}{a\pi} \left\{ \int_{-\infty}^\infty \frac{1}{t^2} e^{itx} dt \right. \\ &- \left. \frac{1}{2} \int_{-\infty}^\infty \frac{1}{t^2} (e^{i(a+x)t} + e^{i(x-a)t} dt \right\} \\ &= \frac{-1}{a\pi} \left\{ \pi|x| - \frac{\pi}{2}(|x + a| + |x - a|) \right\} \\ &= \frac{-1}{a} \left\{ |x| - \frac{1}{2}(|x + a| + |x - a|) \right\} \\ &= \begin{cases} 1 + \frac{x}{a} & (-a < x < 0) \\ 1 - \frac{x}{a} & (0 < x < a) \end{cases} \end{aligned}$$

This is the required expression:

$$\Lambda\left(\frac{x}{a}\right) \Longleftrightarrow a\ \mathrm{sinc}^2 \frac{at}{2}$$

From Fourier's integral formula, it follows that

$$f(x) = \int_{-\infty}^\infty \frac{1}{t^2}(1 - \cos at) e^{itx} dt$$

$$= \begin{cases} 1 - \frac{|x|}{a} & (|x| < a) \\ 0 & (|x| > a) \end{cases}$$

Putting $x = 0$ into the above equation, then we get

$$\int_{-\infty}^\infty \frac{1}{t^2}(1 - \cos at) dt = a\pi$$

Further putting $a = 1$, we have

$$\int_{-\infty}^{\infty} \frac{1}{t^2}(1-\cos t)dt = \pi$$

(4) Consider Fourier transform for the function;

$f(x) = e^{-a|x|}$

$$\begin{aligned} F(t) &= \int_{-\infty}^{\infty} e^{-a|x|}e^{-itx}dx \\ &= \int_{-\infty}^{0} e^{ax}e^{-itx} + \int_{0}^{\infty} e^{-ax}e^{-itx}dx \\ &= \frac{1}{a-it} + \frac{1}{a+it} = \frac{2a}{a^2+t^2} \qquad (27) \end{aligned}$$

An another method of calculation by the use of Laplace transformation has shown

$$F(t) = \int_{-\infty}^{\infty} e^{-a|x|}e^{-itx}dx$$

$$= 2\int_{0}^{\infty} e^{-ax}\cos txdx = \frac{2a}{t^2+a^2}$$

Verify inversion formula:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2a}{t^2+a^2}e^{itx}dt$$

$$= \frac{a}{\pi}\int_{-\infty}^{\infty} \frac{1}{t^2+a^2}e^{itx}dt$$

By using the theory of Residues, it is found that when $a > 0$

$$\int_{-\infty}^{\infty} \frac{1}{t^2+a^2}e^{itx}dt = 2\pi i Res(t = ia)$$

$$= 2\pi i \cdot \frac{1}{2ia}e^{i(ia)x} = \frac{\pi}{a}e^{-ax}$$

In a similar manner, when $a < 0$

$$\int_{-\infty}^{\infty} \frac{1}{t^2+a^2}e^{itx}dt = -2\pi i Res(t = -ia)$$

$$= -2\pi i \cdot \frac{1}{-2ia}e^{i(-ia)x} = \frac{\pi}{a}e^{ax}$$

Hence, for all values of x, we obtain the pair

$$e^{-a|x|} \Longleftrightarrow \frac{2a}{t^2+a^2}$$

(5) Consider Fourier transform for the function; $f(x) = e^{-x^2}$, which is called Gauss function or Normal distribution in the probability thoery:

$$\begin{aligned} F(t) &= \int_{-\infty}^{\infty} e^{-x^2}e^{-itx}dx \\ &= \int_{-\infty}^{\infty} e^{-(x+i\frac{t}{2})^2-\frac{t^2}{4}}dx \end{aligned}$$

By using Cauchy's theorem, $\oint f(z)dz = 0$ since $f(z) = e^{-z^2}$ is analytic with respect to $z = x+i\frac{t}{2}$,

$$\begin{aligned} F(t) &= e^{-\frac{t^2}{4}}\int_{-\infty}^{\infty} e^{-(x+i\frac{t}{2})^2}dx \\ &= e^{-\frac{t^2}{4}}\int_{-\infty}^{\infty} e^{-x^2}dx = e^{-\frac{t^2}{4}}\sqrt{\pi} \end{aligned}$$

An another method by means of the theory of differential equation is shown:

$$\begin{aligned} \frac{dF(t)}{dt} &= \int_{-\infty}^{\infty} e^{-x^2}(-ix)e^{-itx}dx \\ &= \frac{i}{2}\int_{-\infty}^{\infty} \left(\frac{de^{-x^2}}{dx}\right)e^{-itx}dx \\ &= \frac{i}{2}\left\{\left[e^{-x^2}e^{-itx}\right]_{-\infty}^{\infty}\right. \\ &\quad \left. - \int_{-\infty}^{\infty} e^{-x^2}(-it)e^{-itx}dx\right\} \\ &= -\frac{t}{2}\int_{-\infty}^{\infty} e^{-x^2}e^{-itx}dx \\ &= -\frac{t}{2}F(t) \qquad (28) \end{aligned}$$

In order to solve the differential equation $F'(t) = -\frac{t}{2}F(t)$ by separated variable,

$$\log F(t) = -\frac{t^2}{4} + c \quad (c = \text{constant})$$

Consequently, we obtain

$$F(t) = e^{-\frac{t^2}{4}+c} = \sqrt{\pi}e^{-\frac{t^2}{4}} \quad (F(0) = \sqrt{\pi} = e^c)$$

Verify inversion formula:

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \sqrt{\pi}e^{-\frac{t^2}{4}}e^{itx}dt \\ &= \frac{1}{2\sqrt{\pi}}e^{-x^2}\int_{-\infty}^{\infty} e^{-\frac{1}{4}(t+2ix)^2}dt \\ &= \frac{1}{2\sqrt{\pi}}e^{-x^2}\int_{-\infty}^{\infty} e^{-\frac{1}{4}x^2}dt \\ &= \frac{1}{2\sqrt{\pi}}e^{-x^2}2\sqrt{\pi} = e^{-x^2} \end{aligned}$$

However, it is convenient for the expression of Fourier transform for Gauss function that the exponent has $-\frac{x^2}{2}$ instead of $-x^2$. We write the pair

$$e^{-\frac{x^2}{2}} \Longleftrightarrow e^{-\frac{t^2}{2}}$$

1.3 Some properties of Fourier transformation

In order to discuss the properties of Fourier transformation, we shall again write the definitions of Fourier transformation for the pair $f(x) \Longleftrightarrow F(t)$:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(t)e^{itx}dt$$

and

$$F(t) = \int_{-\infty}^{\infty} f(x)e^{-itx}dx$$

(1) Symmetry and Conjugate

By the substitution $x = -x$ in the definition (1), $f(x)$ becomes

$$f(-x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(t)e^{it(-x)}dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(t)e^{-ixt}dt$$

From this, changing the variable $x \to t$, we have the pair of Fourier transformation

$$F(x) \Longleftrightarrow 2\pi f(-t)$$

Consider Fourier transformation for conjugate functions:

$$\mathcal{F}(\overline{f(x)}) = \int_{-\infty}^{\infty} \overline{f(x)}e^{-itx}dx = \overline{\int_{-\infty}^{\infty} f(x)e^{itx}dx}$$

$$= \overline{\int_{-\infty}^{\infty} f(x)e^{-i(-t)x}dx} = \overline{F(-t)} \tag{29}$$

Then, we may write the pair of Fourier transformation as

$$\overline{f(x)} \Longleftrightarrow F(-t) \text{ and } \overline{f(x)} \Longleftrightarrow \overline{F(-t)}$$

Putting $x = -x$ in the above equation, we get

$$\mathcal{F}(\overline{f(-x)}) = \int_{-\infty}^{\infty} \overline{f(-x)}e^{-itx}dx$$

$$= \overline{\int_{-\infty}^{\infty} f(-x)e^{itx}dx}$$

$$\overset{-x=y}{=} \overline{\int_{-\infty}^{\infty} f(y)e^{-ity}dy} = \overline{F(t)} \tag{30}$$

Then, we may write

$$\overline{f(x)} \Longleftrightarrow \overline{F(-t)} \text{ and } \overline{f(-x)} \Longleftrightarrow \overline{F(t)}$$

(2) Linearity

2.1) scalar product (similarity)

Consider the integral,

$$\mathcal{F}(f(\pm ax)) = \int_{-\infty}^{\infty} e^{-itx}f(\pm ax)dx$$

Substituting $ax = y$ $(a > 0)$, we have

$$\mathcal{F}(f(ax)) = \int_{-\infty}^{\infty} e^{-it\frac{y}{a}}f(x)\frac{dy}{a} = \frac{1}{a}F\left(\frac{t}{a}\right)$$

In a similar manner, when $a < 0$, then,

$$\mathcal{F}(f(ax)) = \int_{\infty}^{-\infty} e^{-it\frac{y}{a}}f(x)\frac{dy}{a}$$

$$= -\int_{-\infty}^{\infty} e^{-it\frac{y}{a}}f(x)\frac{dy}{a} = -\frac{1}{a}F\left(\frac{t}{a}\right)$$

Therefore, for both cases we may write

$$\mathcal{F}(f(ax)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}f(ax)dx$$

$$= \frac{1}{|a|}F\left(\frac{t}{a}\right)$$

The multiplication to x of the function $f(x)$ by a corresponds to the division of Fourier transform $F\left(\frac{t}{a}\right)$ by $|a|$.

$$f(\pm ax) \Longleftrightarrow \frac{1}{|a|}F\left(\frac{t}{a}\right) \tag{31}$$

2.2) Addition (Shifting)

Consider the integral:

$$\mathcal{F}(f(x \pm b)) = \int_{-\infty}^{\infty} e^{-itx}f(x \pm b)dx$$

On replacing $x \pm b$ by y, the integral reduces to

$$\mathcal{F}(f(x \pm b)) = \int_{-\infty}^{\infty} e^{-it(y \mp b)}f(y)dy$$

$$= e^{\pm itb}\int_{-\infty}^{\infty} e^{-ity}f(y)dy = e^{\pm itb}F(t)$$

That is to say, the translation of x in $f(x)$ by $a \pm b$ corresponds to the multiplication of Fourier transform $F(t)$ by $t \mp b$.

The Fourier transform $\mathcal{F}(e^{\pm ibx}f(x))$ may be given:

$$\mathcal{F}(e^{\pm ibx}f(x)) = \int_{-\infty}^{\infty} e^{-itx}\left\{e^{\pm ibx}f(x)\right\}dt$$

$$= \int_{-\infty}^{\infty} e^{-i(t\mp b)x}f(x)dx = F(t\mp b) \qquad (32)$$

The multiplication to the function $f(x)$ by $e^{\pm itb}$ corresponds to the translation of t in $F(t)$ by $t\mp b$.

$$f(x\pm b) \Longleftrightarrow e^{\pm itb}F(t) \text{ and } e^{\pm itb} \Longleftrightarrow F(t\mp b)$$

Combining the equations (31) and (32) furthermore, we have

$$\mathcal{F}(f(ax+b)) = \frac{1}{|a|}e^{-it\frac{b}{a}}F\left(\frac{t}{a}\right)$$

(3) Fourier integrals for derivatives and integrals

3-1) Consider the integral for the first derivative of function $f(t)$

$$\mathcal{F}\left(\frac{df(x)}{dx}\right) = \int_{-\infty}^{\infty} e^{-itx}f'(x)dx$$

$$= \left[e^{-itx}f(x)\right]_{-\infty}^{\infty} + it\int_{-\infty}^{\infty} e^{-itx}f(x)dx = itF(t)$$

Differentiation of the function $f(x)$ corresponds to the multipication of Foutier transform by it. In like manner, we have for the second derivative:

$$\mathcal{F}\left(\frac{d^2f(x)}{dx^2}\right) = \int_{-\infty}^{\infty} e^{-itx}\frac{d^2f(x)}{dx^2}dx$$

$$= \left[e^{-itx}\frac{df(x)}{dx}\right]_{-\infty}^{\infty} + it\int_{-\infty}^{\infty} e^{-itx}f'(x)dx$$

$$= (it)^2F(t)$$

In general, we may expect

$$\mathcal{F}(f^{(n)}(x)) = (it)^nF(t) \qquad (33)$$

This relation is of great importance to apply Fourier transformation to solving differential equations.

Consider Fourier transform for the function multiplying $-ix$:

$$\mathcal{F}(-ix\cdot f(x)) = \int_{-\infty}^{\infty} -ixf(x)e^{-itx}dx$$

$$= \int_{-\infty}^{\infty} f(x)\frac{\partial}{\partial t}\left(e^{-itx}\right)dx$$

$$= \frac{d}{dt}\int_{-\infty}^{\infty} f(x)e^{-itx}dx = \frac{d}{dt}F(t)$$

The multiplication to the function by $-ix$ correponds to differentiation of Fourier transform. In general, we may expect

$$\mathcal{F}((-ix)^nf(x)) = \frac{d^n}{dt^n}F(t)$$

$$\text{or } F(x^nf(x)) = i^n\frac{d^n}{dt^n}F(t)$$

Symbolically,

$$f^{(n)}(x) \Longleftrightarrow (it)^nF(t)$$

or

$$(-ix)^nf(x) \Longleftrightarrow \frac{d^n}{dt^n}F(t)$$

3-2) Consider the following integral:

$$\mathcal{F}\left(\int_0^x f(y)dy\right) = \int_{-\infty}^{\infty} e^{-itx}\left(\int_0^x f(y)dy\right)dx$$

$$= \left[-\frac{1}{it}e^{-itx}\int_0^x f(y)dy\right]_{-\infty}^{\infty}$$

$$+\frac{1}{it}\int_{-\infty}^{\infty} e^{-itx}f(x)dx = \frac{1}{it}F(t)$$

As usual, we write

$$\mathcal{F}\left(\int_0^x f(y)dy\right) \Longleftrightarrow \frac{1}{it}F(t) \qquad (34)$$

(4) Convolution

The convolution is defined

$$f*g \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} f(x-y)g(y)dy \qquad (35)$$

Consider Fourier transformation of the convolution:

$$\mathcal{F}(f*g) = \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} f(x-y)g(y)dy\right\}e^{-itx}dx$$

$$= \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} f(x-y)e^{-it(x-y)}d(x-y)\right\}$$

$$\times\, g(y)e^{-ity}dy = F(t)\int_{-\infty}^{\infty} g(y)e^{-ity}dy$$

$$= F(t)G(t) \qquad (36)$$

That is to say, Fourier transform of the convolution is the product of each of Fourier transforms.

$$f * g \Longleftrightarrow F(t)G(t)$$

Using the formula (36), we shall show

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(t)|^2 dt \qquad (37)$$

which is called Parseval's equality.

By means of the definition of inverse formula, it follows that

$$f * g = \int_{-\infty}^{\infty} f(x)g(y-x)dx$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(t)G(t)e^{ity} dt$$

By the substitution $y = 0$ in the above equation, we have

$$\int_{-\infty}^{\infty} f(x)g(-x)dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(t)G(t)dt$$

Providing that $g(-x) = f(x)$, it follows that

$$G(t) = \mathcal{F}(g(x)) = \mathcal{F}(\overline{f(-x)}) = \overline{F(t)}$$

Hence we obtain

$$\int_{-\infty}^{\infty} f(x)\overline{f(x)}dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(t)\overline{F(t)}dt$$

This is the formula required.

2 Applications of Fourier Transformation

2.1 Ordinary Differential Equations

As an example, we wish to solve the general type of ordinary differential equations with constant coefficients of the second order

$$aD^2y(x) + bDy(x) + cy(x) = f(x) \qquad (38)$$

under the initial conditions;

$$y(0) = p_0 \text{ and } y'(0) = q_0$$

where $D = \frac{d}{dx}$ and $D^2 = \frac{d^2}{dx^2}$ are the differential operators and p_0 and q_0 are constants.

We need to extend the domain of the parameter t to $-\infty$ due to the use of Fourier transformation method as described below. Hence the function $f(x)$ is assumed to be zero in the domain for $0 > t > -\infty$. The Initial conditions are rewritten due to the discontinuity at $t = 0$:

$$y(0) = p_0\delta'(x) \text{ and } y'(0) = q_0\delta(x)$$

We introduce here notations to represent the pair of Fourier transformation. Let us define Fourier transform of $y(x)$

$$Y(t) = \mathcal{F}(y(x)) = \int_{-\infty}^{\infty} y(x)e^{-itx} dt$$

and define inverse Fourier transform

$$y(x) = \mathcal{F}^{-1}(Y(t)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(t)e^{itx} dt$$

The suggested procedure for solving the differential equations by means of Fourier transformation is as follows:

Step **1**. Make both members of the equation of interest Fourier transform.

Step **2**. Solve an algebraic equation to be transformed in Step **(1)**.

Step **3**. Make the solution of the equation in Step **(2)** inverse transform.

To solve the equation (38), We shall first solve the following differential equation

$$aD^2g(x) + bDg(x) + cg(x) = \delta(x) \qquad (39)$$

where the solution $g(x)$ of the equation (39) is in general called fundamental solution.

Taking Fourier transforms on both members in the equation (39), a desired algebraic equation as a function of t may be written

$$a(it)^2G + b(it)G + cG = 1$$

Consequently,

$$Y = -\frac{1}{a} \frac{1}{t^2 - i\frac{b}{a}t - \frac{c}{a}} = -\frac{1}{a} \frac{1}{(t-\alpha)(t-\beta)}$$

$$= -\frac{1}{a} \frac{1}{\alpha - \beta} \left\{ \frac{1}{t-\alpha} - \frac{1}{t-\beta} \right\}$$

where α and β are the roots in the quadratic equation, $t^2 - i\frac{b}{a}t - \frac{c}{a} = 0$

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{ib \pm \sqrt{-b^2 + 4ac}}{2a} = \frac{ib \pm \sqrt{D}}{2a}$$

Case (1); $D = -b^2 + 4ac > 0$

$$\begin{aligned} g(x) &= \mathcal{F}^{-1}(G(t)) \\ &= \mathcal{F}^{-1}\left\{-\frac{1}{a}\frac{1}{\alpha-\beta}\left(\frac{1}{x-\alpha}-\frac{1}{x-\beta}\right)\right\} \\ &= -\frac{1}{a}\frac{1}{\frac{\sqrt{D}}{a}}\frac{1}{2\pi}2\pi i\{Res(x=\alpha) \\ &\quad - Res(x=\beta)\} \\ &= \frac{-i}{\sqrt{D}}\{e^{i\alpha x}-e^{i\beta x}\} \\ &= \frac{-i}{\sqrt{D}}e^{-\frac{b}{2a}x}\left\{e^{i\frac{\sqrt{D}}{2a}x}-e^{-i\frac{\sqrt{D}}{2a}x}\right\} \\ &= \frac{-i}{\sqrt{D}}e^{-\frac{b}{2a}x}\left\{2i\sin\left(\frac{\sqrt{D}}{2a}x\right)\right\} \\ &= \frac{2}{\sqrt{D}}e^{-\frac{b}{2a}x}\sin\left(\frac{\sqrt{D}}{2a}x\right) \end{aligned}$$

The required solution $y(x)$ may be written

$$\begin{aligned} y(x) &= \frac{2}{\sqrt{D}}e^{-\frac{b}{2a}x}\sin\left(\frac{\sqrt{D}}{2a}x\right)H(x)*f(x) \\ &+ p_0\delta'(x)+q_0\delta(x) \end{aligned}$$

Case (2); $D < 0$

$$\begin{aligned} g(x) &= \mathcal{F}^{-1}(G(t)) \\ &= \mathcal{F}^{-1}\left\{-\frac{1}{a}\frac{1}{\alpha-\beta}\left(\frac{1}{x-\alpha}-\frac{1}{x-\beta}\right)\right\} \\ &= -\frac{1}{a}\frac{1}{i\frac{\sqrt{D}}{a}}\frac{1}{2\pi}2\pi i\{Res(x=\alpha) \\ &\quad - Res(x=\beta)\} \\ &= \frac{-1}{\sqrt{D}}\{e^{i\alpha x}-e^{i\beta x}\} \\ &= \frac{1}{\sqrt{D}}e^{-\frac{b}{2a}x}\left\{e^{-\frac{\sqrt{D}}{2a}x}-e^{\frac{\sqrt{D}}{2a}x}\right\} \\ &= \frac{1}{\sqrt{D}}e^{-\frac{b}{2a}x}\left\{2\sinh\left(\frac{\sqrt{D}}{2a}x\right)\right\} \\ &= \frac{2}{\sqrt{D}}e^{-\frac{b}{2a}x}\sinh\left(\frac{\sqrt{D}}{2a}x\right) \end{aligned}$$

The required solution $y(x)$ may be written

$$y(x) = \frac{2}{\sqrt{D}}e^{-\frac{b}{2a}x}\sinh\left(\frac{\sqrt{D}}{2a}x\right)H(x)*f(x)$$

$$+p_0\delta'(x)+q_0\delta(x)$$

Case (3); $D = 0$

$$g(x) = \mathcal{F}^{-1}(g(t)) = \mathcal{F}^{-1}\left\{-\frac{1}{a}\frac{1}{(x-\alpha)^2}\right\}$$

$$= -\frac{1}{a}\frac{1}{2\pi}\pi i Res(x=\alpha) = -\frac{i}{2}(ix)e^{-\frac{b}{2a}x}$$

$$= \frac{1}{2}xe^{-\frac{b}{2a}x}$$

The required solution $y(x)$ may be written

$$y(x) = -\frac{i}{2}(ix)e^{-\frac{b}{2a}x}$$

$$= \frac{1}{2}xe^{-\frac{b}{2a}x}H(x)*f(x)+p_0\delta'(x)+q_0\delta(x)$$

Example 1. Solve the following differential equations.

(1) $Dy(x) = \delta(x)$ (2) $Dy(x) + ay(x) = \delta(x)$

(3) $Dy(x) - ay(x) = \delta(x)$ (4) $D^2y(x) = \delta(x)$

(5) $D^2y(x) + a^2Dy(x) = \delta(x)$

(6) $D^2y(x) - a^2Dy(x) = \delta(x)$

Solution.

(1) The equation in problem is to be thought the definition of $Dy(x) = \delta(x)$. Hence, the general solution may be written $y(x) = \delta(x) + \text{constant term}$. However, the constant term is omitted for simplicity. it is not so easy to find the solution by means of Fourier transformation.

Proceed as above,

$$(it)Y = 1 \text{ then, } Y = \frac{1}{it} \Longleftrightarrow \frac{1}{2}sign\ x$$

This is wrong. The correct answer is

$$Y = \frac{1}{it} + \pi\delta(t) \Longleftrightarrow H(x)$$

Hereafter, apply the method of Fourier transformation described above.

(2) $\{it + a\}Y = 1$ then, $Y = \dfrac{1}{it+a} \Longleftrightarrow e^{-ax}$

The solution is $e^{-ax}H(x)$.

(3) $\{it - a\}Y = 1$ then, $Y = \dfrac{1}{it-a} \Longleftrightarrow e^{ax}$

The solution is $e^{ax}H(x)$.

(4) $(it)^2Y = 1$ then, $Y = -\dfrac{1}{t^2} \Longleftrightarrow \dfrac{1}{2}|x|$

The solution is $\frac{1}{2}|x|$.

(5) $\{(it)^2 + a^2\}Y = 1$ then,

$$Y = -\frac{1}{t^2-a^2} \Longleftrightarrow \frac{\sin ax}{a}$$

The solution is $\dfrac{\sin ax}{a}$.

(6) $\{(it)^2 - a^2\}Y = 1$ then,

$$Y = -\frac{1}{t^2+a^2} \Longleftrightarrow e^{-a|x|}$$

The solution is $e^{-a|x|}$.

2.2 Partial Differential Equations

In this section, we shall be concerned with the method solving the one-dimensional partial differential equations of the second order, which are closely connected with physical problems.

(1) Consider Laplace equation

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = \triangle u(x,y) = 0$$

under the boundary conditions ;

$$u(x,0) = p(x)$$

$$\text{and} \quad u(x,y) \to 0 \quad \text{when} \quad x^2 + y^2 \to \infty$$

In order to solve this, we shall first find the fundamental solution for the following differential equation

$$\triangle g(x,y) = \delta(x,y)$$

As in the proceeding procedure, we shall make both members of the equation transform with respect to x,

$$\begin{aligned} \mathcal{F}\left(\frac{\partial^2 g(x,y)}{\partial x^2}\right) &= \int_{-\infty}^{\infty} \frac{\partial^2 g(x,y)}{\partial x^2} e^{-i\xi x} dx \\ &= (i\xi)^2 \int_{-\infty}^{\infty} g(x,y) e^{-i\xi x} dx \\ &= (i\xi)^2 G(\xi,y) = -\xi^2 G(\xi,y) \end{aligned}$$

$$\mathcal{F}\left(\frac{\partial^2 g(x,y)}{\partial y^2}\right) = \frac{d^2}{dy^2}\mathcal{F}(g(x,y)) = \frac{d^2 G(\xi,y)}{dy^2}$$

$$\begin{aligned} \mathcal{F}(\delta(x,y)) &= \mathcal{F}(\delta(x)\delta(y)) = \delta(y)\mathcal{F}(\delta(x)) \\ &= \delta(y) \end{aligned}$$

From these, an ordinary differential equation to be solved is

$$\frac{d^2 G}{dy^2} - \xi^2 G = \delta(y)$$

The solution of this equation has been known as a foundamental solution

$$G(t,y) = e^{-y|\xi|}$$

According to inverse transformation with respect to ξ, the fundamental solution is written

$$\mathcal{F}^{-1}(G(\xi,y)) = g(x,y) = \frac{1}{2\pi}\frac{y}{x^2+y^2}$$

then, by use of the convolution, the required solution is written

$$\begin{aligned} u(x,y) &= u(x,0)\delta(x) * \frac{1}{2\pi}\frac{y}{x^2+y^2} \\ &= p(x) * \frac{1}{2\pi}\frac{y}{x^2+y^2} \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} p(\xi)\frac{y}{(x-\xi)^2+y^2} d\xi \end{aligned}$$

(2) Consider one-dimensional wave equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} - c^2\frac{\partial^2 u(x,t)}{\partial x^2} = \Box u(x,t) = 0$$

under the intial conditions;

$$u(x,0) = p(x) \quad \text{and} \quad u_t(x,0) = q(x)$$

In order to solve this, we shall first find the fundamental solution for the following differential equation

$$\Box g(x,t) = \delta(x,t)$$

As in like manner (1), a desired ordinary differential equation reduces to

$$\frac{d^2 G(\xi,t)}{dt^2} + c^2 t^2 G(\xi,t) = \delta(t)$$

The solution has been expressed from the formula as a foundamental solution

$$G(\xi,t) = \frac{\sin ct\xi}{ct}$$

By inverse transformation, it follows that

$$\mathcal{F}^{-1}(G(\xi,t)) = g(x,t) = H(x - |ct|)$$

The required solution is written

$$\begin{aligned} u(x,t) &= \{u(x,0)\delta'(x) + u_t(x,0)\delta(x)\} * g(x,t) \\ &= p(x) * g'(x,t) + q(x) * g(x,t) \\ &= p(x) * \{\delta(x+ct) + \delta(x-ct)\} \\ &+ \frac{1}{c}\int_{x-ct}^{x+ct} q(\zeta)d\zeta \\ &= \{p(x+ct) + p(x-ct)\} \\ &- \frac{1}{c}\int_{x-ct}^{x+ct} g(\zeta)d\zeta \end{aligned}$$

(3) Consider one-dimensional heat equation

$$\frac{\partial u(x,t)}{\partial t} - c^2\frac{\partial^2 u(x,t)}{\partial x^2} = \Diamond u(x,t) = 0$$

under the initial condition;

$$u(x,0) = p(x)$$

In order to solve this, we shall first find the fundamental solution for the following differential equation

$$\Diamond g(x,t) = \delta(x,t)$$

As in similar manner above, an ordinary differential equation solved reduces to

$$\frac{dG(\xi,t)}{dt} + c^2\xi^2 G(\xi,t) = \delta(t)$$

The solution has been expressed as a foundamental solution

$$G(\xi,t) = e^{-c^2\xi^2 t}$$

By inverse transformation, it follows that

$$\mathcal{F}^{-1}(G(\xi,t)) = g(x,t) = \sqrt{\frac{\pi}{c^2 t}} e^{-\frac{x^2}{\sqrt{4c^2 t}}}$$

The required solution is written

$$\begin{aligned} u(x,t) &= p(x) * g(x,t) \\ &= \frac{1}{2\pi}\sqrt{\frac{\pi}{c^2 t}} \int_{-\infty}^{\infty} p(\xi) e^{-\frac{|x-\xi|^2}{\sqrt{4c^2 t}}} d\xi \end{aligned}$$