On a Lévy's formula for Brownian motion

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Abstract. Consider Brownian motion $\{B(t)\}$ on a probability space (Ω, a, P). Then $(dB)^2 = dt$, that is, $|dB| = \sqrt{dt}$ leads to a contradiction so that this famous formula does not hold.

Proof. Let $0 \leqq t \leqq 1$ and let $\Delta : 0 = t_0 < t_1 < t_2 < \cdots < t_{n+1} = 1$.

Suppose

$$(*) \quad |dB(t)| = \sqrt{dt}.$$

Then

$$\sum_j \left|B(t_{j+1}) - B(t_j)\right| = \sum_j \sqrt{t_{j+1} - t_j}.$$

So

$$\sup_\Delta \sum_j \left|B(t_{j+1}) - B(t_j)\right| = \sup_\Delta \sum_j \sqrt{t_{j+1} - t_j}.$$

Now, from

$$\sum_{j=0}^n \sqrt{t_{j+1} - t_j} = \sum_{j=0}^n \frac{t_{j+1} - t_j}{\sqrt{t_{j+1} - t_j}}$$

and $(0 <) \sqrt{t_{j+1} - t_j} \leqq \operatorname*{Max}_j \sqrt{t_{j+1} - t_j}$

it holds that

$$\frac{1}{\sqrt{t_{j+1} - t_j}} \geqq \frac{1}{\operatorname*{Max}_j \sqrt{t_{j+1} - t_j}}.$$

So

$$\frac{t_{j+1} - t_j}{\sqrt{t_{j+1} - t_j}} \geqq \frac{t_{j+1} - t_j}{\operatorname*{Max}_j \sqrt{t_{j+1} - t_j}} \ (>0) \text{ because } t_{j+1} - t_j > 0$$

and so that

$$\sum_{j=0}^n \frac{t_{j+1} - t_j}{\sqrt{t_{j+1} - t_j}} \geqq \sum_{j=0}^n \frac{t_{j+1} - t_j}{\operatorname*{Max}_j \sqrt{t_{j+1} - t_j}}$$

Here put $\varepsilon \fallingdotseq \operatorname*{Max}_j \sqrt{t_{j+1} - t_j} \left(= \sqrt{\operatorname*{Max}_j (t_{j+1} - t_j)} \right)$

then

$$\sum_{j=0}^n \sqrt{t_{j+1} - t_j} \geqq \sum_{j=0}^n \frac{t_{j+1} - t_j}{\varepsilon} = \frac{\sum (t_{j+1} - t_j)}{\varepsilon} = \frac{1}{\varepsilon} \to \infty \ (\varepsilon \to 0)$$

Thus, by letting $|\Delta| \fallingdotseq \operatorname*{Max}_j (t_{j+1} - t_j) \to 0$

$\varepsilon \to 0$ holds since $\sqrt{t}$ is monotone increasing and continuous in t

so that $\sum_{j=0}^n \sqrt{t_{j+1} - t_j} \to \infty$ as $|\Delta| \to 0$.

Therefore

$$\sup_\Delta \sum_j \sqrt{t_{j+1} - t_j} = \infty \quad \text{on } \Omega.$$

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On the other hand, by Cauchy-Schwarz's inequality

$$(**)\quad E\left[\sum_{j=0}^{n}\left|B(t_{j+1})-B(t_j)\right|\right]$$
$$\leqq\left\{E\left[\left(\sum_{j=0}^{n}\left|B(t_{j+1})-B(t_j)\right|\right)^2\right]\right\}^{1/2}$$
$$<\infty.$$

Because,

$$E\left[\left(\sum_{j}\left|B(t_{j+1})-B(t_j)\right|\right)^2\right]$$
$$=E\left[\sum_{j}\left(B(t_{j+1})-B(t_j)\right)^2+2\cdot\sum_{j>i}\left|B(t_{j+1})-B(t_j)\right|\left|B(t_{i+1})-B(t_i)\right|\right].$$

Here by Cauchy-Schwarz's inequality

$$\sum_{j>i}\left|B(t_{j+1})-B(t_j)\right|\left|B(t_{i+1})-B(t_i)\right|$$
$$\leqq\left\{\sum_{j}\left|B(t_{j+1})-B(t_j)\right|^2\right\}^{1/2}\cdot\left\{\sum_{i}\left|B(t_{i+1})-B(t_i)\right|^2\right\}^{1/2}$$

and $\sum_{j}\left|B(t_{j+1})-B(t_j)\right|^2=\sum_{j}(t_{j+1}-t_j)=1$ by $(dB)^2=dt$

so that

$$E\left[2\cdot\sum_{j>i}\left|B(t_{j+1})-B(t_j)\right|\left|B(t_{i+1})-B(t_i)\right|\right]$$
$$\leqq E(2\cdot 1)=2.$$

Moreover, $E\left[\sum_{j}\left(B(t_{j+1})-B(t_j)\right)^2\right]=E\left(\sum_{j}(t_{j+1}-t_j)\right)=1.$

Therefore $E\left[\sum_{j=0}^{n}\left|B(t_{j+1})-B(t_j)\right|\right]\leqslant\sqrt{3}$ for all $n=1,2,3,\cdots$.

It may be that this means $\sup_{\Delta}\sum_{j}^{n}\left|B(t_{j+1})-B(t_j)\right|<\infty$ a. e.

Since $(**)$ holds, for each $n=1,2,3,\cdots$

$$\sum_{j=0}^{n}\left|B(t_{j+1})-B(t_j)\right|<\infty \text{ a. e.}$$

Here notice that, since $|dB|=\sqrt{dt}$,

$$\sum_{j=0}^{n}\left|B(t_{j+1})-B(t_j)\right|\in L^1(=L^\infty)\,(n=1,2,3,\cdots).$$

So there is a constant K_Δ^n for any Δ and for each $n=1,2,3,\cdots$
such that

$$\sum_{j=0}^{n}\left|B(t_{j+1})-B(t_j)\right|\leqslant K_\Delta^n<\infty \text{ a. e., i, e., } K_\Delta^n<\infty \text{ on } \Omega\backslash e_n^\Delta,\ P(e_n^\Delta)=0.$$

Set $S=\left\{K_\Delta^n<\infty \text{ a. e.};\ \forall\Delta(n=1,2,3,\cdots)\right\}(\subset R_+)$.

If S is not of upper bounded then for any large number $G>0$ there is some finite partition Δ such that $G<K_\Delta^n$ on $\Omega\backslash e_n^\Delta$ so that there is some Δ such that $K_\Delta^n=\infty$ on $\Omega\backslash e_n^\Delta$ and this contradicts to $K_\Delta^n<\infty$ on $\Omega\backslash e_n^\Delta$ for all Δ. That is, S is upper bounded subset of R_+. Since S is a set of upper bounded, by Zorn's lemma, S has the maximal element $K<\infty$ on some subset $\Omega\backslash\Omega'$ of Ω such that $\Omega'\supset\bigcup_n e_n^\Delta\neq\phi$ and Ω' is the union of all negligible sets e_n^Δ so that $\sup_{\Delta}K_\Delta^n\leqslant K<\infty$ on $\Omega\backslash\Omega'$.

If $P(\Omega)=P(\Omega')(=1)$ then $P(\Omega\backslash\Omega')=0$ so that

$\sup_{\Delta}K_\Delta^n\leqslant K<\infty$ on $\Omega\backslash\Omega'$ with Probability 0.

So $K_\Delta^n < \infty$ on $\Omega \backslash \Omega'$ with Probability 0.
So $K_\Delta^n = \infty$ on Ω' with Probability 1. This contradicts to $K_\Delta^n < \infty$ a. e.
So $\phi \neq \Omega' \subsetneqq \Omega$ and $0 \leqq P(\Omega') < 1$.
This means that

$\sup_\Delta \sum_j |B(t_{j+1}) - B(t_j)|$ ($=$ a function of $\omega \in \Omega$) $< \infty$ on $\Omega \backslash \Omega'$ and notice that $\Omega' \supset \bigcup_n e_n^\Delta$.

Since $\Omega' \supset \bigcup_n e_n^\Delta$, $\Omega' \backslash \bigcup_n e_n^\Delta \supset \Omega \backslash \Omega'$ holds.
Therefore

$\sup_\Delta \sum_j |B(t_{j+1}) - B(t_j)| \neq \infty$ on $\Omega \backslash \bigcup_n e_n^\Delta (\subset \Omega)$, $P(\bigcup_n e_n^\Delta) = 0$,

that is, $\sup_\Delta \sum_j |B(t_{j+1}) - B(t_j)| \neq \infty$ a. e.

Therefore $(dB)^2 = dt$, that is, $|dB| = \sqrt{dt}$ leads to a contradiction.

This means that $(dB)^2 = dt$ does not hold. (q. e. d.)

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