Guide to Applied Mathematics for Foreign Students IV

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(Received on 30 Nov. 2001)

Abstract

This article is self-study for foreign students in our college. The content is preliminary illustration about complex numbers and complex functions with one variable

1 Complex numbers

Complex number α is defined by $a + ib$, where a and b are real numbers and i is a number such that $i^2 = -1$[1]. The numbers a and b are called the real part and the imaginary part of the complex number, respectively. Each term is designated as $a = \Re\alpha$ or $b = \Im\alpha$. The complex number $\alpha = ib$ is called a pure imaginary. The zero of the complex number is zeros of real and imaginary parts of the complex number; ($\alpha = a + ib = 0 \longrightarrow a = b = 0$).

The complex number of $a + ib$ corresponds to a point in the plain, which has a rectangular Cartesian coordinate system, while the real number a corresponds to a point on the line (x-axis):

A pair of real numbers (a, b) can also be interpretted as two dimensional vector. Thus, we can use both the vector and a point representation of a complex number in the following.

The absolute value of $\alpha = a + ib$ is the length of vector $\vec{0\alpha} = \vec{r}$, which is denoted as

$$|\vec{\alpha}| = r = |a + ib| = \sqrt{a^2 + b^2}$$

The operations of addition, subtraction, multiplication and division of complex numbers are defined in terms of the corresponding ones for real numbers, where $\alpha = a + ib$ and $\beta = c + id$ (a, b, c, d are real numbers);

$$\text{addition} \quad \alpha + \beta = (a + ib) + (c + id)$$

$$= (a + c) + i(b + d)$$

$$\text{subtraction} \quad \alpha - \beta = (a + ib) - (c + id)$$

$$= (a - c) + i(b - d)$$

where $|\alpha - \beta|$ means the distance between two points α and β.

$$\text{muptiplication} \quad \alpha\beta = (a + ib)(c + id)$$

$$= (ac - bd) + i(bc + ad)$$

$$c\alpha = c(a + ib) = ca + icb \quad (\text{when } \beta = c)$$

$$\text{division} \quad \frac{\alpha}{\beta} = \frac{(a + ib)}{(c + id)} = \frac{(ac + bd) + i(bc - ad)}{c^2 + d^2}$$

[Example 1] Show expressions of the above operations of complex numbers by a pair of real numbers.

[Solution.]

$$\alpha + \beta = (a + c, b + d), \ \alpha - \beta = (a - c, b - d)$$

$$\alpha\beta = (ac - bd, bc + ad), \frac{\alpha}{\beta} = \left(\frac{ac + bd}{c^2 + d^2}, \frac{bc - ad}{c^2 + d^2}\right)$$

The complex number $a - ib$ is called the conjugate of $a + ib$, where we denote the conjugate complex number as $\overline{\alpha} = \overline{a + ib}$. These complex numbers differ only in the signs of their imaginary parts. Since the imaginary part of a complex number is the y-coordinate in the plain, two conjugate complex numbers are the images each other in the x-axis.

The addition and subtraction of α and $\overline{\alpha}$ give $\begin{cases} \alpha + \overline{\alpha} = 2a \\ \alpha - \overline{\alpha} = i2b \end{cases}$. Solving these equations for a and b, we have

$$a = \Re\alpha = \frac{\alpha + \overline{\alpha}}{2} \tag{1}$$

[1]The operation of i means the rotation of $\frac{\pi}{2}$: $1 \xrightarrow{i} i \xrightarrow{i} i^2 = -1$

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$$b = \Im\alpha = \frac{\alpha - \overline{\alpha}}{2i} \qquad (2)$$

From these formulas,

$$\overline{\alpha} = \alpha \iff \alpha \text{ is a real number} \qquad (3)$$

$$\overline{\alpha} = -\alpha \iff \alpha \text{ is a pure imaginary} \qquad (4)$$

The product of α and $\overline{\alpha}$ is $|\alpha|^2$;

$$\alpha\overline{\alpha} = (a+ib)\overline{(a+ib)} = (a+ib)(a-ib)$$

$$= a^2 + b^2 = |\alpha|^2 \longrightarrow |\alpha| = \sqrt{\alpha\overline{\alpha}}$$

[**Problem.**] Verify the following formuras.

(1) $\overline{\alpha+\beta} = \overline{\alpha} + \overline{\beta}$ (2) $\overline{\alpha-\beta} = \overline{\alpha} - \overline{\beta}$

(3) $\overline{\alpha\beta} = \overline{\alpha}\overline{\beta}$ (4) $\overline{\left(\frac{\alpha}{\beta}\right)} = \frac{\overline{\alpha}}{\overline{\beta}}$

A differrent form of complex numbers is trigonometric one, which is

$$\alpha = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

where θ is called the argument denoted as $\theta = Arg\ \alpha$. This coordinate system (r, θ) is called a polar one. It is convinient to call r and θ radius vector and vectorial angle of the point α in the rectangular Cartesian coordinate system:

$$\begin{cases} x = \cos\theta \\ y = r\sin\theta \end{cases} \longleftrightarrow \begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}\frac{y}{x} \end{cases}$$

With the use of the argument we may rewrite the above statements (3) and (4) as

$$\alpha \text{ is a real number} \iff Arg\ \theta = 0 \text{ or } \pi$$

$$\overline{\alpha} = -\alpha \iff \alpha \text{ is a pure imaginary}$$

$$\iff Arg\ \theta = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}$$

[**Example 2**] Calculate the product and division of two complex numbers α and β with the use of trigonometric forms and show them in the plain.

[**Solution**] as $\alpha = r(\cos\theta + i\sin\theta)$ and $\beta = r'(\cos\theta' + i\sin\theta')$,

$$\alpha\beta = rr'(\cos\theta + i\sin\theta)(\cos\theta' + i\sin\theta')$$

$$= rr'\{\cos(\theta+\theta') + i\sin(\theta+\theta')\}$$

where $Arg\ \alpha\beta = Arg\ \alpha + Arg\ \beta = \theta + \theta'$.

$$\frac{\alpha}{\beta} = \frac{r(\cos\theta + i\sin\theta)}{r'(\cos\theta' + i\sin\theta')} = \frac{r}{r}\{\cos(\theta-\theta') + i\sin(\theta-\theta')$$

where $Arg\ \frac{\alpha}{\beta} = Arg\ \alpha - Arg\ \beta = \theta - \theta'$.

[**Example 3**] Solve the following equation (**roots of the powers**).

$$z^n = \alpha \quad \text{(n is any integer)}$$

[**Solution.**] Putting $z = Re^{i\Theta}$ and $\alpha = re^{i\theta}$ into the above equation, we write

$$R^n e^{in\Theta} = re^{i\theta}$$

Then, we obtain

$$R^n = r$$

$$\cos(n\Theta) = \cos\theta, \quad \sin(n\Theta) = \sin\theta$$

Therefore,

$$n\Theta = \theta + 2k\pi \longrightarrow \Theta = \frac{\theta + 2k\pi}{n} \quad (o \le k \le n-1)$$

The solution is

$$z = r^{\frac{1}{n}}\left(\cos\frac{\theta+2k\pi}{n} + i\sin\frac{\theta+2k\pi}{n}\right)$$

For the equation $z^n = 1$, the solution for z are $1, z_1, z_2, \cdots, z_{n-1}$ where $z = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$ (power roots of 1).

Let us consider a different form of the complex number by matrix:

$$\alpha = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

and the unit matrices of real and imaginary parts are

$$\mathrm{E} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathrm{I} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

where E·E=E^2=E, which is called unit matrix and I·I=I^2 = −E. With the use of these units, α is written as

$$\alpha = a\mathrm{E} + b\mathrm{I} = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + \begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

The operations by matrices are satisfied with the four ones of complex numbers;

$$\alpha \pm \beta = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \pm \begin{pmatrix} c & -d \\ d & c \end{pmatrix}$$

$$= \begin{pmatrix} a \pm c & -(b \pm d) \\ b \pm d & a \pm c \end{pmatrix} = (a \pm c)\mathrm{E} + (b \pm d)\mathrm{I}$$

$$\alpha\beta = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} c & -d \\ d & c \end{pmatrix}$$

$$= \begin{pmatrix} ac - bd & -bc - ad \\ bc + ad & ac - bd \end{pmatrix} = (ac - bd)\mathrm{E} + (bc + ad)\mathrm{I}$$

$$\frac{\alpha}{\beta} = \alpha\beta^{-1} = \frac{1}{c^2 + d^2} \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} c & d \\ -d & c \end{pmatrix}$$

$$= \frac{1}{c^2 + d^2} \begin{pmatrix} ac + bd & -bc + ad \\ bc - ad & ac + bd \end{pmatrix}$$

$$= \frac{1}{c^2 + d^2}\{(ac + bd)\mathrm{E} + (bc - ad)\mathrm{I}\}$$

$$(c^2 + d^2 \neq 0)$$

Here $\beta^{-1} = \frac{1}{c^2 + d^2} \begin{pmatrix} c & d \\ -d & c \end{pmatrix}$ is the inverse matrix of β. (The definition of the inverse matrix β is $\beta\beta^{-1} = \beta^{-1}\beta = \mathrm{E}$.) $c^2 + d^2 = \begin{vmatrix} c & d \\ -d & c \end{vmatrix}$ is called determinant of the matrix β. When $c^2 + d^2 = 0 \Longleftrightarrow c = d = 0$, $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is called the zero matrix.

The product of $\alpha (= a + ib = re^{i\theta})$ and $z (= x + iy)$ is $\alpha z = z' = x' + iy'$. This may be rewritten as the transformation between vectors; $(x, y) \to (x', y')$.

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= r \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

This transformation is a rotation through angle *Arg* θ conbined with a uniform expansion ($r > 1$) or contraction ($0 < r < 1$).

[Problem 1] Show that a condition such that three points α, β and γ is on the line may be written in the form; $\frac{\beta - \alpha}{\gamma - \alpha} = k$ (k is real).

[Problem 2] Show that a condition such that each segments of lines $\vec{\beta\alpha}$ and $\vec{\gamma\alpha}$ are perpendicular may be written in the form; $\frac{\beta - \alpha}{\gamma - \alpha} = ik$ (ik is a pure imaginary).

[Problem 3] Show that

$$\Delta\alpha\beta\gamma \propto \Delta\alpha'\beta'\gamma' \Longleftrightarrow \frac{\beta - \alpha}{\gamma - \alpha} = \frac{\beta' - \alpha'}{\gamma' - \alpha'}$$

and that

$\Delta\alpha\beta\gamma$ is equilateral triangle

$$\Longleftrightarrow \alpha^2 + \beta^2 + \gamma^2 - \alpha\beta - \beta\gamma - \gamma\alpha = 0$$

[Problem 4] Show that a condition such that four points α, β, γ and δ is on a circle or on the same line may be written in the form; $\frac{\alpha - \gamma}{\beta - \gamma} \Big/ \frac{\alpha - \delta}{\beta - \delta} = k$ (k is real).

[Problem 5] Verify the following statements:

(1) The equation of a line is

$$\alpha z + \overline{\alpha z} + c = 0 (\text{c is real})$$

(2) The equation of a line such that two points α and β are on the line is

$$z = cz_1 + (1 - c)z_2 (\text{c is real})$$

(3) The equation of a circle is

$$z\overline{z} + \alpha z + \overline{\alpha z} + c = 0 \quad (\alpha\overline{\alpha} - c > 0)$$

(4) the equation of a circle such that two points z_1 and z_2 are terminal points of diameter on the circle is $z = \frac{z_1 - icz_2}{1 - ic}$.

2 Differentiation of complex functions

2.1 Limit, Derivative and Holomorphic

Let $\mathcal{D}$ and $\mathcal{B}$ be two sets of complex numbers in the complex plane such that one number z in $\mathcal{D}$ corresponds to each number ω in $\mathcal{B}$ and each number in $\mathcal{D}$ corresponds to at least one number in $\mathcal{D}$. This correpondence defines a function, symbolically shown as

$$f : z \longrightarrow \omega \quad \text{or} \quad \omega = f(z)$$

Sets $\mathcal{D}$ and $\mathcal{B}$ of complex numbers are called 'domain' and 'range', respectively.

The limit of a complex function is expressed like that of real functions

$$\lim_{z \to z_0} f(z) = \alpha \quad \text{or} \quad f(z) \longrightarrow \alpha \quad (z \to z_0)$$

which represents that the limit of the function $f(z)$ is α when z tends to z_0.

[Example]

- $\lim_{z \to 1+i} f(z) = \frac{1}{1+i}$ when $f(z) = \frac{1}{z}$
- $\lim_{z \to 1+i} f(z) = 1$ when $f(z) = \frac{1}{z-i}$

The limit of $\lim_{z \to 0} \frac{1}{z}$ is not still defined. Introducing the infinity ∞ in the complex plane formally, we write $\lim_{z \to 0} \frac{1}{z} = \infty$. Further, we can write $\lim_{z \to \infty} \frac{1}{z} = 0$.

The complex plane attached with the infinite ∞ is called the extended complex plane. However, in this study the extended complex plane is used as the complex plane, simply as z-plane[2].

A function $f(z)$ is continuous at $z = z_0$ if $\lim_{z \to z_0} f(z) = f(z_0)$ $(z_0 \in \mathcal{D})$ and a function $f(z)$ is continuous in $\mathcal{D}$ if $f(z)$ is continuous at all points in $\mathcal{D}$.

[2]Properties of operations including the infinity are (c is a constant)

$$\frac{c}{\infty} = 0 \quad c + \infty = \infty$$
$$\frac{c}{0} = \infty \quad c * \infty = \infty$$

Operations as $\infty + \infty, 0 * \infty, \frac{\infty}{\infty}, \frac{0}{0}$ are not defined.

[Example]

- $f(z) = z$ is continuous at all points in the z-plane.
- $f(z) = \frac{1}{z}$ is continuous at all points except for $z = 0$ in the z-plane. The particular point $z = 0$ in this case is called the isolated singular one or the singurality.

The term, holomorphic or regurality of complex function, is related with differentiability of complex function. Consider a function $f'(z)$, which is called the derivative of $f(z)$. We define $f'(z)$ as usual

$$f'(z) = \frac{df(z)}{dz} = \lim_{\Delta z \to 0} \frac{\Delta f(z)}{\Delta z}$$
$$= \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

which exists and is independent of the every path in which $\Delta z \to 0$ or $z \to z_0$. This strict condition of differentiability of complex function is essential as compared with that of real function.

Let us consider and discuss differentiability of complex function by using simple functions, which are $f(z) = z$ and $f(z) = \overline{z}$.

According to the definition of the derivative, we have

$$\frac{(z + \Delta z) - z}{\Delta z} = \frac{\{(x + \Delta x) - x\} + i\{(y + \Delta y) - y\}}{\Delta x + i\Delta y}$$
$$= \frac{\Delta x + i\Delta y}{\Delta x + i\Delta y} = 1$$

Therefore $f(z) = z$ is said to be differentiable at any point in z-plane.

On the other hand, for $f(z) = \overline{z} = x - iy$ it follows that

$$\frac{(z + \Delta z) - z}{\Delta z} = \frac{\{x + \Delta x) - x\} - i\{y + \Delta y) - y\}}{\Delta x + i\Delta y}$$
$$= \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

In this case, the limit depends on the path. For example, take $\Delta y = m\Delta x$ $(m = const.)$, as path, then we get

$$\frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} = \frac{\Delta x - im\Delta x}{\Delta x + im\Delta x}$$

$$= \frac{1-im}{1+im}$$

The limit is dependent on m. Hence, $f(z) = \overline{z}$ is not differentiable. A differentiable function is called holomorphic or regular function. Thus, holomorphic or regularity of complex function is said to be continuous and defferentiable at any point in the z-plane.

Now let us investigate the differentiable condition of complex function.

Let $f(z) = u(x,y) + iv(x,y)$. By the definition of the derivative of $f(z)$, we have

$$\lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = \lim_{\left\{\begin{array}{l}\Delta x \to 0 \\ \Delta y \to 0\end{array}\right.} \frac{\Delta u(x,y) + i\Delta v(x,y)}{\Delta x + i\Delta y}$$

$$= \alpha + i\beta$$

where α and β to be determined as follows are the real part and imaginary one of the derivative $f'(z)$, respectively. From these equations, we get

$$\Delta u(x,y) + i\Delta v(x,y) = (\alpha + i\beta)(\Delta x + i\Delta y)$$

$$= (\alpha\Delta x - \beta\Delta y) + i(\beta\Delta x + \alpha\Delta y)$$

From these, we can determine α and β. First, in the case of $\Delta z = \Delta x$ it follows that

$$\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \frac{\partial u}{\partial x} = u_x, \quad \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} = \frac{\partial v}{\partial x} = v_x$$

Consequently, we write

$$f'(z) = \alpha + i\beta = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = u_x + iv_x \qquad (5)$$

Next in the case of $\Delta z = i\Delta y$, it follows that

$$f'(z) = \alpha + i\beta = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = v_y - iu_y \qquad (6)$$

From the equations (5) and (6), we obtain

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad (u_x = v_y) \qquad (7)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad (u_y = -v_x) \qquad (8)$$

This relation is called Cauchy-Riemann's equations and we denote these as C-R for simplisity. From C-R, we obtain

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ and } \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \qquad (9)$$

where the functions $u(x,y)$ and $v(x,y)$ is called harmonic functions. Introducing the differential operator $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, which is called Laplacian, we can write these equations

$$\Delta u = \Delta v = 0 \qquad (10)$$

These equations are (two dimensional) Laplace equation or potential equation in math-physical science.

[**Example 1**] Express the form of C-R by the polar coordinate system (r, θ).

[**Solution**] Putting $x + iy = re^{i\theta}$ and then $dz = dre^{i\theta} + ire^{i\theta}d\theta$, we have

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

For $u + iv = Re^{i\Theta}$,

$$\frac{r}{R}\frac{\partial R}{\partial r} = \frac{\partial \Theta}{\partial \theta}, \quad \frac{1}{R}\frac{\partial R}{\partial \theta} = -r\frac{\partial \Theta}{\partial r}$$

[**Example 2**] Verify that $\frac{\partial f}{\partial \overline{z}} = 0 \Longleftrightarrow \frac{\partial \overline{f}}{\partial z} = 0$.

[**Solution**] Let $z = x + iy$ and $\overline{z} = x - iy$, then $x = \frac{z + \overline{z}}{2}$, $y = \frac{z - \overline{z}}{2i}$. Here z and $\overline{z}$ are formally treated independently though they are really dependent.

$$\frac{\partial x}{\partial z} = \frac{\partial x}{\partial \overline{z}} = \frac{1}{2}, \quad \frac{\partial y}{\partial z} = -\frac{\partial y}{\partial \overline{z}} = \frac{1}{2i}$$

$$\frac{\partial}{\partial z} = \partial_z = \frac{\partial x}{\partial z}\frac{\partial}{\partial x} + \frac{\partial y}{\partial z}\frac{\partial}{\partial y} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

$$\frac{\partial}{\partial \overline{z}} = \partial_{\overline{z}} = \frac{\partial x}{\partial \overline{z}}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \overline{z}}\frac{\partial}{\partial y} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

By substututing $f(z) = u(x,y) + iv(x,y)$ and $\overline{f(z)} = u(x,y) - iv(x,y)$ into the above expressions, respectively and then equalify, we get the required results:

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u + iv)$$

$$= \frac{1}{2}\{(u_x - v_y) + i(u_y + v_x)\}$$

$$\frac{\partial \overline{f}}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(u - iv)$$

$$= \frac{1}{2}\{(u_x + v_y) + i(-u_y + v_x)\}$$

These results show that if a function $f(z, \overline{z})$ is holomorphic, it depends on only z: $\partial_{\overline{z}} f = 0$. If a function $f(z)$ is holomorphic, the derivative of $f(z)$ is $\frac{\partial f}{\partial z} = \partial_z f = \frac{df}{dz}$.

[**Example** 3] Prove that $\Delta f = 0 \Longleftrightarrow \dfrac{\partial^2 f}{\partial z \partial \overline{z}} = 0$

[**Solution**] By using the proceding results, it follows that

$$\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

$$\frac{\partial^2}{\partial z \partial \overline{z}} = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = \frac{1}{4}\Delta$$

It should be noted that this statement does not hold for n-variables.

[**Example** 4] Verify that a function $f(z)$ is a constant if $|f(z)| = const.$.

[**Solution**] Let $f(z) = u(x,y) + iv(x,y)$ and $|f(z)| = \sqrt{u^2+v^2}$, then $u^2+v^2 = c^2$, $(c = const.)$. Differntiating both members of $u^2+v^2 = c^2$ by x (or y), we get

$$\begin{aligned} uu_x + vv_x &= 0 \\ uu_y + vv_y &\overset{\text{C-R}}{=} vu_x - uv_x = 0 \end{aligned}$$

The coefficients of u_x and v_x is not zero since $\begin{vmatrix} u & v \\ -u & v \end{vmatrix} = u^2+v^2 \neq 0$. Then, $u_x = v_x = 0$. In a similar manner, $u_y = v_y = 0$. From these, we obtain $f(z) = const.$

2.2 Comformal mapping

The differential coefficient of a differentiable real function which represents a curve C has the meaning of the tangent of that curve C. Here we shall investigate a geometrical meaning of the derivative of complex function.

From the definition of the derivative of complex function $f(z)$, we may write

$$\begin{aligned} f(z+h) - f(z) &= h(f'(z) + A(h)) \\ f(z+k) - f(z) &= k(f'(z) + B(k)) \end{aligned}$$

where $A(h)$ and $B(k) \longrightarrow 0$ when $h \longrightarrow 0$ and $k \longrightarrow 0$, respectively.

By taking the ratio of the above equations, it follows that

$$\frac{f(z+h) - f(z)}{f(z+k) - f(z)} = \frac{h}{k}\frac{f'(z) + A(h)}{f'(z) + B(k)}$$

Equating arguments of both members of the above equations, we have

$$\phi = \theta + Arg\left(\frac{f'(z) + A(h)}{f'(z) + B(k)}\right)$$

where

$$\frac{f'(z) + A(h)}{f'(z) + B(k)} \overset{\left\{ \substack{h \to 0 \\ k \to 0} \right.}{\longrightarrow} \frac{f'(z)}{f'(z)} = 1$$

and therefore

$$Arg\left(\frac{f'(z) + A(h)}{f'(z) + B(k)}\right) \longrightarrow 0$$

Consequently we get the required result: $\phi = \theta$.

The function or the mapping, $f : z \to \omega$ or $\omega = f(z)$ with angle-preserving is called 'comformal mapping'. In other words, the mapping defined by a holomorphic function $f(z)$ is comformal except at points where $f'(z) = 0$

Let us illustrate an important and interesting example of comformal mapping. Let

$$\omega = f(z) = \frac{\alpha z + \beta}{\gamma z + \delta} \quad \left(\alpha\delta - \beta\gamma = \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} \neq 0\right) (11)$$

ω is called a linear-fractional function or customarily a linear function. This mapping is comformal if $f'(z) = \dfrac{\alpha\delta - \beta\gamma}{(\gamma z + \delta)^2} \neq 0$.

It is convinient to understand the mapping of the linear fractional function that we may divide the equation (11) into the following parts

$$\omega = \frac{\alpha}{\gamma} - \frac{\alpha\delta - \beta\gamma}{\gamma^2}\frac{1}{z + \dfrac{\delta}{\gamma}} = C_1 + \frac{C_2}{z + C_3}$$

$$\left(C_1 = \frac{\alpha}{\gamma}, C_2 = -\frac{\alpha\delta - \beta\gamma}{\gamma^2}, C_3 = \frac{\delta}{\gamma}\right)$$

This mapping is a composite of the following four mapping

$$z \overset{(1)}{\longrightarrow} z + C_3 \overset{(2)}{\longrightarrow} \frac{1}{z + C_3}$$

$$\overset{(3)}{\longrightarrow} \frac{C_2}{z + C_3} \overset{(1)}{\longrightarrow} C_1 + \frac{C_2}{z + C_3}$$

where basic mappings which are denoted as (1), (2) and (3) represent shifting or translation, inversion and similarity or rotation, respectively. Of course, each basic mapping is comformal.

The forms of the matrices which correspond these basic mappings are given by

$$(1) \to \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \ (2) \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$(3) \to \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$$

where a is some constant.

To understand the meaning of linear mapping of linear fractional functions, let $z = \dfrac{z_1}{z_2}$ be $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ and let $\omega = \dfrac{\omega_1}{\omega_2}$ be $\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$. The mappings or transformations of the points between z and ω are defined by

$$\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \tag{12}$$

which is equivalent to the equation (11). By multiplying matrices which correspond to the composite of basic mappings described in (2.2), it follows that

$$\begin{pmatrix} 1 & C_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} C_2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & C_3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

[**Problem**] The cross ratio of four points z_1, z_2, z_3 and z_4 which are ordered, is defined as

$$\frac{\dfrac{z_1 - z_3}{z_1 - z_4}}{\dfrac{z_2 - z_3}{z_2 - z_4}} = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}$$

Verify that

$$\frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)} = \frac{(\omega_1 - \omega_3)(\omega_2 - \omega_4)}{(\omega_1 - \omega_4)(\omega_2 - \omega_3)}$$

where $\omega_i = f(z_i), \ i = 1, 2, 3, 4$.

2.3 Elementary holomorphic functions

Let a function $f(z)$ be holomorphic and let $f'(z) \neq 0$. If an equation $\omega = f(z)$ can be solved for z as a function of ω and this solution is written as $z = g(\omega)$, the function $g(z)$, where $g(\omega)$ and $g(z)$ really indicate the same function although the variables differ, is called an inverse function of $f(z)$ and is denoted as $g(z) = f^{-1}(z)$. The derivative of $f^{-1}(z)$ exists if $|f(z)| \neq 0$:

$$\frac{dg(z)}{dz} = \frac{df^{-1}(z)}{dz} = \frac{1}{\dfrac{df(\omega)}{d\omega}} \tag{13}$$

A function $f(z)$ is called biholomorphic when both functions $f(z)$ and $f^{-1}(z)$ are holomorphic.

As an example, the inverse function of $f(z) = \dfrac{\alpha z + \beta}{\gamma z + \delta}$ exists since $f'(z) \neq 0$ and its expression is

$$\omega \longrightarrow z = f^{-1}(\omega) = \frac{\delta\omega - \beta}{-\gamma\omega + \alpha} \tag{14}$$

Consider the function defined by $\omega = e^z$, which is called exponential function of z. The properties of the function e^z are in the following:

- (1) Law of exponent $e^{z_1 + z_2} = e^{z_1} e^{z_2}$
- (2) Periodicity $e^z = e^{z + i2n\pi}$ (n is any integer)
- (3) Derivative $\dfrac{de^z}{dz} = (e^z)' = e^z$

The above equation $\omega = e^z$ may be solved for z to give $z = \log \omega$. Therefore, the inverse function of e^z is $\log z$, which is called logarithmic function. According to the definition (13), the derivative of $\log z$ is

$$\frac{d \log z}{dz} = \frac{1}{\dfrac{de^\omega}{d\omega}} = \frac{1}{e^\omega} = \frac{1}{z}$$

Putting $z = re^{i(\theta + 2n\pi)}$ into $\log z$, we have

$$\omega = \log z = \log re^{i(\theta + 2n\pi)}$$

$$= \log r + i(\theta + 2n\pi)$$

which means that the logarithmic function is an infinite-valued one. However, if n fixes, the mapping $f : z \to \omega$ becomes one to one correspondense. Each is called branch. In like manner, writing $z = x + iy$ we have

$$\omega = \log z = \log(x+iy)$$
$$= \frac{1}{2}\log(x^2+y^2) + i\left(\tan^{-1}\frac{y}{x} + 2n\pi\right)$$

[**Example 1**] Find the following logarithm.

(1) $\log(-1)^2$ (2) $2\log(-1)$

[**Solution.**] From the above formula of the logarithm,

$$\begin{aligned}\log(-1)^2 &= \log 1 = \log e^{i2n\pi} = i2n\pi \\ 2\log(-1) &= 2\log e^{i(\pi+2n\pi)} = 2(i\pi + 2n\pi) \\ &= i(2+4n)\pi\end{aligned}$$

This result shows $\log(-1)^2 \neq 2\log(-1)$. In general, $\log z_1 z_2 \neq \log z_1 + \log z_2$.

Trigonometric and hyperbolic functions are writen by exponential functions;

$$\cos z = \frac{e^{iz}+e^{-iz}}{2} \quad \sin z = \frac{e^{iz}-e^{-iz}}{2i}$$

$$\cosh z = \frac{e^{z}+e^{-z}}{2} \quad \sinh z = \frac{e^{z}-e^{-z}}{2}$$

The some properties of trigonometric anf hyperbolic functions are in the following:

	trigonometric function
addition theorem	$\sin(z_1+z_2) = \sin z_1 \cos z_2 + \sin z_2 \cos z_2$
	$\cos(z_1+z_2) = \cos z_1 \cos z_2 - \sin z_2 \sin z_2$
derivative	$(\cos z)' = -\sin z$
squares	$\sin^2 z + \cos^2 z = 1$
periodicity	2π
zeros	$n\pi$ for $\sin z$ (n is any integer)
	$2n\pi \pm \frac{\pi}{2}$ for $\cos z$

	hyperbolic function
addition theorem	$\sinh(z_1+z_2) = \sinh z_1 \cosh z_2 + \sinh z_2 \cosh z_2$
	$\cosh(z_1+z_2) = \cosh z_1 \cosh z_2 + \sinh z_2 \sinh z_2$
derivative	$(\sinh z)' = \cosh z$
	$(\cosh z)' = \sinh z$
squares	$\sinh^2 z - \cosh^2 z = 1$
periodicity	2π
zeros	$in\pi$ for $\sinh z$
	$i\left(2n\pi \pm \frac{\pi}{2}\right)$ for $\cosh z$

We wish to indicate relations among some elementary functions in the following:

Trigonometric function (hyperbolic function)	$\longleftrightarrow$	Exponential functio
$\Updownarrow$		$\Updownarrow$
Inverse trigonometric function (Inverse hyperbolic function)	$\longleftrightarrow$	Logarithmic functio

[**Problem 1**] Solve the equations for z.

(1) $\cos z = 1$ (2) $\sin z = 2$ (3) $\tan z = 1$

[**Problem 2**] Show the following formulas.

(1) $\cos(iz) = \cosh(z)$ (2) $\sin(iz) = i\sinh(z)$

(3) $\tan(iz) = i\tanh(z)$ (4) $\cosh(iz) = \cos z$

(5) $\sinh(iz) = i\sin z$ (6) $\tanh(iz) = -i\tan z$

[**Problem 3**] Show the following formulas.

(1) $\cos^{-1} z = \frac{1}{i}\log(z \pm \sqrt{1-z^2})$ $\quad \frac{d\cos^{-1} z}{dz} = -\frac{1}{\sqrt{1-z^2}}$

(2) $\sin^{-1} z = \frac{1}{i}\log(iz \pm \sqrt{1-z^2})$ $\quad \frac{d\sin^{-1} z}{dz} = \frac{1}{\sqrt{1-z^2}}$

(3) $\tan^{-1} z = \frac{1}{2i}\log\frac{1+iz}{1-iz}$ $\quad \frac{d\tan^{-1} z}{dz} = -\frac{1}{1+z^2}$

(4) $\cosh^{-1} z = \log(z \pm \sqrt{z^2-1})$ $\quad \frac{d\cosh^{-1} z}{dz} = -\frac{1}{\sqrt{z^2-1}}$

(5) $\sinh^{-1} z = \log(z \pm \sqrt{z^2+1})$ $\quad \frac{d\sinh^{-1} z}{dz} = \frac{1}{\sqrt{z^2+1}}$

(6) $\tanh^{-1} z = \frac{1}{2}\log\frac{1+z}{1-z}$ $\quad \frac{d\tanh^{-1} z}{dz} = -\frac{1}{1-z^2}$

Let us show some examples of the mappings of elementary functions:

[Example 2] Find the mappings of the following functions and draw graphs of them.

$$(1)\ \omega = z\ (2)\ \omega = \frac{1}{z}\ (3)\ \omega = e^z\ (4)\ \omega = \cos z$$

[Solution]

(1) Let z and ω be $x+iy$ and $u(x,y)+iv(x,y)$, respectively. Rewriting the equation $\omega = z$ gives $u+iv = x+iy$. Then, we obtain

$$u = x,\quad v = y$$

(2) $u+iv = \dfrac{1}{x+iy}$. By similar method in (1), we have

$$\begin{cases} u = \dfrac{x}{x^2+y^2} \\ v = \dfrac{-y}{x^2+y^2} \end{cases} \text{and} \begin{cases} x = \dfrac{u}{u^2+v^2} \\ y = \dfrac{-v}{u^2+v^2} \end{cases}$$

$$a = \frac{u}{u^2+v^2} \longrightarrow \left(u+\frac{1}{2a}\right)^2 + v^2 = \left(\frac{1}{2a}\right)^2$$
$$b = \frac{-v}{u^2+v^2} \longrightarrow u^2 + \left(v-\frac{1}{2b}\right)^2 = \left(\frac{1}{2b}\right)^2$$

(3) $u+iv = e^{x+iy} = e^x(\cos y + i\sin y)$

$$\begin{cases} u = e^x\cos y \\ v = e^x \sin y \end{cases}$$

when $x = a \longrightarrow u^2+v^2 = (e^a)^2 = e^{2a}$

when $y = b \longrightarrow v = u\tan b$

(4) $\cos z = \dfrac{1}{2}(e^{iz}+e^{-iz})$

$$u+iv = \cos x\cosh y - i\sin x\sinh y$$

$$u = \cos x\cosh y,\ v = -\sin x\sinh y$$

when $x = a \longrightarrow \left(\dfrac{u}{\cos a}\right)^2 - \left(\dfrac{v}{\sin a}\right)^2 = 1$

when $y = b \longrightarrow \left(\dfrac{u}{\cosh b}\right)^2 + \left(\dfrac{v}{\sinh b}\right)^2 = 1$

[Problem 4] Find the mappings of the following functions and draw graphs of them.

$$(1)\omega = z+\frac{1}{z}\ (2)\ \omega = \log z\ (3)\ \omega = \sin z$$

3 Integrals of complex functions

3.1 Line intergals and Cauchy's integral theorem

We shall first define line integral, denoted as $\int_C f(z)dz$, of a complex function $f(z)$ in the z-plane. $\int_C \cdots dz$ represents the integral along a curve C. Now divide C into n portions in the range $[z_0, z_n]$, where z_0 and z_n are terminal points, e.g. starting-point and end-point, respectively. The curve C here has no crossing and then is called a simple conncted one. Let ζ_i be between each portion, z_{i-1} and z_i. we consider a sum

$$\sum_{i=1}^{n} f(\zeta_i(z_i - z_{i-1}) = \sum_{i=1}^{n} f(\zeta_i)\Delta z_i$$

$$\overset{n\to\infty}{\Longrightarrow} \oint_C f(z)dz \qquad (1)$$

The curve C is called the path of integration. In a particular case, when C is closed, it is called a contour integral and its notation of integral along C is $\oint_C \cdots dz$, or simply $\oint \cdots dz$.

The value of integral (1) depends on the form of $f(z)$, and not on the particular way of dividing the range into n portions. Then we may write

$$\int_C f(z)dz = \int_{z_0}^{z_n} f(z)dz = F(z_n) - F(z_0)$$

where $F(z)$ is indenifite integral of $f(z)$; $\dfrac{dF(z)}{dz} = f(z)$.

Next, we shall make real forms of $\int_C f(z)dz$, where $f(z) = u(x,y)+iv(x,y)$.

$$\int_C f(z)dz = \int_C (u(x,y)+iv(x,y))(dx+idy)$$

$$= \int_C (udx - vdy) + i\int_C (udy+vdx) \qquad (2)$$

Let the equation with the parameter t of the curve C be

$$z(t) = x(t) + iy(t)$$

and then

$$\int_C f(z)dz = \int_C f(z(t))z'(t)dt$$

When $f(z) = 1$, then

$$\int_C z'(t)dt = \int_{z_0}^{z_n} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

which is the length of the curve C.

Let us show and discuss some properties of line integrals:

(1) Liniality

$$\int_C (af(z)+bg(z))dz = a\int_C f(z)dz + b\int_C g(z)dz \quad (3)$$

(2) Distribution low

$$\int_{C_1+C_2} f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz \quad (4)$$

(3) Inverse element

$$\int_{-C} f(z)dz = -\int_C f(z)dz \quad (5)$$

(4) Cauchy's integral theorem

$$\oint_C f(z)dz = 0 \quad (6)$$

where $f(z)$ has no singular points inside C.

(5) Extension of Cauchy's integral theorem

$$\oint_C f(z)dz = \sum_{i=1}^{n} \oint_{C_i} f(z)dz \quad (7)$$

where $\oint_{C_i}$ denote integrals taken counter-clockwise along the circles C_i.

(1), (2) and (3) are obvious. About (4), let $f(z)$ be holomorphic in a domain D and then from (2),

$$\int_C f(z)dz = \int_C (udx - vdy) + i\int_C (udy + vdx)$$

$$= -\int\int \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) dxdy$$

$$+i\int\int \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy \overset{\text{C-R}}{=} 0$$

Reversely, if $\oint f(z)dz = 0$, then it can be said $f(z)$ is holpmorphic (Morera's theorem).

Let rewrite in a manner by differential form. It is more convienient to understand it.

$$\phi = f(z)dz = (u+iv)(dx+idy) = \phi_1 + i\phi_2$$

where

$$\phi_1 = udx - vdy, \quad \phi_2 = vdx + udy$$

Then, it follows that

$$\begin{aligned} d\phi &= d\phi_1 + id\phi_2 \\ &= -\int\int \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) dxdy \\ &+ i\int\int \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy \end{aligned}$$

where $dxdx - dydy = 0$ and $dxdy = -dydx$. According to the C-R relation, $d\phi$ has to be zero, that is, $d\phi = 0$. Hence, we have

$$\int_C f(z)dz = \int_C \phi = \int_D d\phi = 0$$

On the other hand, we use $f(z, \bar{z})$ instead of $f(z)$

$$df(z,\bar{z}) = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z}$$

$$\int_C f(z)dz = \int_D df dz = \int_D \frac{\partial f}{\partial \bar{z}} dz d\bar{z} \overset{\text{C-R}}{=} 0$$

where $dzdz = d\bar{z}d\bar{z} = 0$. Then, we get again

$$\int_C f(z)dz = 0$$

This is the desired result. It is obvious that

$$\int_C f(z)dz - \sum_{i=1}^{n} \int_{C_i} f(z)dz = 0$$

Therefore we obtain (5).

3.2 Cauchy's integral formula and Taylor series

Let α be any point inside C in a domain of z-plane. Assuming that a function $f(z)$ is holomorphic at any point inside C, we now consider a function $\dfrac{f(z)}{z-\alpha}$, which is holomorphic on and inside C except the point $z = \alpha$. Such a point is call the isolated singular one.

We can write

$$\int_C \frac{f(z)}{z-\alpha}dz = \int_{C'} \frac{f(z)}{z-\alpha}dz \qquad (8)$$

where C' is a circle with a center α inside C.

Replacing $z-\alpha$ by $re^{i\theta}$, then $dz = ire^{i\theta}$, we obtain

$$\begin{aligned}\int_C \frac{f(z)}{z-\alpha}dz &= \int_0^{2\pi} \frac{f(\alpha + re^{i\theta})}{re^{i\theta}} ire^{i\theta}d\theta \\ &= i\int_0^{2\pi} f(\alpha + re^{i\theta})d\theta \\ &\overset{r\to 0}{\longrightarrow} i2\pi f(\alpha)\end{aligned}$$

Therefore,

$$f(\alpha) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-\alpha}dz \qquad (9)$$

This formula shows that the value of the function $f(\alpha)$ at $z=\alpha$ is represented by the line intergal along the coutour C.

If the point α inside C is arbitrary, the derivative of $f(z)$ at the point $z=\alpha$ can be written

$$\begin{aligned}f'(\alpha) &= \int_C \frac{f(z)}{(z-\alpha)^2}dz \\ f''(\alpha) &= \int_C \frac{f(z)}{(z-\alpha)^3}dz \\ \vdots &= \vdots \\ f^{(n)}(\alpha) &= \int_C \frac{f(z)}{(z-\alpha)^{n+1}}dz\end{aligned}$$

These show that if a function $f(z)$ has a first derivate with respect to the complex variable at any point inside C, it has derivates of all orders at any point inside C, in other words, a function $f(z)$ has successive differentiations.

3.2.1 Taylor's series

Using above notations, we can expand a function $f(z)$ at any point $z=\alpha$ inside C into power series

$$\begin{aligned}f(z) &= \sum_{n=0}^{\infty} a_n(z-\alpha)^n \\ &= a_0 + a_1(z-\alpha) + a_2(z-\alpha)^2 + \cdots\end{aligned}$$

where

$$a_n = \frac{f^{(n)}(\alpha)}{n!} = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-\alpha)^{n+1}}d\zeta$$

Let C' be a circle with a center α inside C and z is inside it. When $|\zeta-\alpha| > |z-\alpha|$ $(\zeta \in C')$, it follows that

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta-\alpha)-(z-\alpha)} = \frac{1}{\zeta-\alpha}\frac{1}{1-\dfrac{z-\alpha}{\zeta-\alpha}}$$

$$= \frac{1}{\zeta-\alpha}\left(1 + \frac{z-\alpha}{\zeta-\alpha} + \left(\frac{z-\alpha}{\zeta-\alpha}\right)^2 + \left(\frac{z-\alpha}{\zeta-\alpha}\right)^3 + \cdots\right)$$

$$= \sum_{n=0}^{\infty} \frac{(z-\alpha)^n}{(\zeta-\alpha)^{n+1}}$$

Put this result into the above series, we have

$$\begin{aligned}f(z) &= \frac{1}{2\pi i}\sum_{n=0}^{\infty}\int_C f(\zeta)\frac{(z-\alpha)^n}{(\zeta-\alpha)^{n+1}}d\zeta \\ &= \sum_{n=0}^{\infty} a_n(z-\alpha)^n \qquad (10) \\ a_n &= \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-\alpha)^{n+1}}d\zeta \qquad (11)\end{aligned}$$

This formula is call Taylor's series. A function $f(z)$, which can be expanded into power series, is called an analytic function.

[**Example 1**] Taylor series is written in the form of the polar coordinate $z-\alpha = re^{i\theta}$

$$f(z) = \sum_{n=0}^{\infty} a_n(re^{i\theta})^n = \sum_{n=0}^{\infty} a_n r^n e^{in\theta}$$

where $a_n = \frac{1}{2\pi}\int_0^{2\pi} f(\alpha + re^{i\theta})e^{-in\theta}d\theta$. Prove the following Perseval's equality

$$\frac{1}{2\pi}\int_0^{2\pi} |f(\alpha + re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |a_n|^2 r^{2n} \qquad (12)$$

[**Solution**]

$$\frac{1}{2\pi}\int_0^{2\pi} |f(\alpha + re^{i\theta}|^2 d\theta = \frac{1}{2\pi}\int_0^{2\pi} f\overline{f}d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \sum_{m=0}^{\infty} a_m(z-\alpha)^m \sum_{n=0}^{\infty} \overline{a_n(z-\alpha)^n}d\theta$$

$$= \frac{1}{2\pi}\sum_{m=0}^{\infty} a_m r^m \sum_{n=0}^{\infty} \overline{a_n} r^n \int_0^{2\pi} e^{im\theta}e^{-in\theta}d\theta$$

$$= \sum_{n=0}^{\infty} |a_n|^2 r^{2n} \quad \left(\int_0^{2\pi} e^{im\theta}e^{-in\theta}d\theta = 2\pi\delta_{mn}\right)$$

which gives the required result.

[**Example 2**] Verify that a function $f(z)$ is constant when $f(z)$ is holomorphic and $|f(z)|$ has maximum in domain $\mathcal{D}$.

[**Solution**] $f(z)$ has the maximum $M = f(\alpha)$ ($z \in \mathcal{D}$) and can be expanded into $\sum_{n=0}^{\infty} a_n(z-\alpha)^n$ near $z = \alpha$. It follows that the square expansion coefficients are

$$|a_0|^2 + \sum_{n=0}^{\infty} |a_m|^2 r^{2n} \leq M^2 = |a_0|^2$$

since $a_0 = f(\alpha) = M$. Hence the coefficients must be zero. Then we get $|f(z)| = a_0$=const..

[**Example 3**] Verify that a function $f(z)$ is constant if $f(z)$ is holomorphic and bounded in the z-plane (Liouville's theorem).

[**Solution**] Take $|f(z)| \leq M$, which is the maximum of $f(z)$, since $f(z)$ is bounded. It follows that

$$|a_n| = \left| \frac{1}{2\pi} \int_0^{2\pi} f(\alpha + re^{i\theta}) e^{-in\theta} d\theta \right|$$

$$\leq \frac{M}{2\pi r^n} 2\pi = \frac{M}{r^n} \overset{r\to\infty}{\longrightarrow} 0$$

Then we obtain $f(z) = a_0 =$ const.. A holomorphic and bounded function $f(z)$ is called "entire function".

[**Example 4**] If an algebraic equation $f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ is not a constant, then $f(z) = 0$, for at least one solution of z ("Fundamental theorem of algebra" by Gauss).

[**Solution**] Assuming that $f(z) \neq 0$ for all z, we make a function $g(z) = \frac{1}{f(z)}$, which is an entire function discribed above. It follows that

$$\lim_{z\to\infty} g(z) = \lim_{z\to\infty} \frac{1}{f(z)} = 0 \quad \text{since,} \quad \lim_{z\to\infty} f(z) = 0$$

According to Liouville's theorem, $f(z)$ must be a constant. However, this conclusion is in contradiction with the assumption. Hence, It is concuded that the algebraic equation has always at least one solution.

3.3 Laurent series and Residue

Taylor's expansion of a function $f(z)$ cannot be applied where $f(z)$ has singular points. Let C_1 and C_2 be two concentric circles with a center. Assuming that a function $f(z)$ is holomorphic in the annulus between C_1 and C_2, we can expand $f(z)$ into Laurent's series, which will be shown as follows:

Let C' be in the annulus between C_1 and C_2. Now consider a function

$$f(z) = \frac{1}{2\pi i} \int_{C'} \frac{f(z)}{\zeta - z} dz$$

According to the theorem (7),

$$\int_{C_1} = \int_{C_2} + \int_{C'} \longrightarrow \int_{C'} = \int_{C_2} - \int_{C_1}$$

Hence, we rewrite

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{C_1} \frac{f(z)}{\zeta - z} dz - \frac{1}{2\pi i} \int_{C_2} \frac{f(z)}{\zeta - z} dz \\ &= (I) + (II) \end{aligned}$$

Here, the expansion of (I) is the same as Taylor series described above. Now consider the expansion of (II), where $|\zeta - \alpha| < |z - \alpha|$. This enequaity is inverse in the case of expansion of (I), Taylor series. The expansion of $\frac{1}{\zeta - z}$ is by a prevoiusly similar expansion

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - \alpha) - (z - \alpha)} = \frac{-1}{z - \alpha} \frac{1}{1 - \frac{\zeta - \alpha}{z - \alpha}}$$

$$= \frac{-1}{z - \alpha} \left(1 + \frac{\zeta - \alpha}{z - \alpha} + \left(\frac{\zeta - \alpha}{z - \alpha} \right)^2 + \left(\frac{\zeta - \alpha}{z - \alpha} \right)^3 + \cdots \right)$$

$$= -\sum_{n=0}^{\infty} \frac{(\zeta - \alpha)^{n-1}}{(z - \alpha)^n}$$

Therefore, it follows that

$$\begin{aligned} (II) &= -\frac{1}{2\pi i} \sum_{n=1}^{\infty} \int_C f(\zeta) \frac{(\zeta - \alpha)^{n-1}}{(z - \alpha)^n} d\zeta \\ &= -\sum_{n=1}^{\infty} b_n \frac{1}{(z - \alpha)^n} \qquad (13) \\ b_n &= \frac{1}{2\pi i} \int_C f(\zeta)(\zeta - \alpha)^{n+1} d\zeta \qquad (14) \end{aligned}$$

Then, it follows that

$$f(z) = (I) + (II) = \sum_{n=0}^{\infty} a_n(z-\alpha)^n - \sum_{n=0}^{\infty} b_n \frac{1}{(z-\alpha)^n}$$

where

$$a_n = \frac{1}{2\pi i}\sum_{n=0}^{\infty}\int_C \frac{f(\zeta)}{(\zeta-\alpha)^{n+1}}d\zeta,$$

$$b_n = \frac{1}{2\pi i}\sum_{n=1}^{\infty}\int_C f(\zeta)(\zeta-\alpha)^{n-1}d\zeta$$

Instead of the above expressions we may simply write

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-\alpha)^n, \qquad (A)$$

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-\alpha)^{n+1}}d\zeta \qquad (B)$$

which is called Laurent series.

[Example 2] Find the values of

$$\oint_C (z-z_0)^n dz = \begin{cases} 2\pi i & (n=-1) \\ 0 & (n \neq -1) \end{cases}$$

where n is any integer and a contour C is the circle with the radius r and the center $z = z_0$.

[Solution.] If $n \geq 0$, we obtain $\oint (z-z_0)^n dz = 0$ accoding to Cauchy's intergal theorem. When $n < 0$, replacing $z - z_0$ by $re^{i\theta}$, and then $dz = ir^{i\theta}d\theta$,

$$\oint (z-z_0)^n dz = \int_0^{2\pi} \frac{ire^{i\theta}}{(re^{i\theta})^n}d\theta = \begin{cases} 2\pi i & (n=-1) \\ 0 & (n \neq -1) \end{cases}$$

This is the required results.

[Example 3] Prove that $\int_0^{\infty} \frac{\sin x}{x}dx = \frac{\pi}{2}$

[Solution] Consider $\oint_C \frac{e^{iz}}{z}dz$, where a contour is taken around the boundary of semi ring-shaped region in upper z-plane as shown in Fig.(). According to Cauchy's integral theorem,

$$\oint_C = \int_{AB} + \int_{\widehat{BC}} + \int_{CD} + \int_{\widehat{DA}} = 0$$

where

$$\int_{\widehat{AB}} \overset{R\to\infty}{\longrightarrow} 0 \quad \text{and} \quad \int_{\widehat{DA}} \overset{r\to 0}{\longrightarrow} -i\pi$$

$$\int_{AB} = \int_r^R \frac{e^{ix}}{x}dx$$

$$\int_{CD} = -\int_{DC} = -\int_{-r}^{-R} \frac{e^{-ix}}{x}dx = -\int_r^R \frac{e^{-ix}}{x}dx$$

Then, we get

$$\int_{AB} + \int_{CD} + \int_{\widehat{DA}} = 0 \; \overset{\begin{cases} R\to\infty \\ r\to 0\end{cases}}{\longrightarrow}$$

$$2i\int_0^{\infty} \frac{\sin x}{x}dx - i\pi = 0$$

Therefore, we reached the required result.

[Example 4] Prove that $\oint_{-\infty}^{\infty} e^{-(x+ic)^2} = \sqrt{\pi}$ and that $\int_0^{\infty} e^{-x^2}\cos 2cx dx = \sqrt{\pi}e^{-c^2}$

[Solution] Consider $\oint_C e^{-z^2}dz$, where a contour is taken around the boundary of the rectangle ABCD. According to Cauchy's integral theorem,

$$\oint_C = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA} = 0$$

where

$$\int_{AB} \text{ and } \int_{CD} \overset{R\to\infty}{\longrightarrow} 0$$

$$\int_{DA} = \int_{-R}^{R} e^{-x^2}dx \overset{R\to\infty}{\longrightarrow} \sqrt{\pi}$$

$$\int_{BC} = -\int_{CB} = -\int_{-R+ic}^{R+ic} e^{-(x+ic)^2}dx$$

$$\overset{\begin{cases} R\to\infty \\ c\to 0\end{cases}}{\longrightarrow} \; -\int_{-\infty}^{\infty} e^{-(x+ic)^2}dx$$

Consequently, we have

$$\int_{-\infty}^{\infty} e^{-(x+ic)^2} = \sqrt{\pi}$$

Comparing between real parts of both members in the above equation, we obtain

$$\int_0^{\infty} e^{-x^2}\cos 2cx dx = \sqrt{\pi}e^{-c^2}$$

[Example 5] Prove that $\int_{-\infty}^{\infty} \begin{matrix}\cos\\ \sin\end{matrix}\, x^2 dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$

[**Solution.**] Consider $\oint_C e^{-z^2}dz$, where a contour is taken around the boundary of the cone-shaped region OAB. According to Cauchy's integral theorem,

$$\oint_C = \int_{OA} + \int_{\widehat{AB}} + \int_{BC} = 0$$

where

$$\int_{\widehat{AB}} \overset{R\to\infty}{\longrightarrow} 0$$

$$\int_{OA} = \int_0^R e^{-x^2}dx \overset{R\to\infty}{\longrightarrow} \frac{\sqrt{\pi}}{2}$$

$$\int_{BO} = -\int_{OB} = -\int_0^R e^{-\left(re^{i\frac{\pi}{4}}\right)^2} e^{i\frac{\pi}{4}}dr$$

$$= \frac{1+i}{\sqrt{2}}\int_0^R e^{-ir^2}dr$$

$$\overset{R\to\infty}{\longrightarrow} -\int_0^\infty \left(\cos r^2 - i\sin r^2\right)dr$$

From the equation $\int_{OA} + \int_{BO} = 0$, we obtain the required result.

3.3.1 Classification of Singularities

The negative expansion of Laurent series $\sum_{n=1}^{\infty} b_n \frac{1}{(z-\alpha)^n}$ is called the principal part of f(z) near $z=\alpha$. Using this term, we may classify singular points as follows:

(I) when there is no negative power series in Laurent series, the point $z=\alpha$ ia called a removable singular one.

(II) when $n=k$ (finite series), the point $z=\alpha$ is called a pole and k represents the order of the pole.

(III) when $n=\infty$ (infinite series), the point $z=\alpha$ is called an essential singular one.

[**Example 1**] Expand the following functions into Laurent series with the point $z=0$ and Consider about classification of singular points.

(1) $\sin\frac{1}{z}$ (2) $\frac{\sin z}{z}$ (3) $\frac{\sin z}{z^2}$

[**Solution**]

$$(1) \quad \sin\frac{1}{z} = \frac{1}{z} - \frac{1}{3!}\frac{1}{z^3} + \frac{1}{5!}\frac{1}{z^5} - \cdots$$

$z=0$ is the essential singular point since the principal part is infinite.

$$(2) \quad \frac{\sin z}{z} = \frac{1}{z}\left(z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 - \cdots\right)$$

$$= 1 - \frac{1}{3!}z^2 + \frac{1}{5!}z^4 - \cdots$$

$z=0$ is the removable singular point since the series has no principal part.

$$(3) \quad \frac{\sin z}{z^2} = \frac{1}{z^2}\left(z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 - \cdots\right)$$

$$= \frac{1}{z} - \frac{1}{3!}z + \frac{1}{5!}z^3 - \cdots$$

$z=0$ is the first order of pole or is called a simple pole.

3.3.2 Residue theorem

By integrating both members in (A), we get

$$\int_C f(z)dz = a_n \sum_{n=-\infty}^{\infty} \int_C (z-\alpha)^n dz$$

$$= \begin{cases} 2\pi i a_{-1} & (n=-1) \\ 0 & (n\neq -1) \end{cases} \tag{15}$$

Here, the expansion coefficient a_{-1} of the principal part is called residue at $z=\alpha$. The residue of a function $f(z)$ is denoted by $Res[f(z),\alpha]$, or simply $Res[\alpha]$.

$$\frac{1}{2\pi i}a_{-1} = \frac{1}{2\pi i}Res[f(z),\alpha] = \int_C f(z)dz \tag{16}$$

This statement can be expanded for many singular points α_i $(i=1,2,\ldots)$ of $f(z)$ inside C;

$$\int_C f(z)dz = 2\pi i \sum a_{-1} = 2\pi i \sum_{n=1} Res[f(z),\alpha_n] \tag{17}$$

which is called residue theorem.

The residue of $f(z)$ with a simple pole is given by

$$a_{-1} = \lim_{z\to\alpha}(z-\alpha)f(z) \tag{18}$$

and with the k-th order of a pole by

$$a_{-1} = \frac{1}{(k-1)!}\lim_{z\to\alpha}\frac{d^{(k-1)}}{dz^{(k-1)}}\left[(z-\alpha)^k f(z)\right] \tag{19}$$

(平成13年11月20日受理)