On the Semiranked Group (I)

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Synopsis

In this paper we will give a definition of the SR-group (namely, Semiranked Group) that is a new notion, and will attempt its general theory.

Introduction:

An abstract space with a mathematical structure[1] S is called S-space. If so, *What is the method of R-spaces?*[2] It is to replace the structure T in the T-space (i.e. Topological space) with the structure R.

In this paper we will define a new notion, SR-group, by the same method as is taken in the definition of the semitopological group.[3] We shall use the same terminology that is introduced in [1] and [2]. And throughout this paper, we shall treat only R-spaces with indicater ω_0. We shall denote the point of an R-space by $x, y, z, \cdots$, the family of neighborhoods of x with rank n by $\mathfrak{B}_n(x)$, and fundamental sequences of neighborhoods with respect to x[4] by $\{u_n(x)\}$, $\{v_n(x)\}, \cdots$.

§1. Continuous, Homeomorphism.

In this section we will define two new notions, r-continuous, and r-homeomorphism.

Definition 1. r-continuous.

Let G and H be two R-spaces. A mapping f of G into H is said to be r-continuous if it satisfy next condition:

(**) for each $x \in G$ and any $\{u_n(x)\}$, there exists a $\{v_n(f(x))\}$ such that $f(u_n(x)) \subseteq v_n(f(x))$.

Remark 1. (**) implies if $x \in \{\lim_n x_n\}$ then $f(x) \in \{\lim_n f(x_n)\}$.

Definition 2. r-homeomorphism, r-homeomorphic.

Let G and H be two R-spaces with same indicater ω_0. A mapping f of G onto H is said to be r-homeomorphism if it satisfies next conditions:

1) f is a bijection.[5]

2) f is (bi-) continuous.

3) for any $\{u_n(x)\}$, $\{v_n(f(x))\}$ (such that $v_n(f(x)) \equiv f(u_n(x))$) is a fundamental sequence of neighborhoods with respect to $f(x) \in H$.

If there is a homeomorphism between two R-spaces, then they are called homeomorphic with each other.

§2. The definition of SR-group and R-group.

Definition 3. (i) An R-space G that is also a group is called a SR-group (i.e. Semiranked group) if the operation $(x, y) \to xy$ is continuous as follows:

(a) Let x, y be $\forall x, y \in G$. Then for any $\{u_n(x)\}$, $\{v_n(y)\}$, there exists a $\{w_n(xy)\}$ such that $u_n(x) v_n(y) \subseteq w_n(xy)$.

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1) [12].

2) [1], [2].

3) [8].

4) [2], II, p. 551.

5) J. Dieudonné: Foundations of Modern Analysis. Academic Press, New York, 1960, p. 45.

(ii) An R-space G that is also a group is called an R-group[6] (i. e. Ranked group) if the mapping $(x,y) \to xy^{-1}$ is continuous as follows:

(b) Let x, y be $\forall x,y \in G$. Then for any $\{u_n(x)\}$, $\{v_n(y)\}$, there exists a $\{w_n(xy^{-1})\}$ such that $u_n(x)v_n(y)^{-1} \subseteq w_n(xy^{-1})$.

Remark 2. (a) implies that, if $x \in \{\lim_n x_n\}$ and $y \in \{\lim_n y_n\}$, then $xy \in \{\lim_n x_n y_n\}$. (b) implies that, if $x \in \{\lim_n x_n\}$, $y \in \{\lim_n y_n\}$, then $xy^{-1} \in \{\lim_n x_n y_n^{-1}\}$.

Remark 3. (b)$\Leftrightarrow$[5] (I)-(II), p. 246.

Evidently, we get following proposition:

Proposition 1. Every R-group is a SR-group. But the converse is not true.

Theorem 1. Let a be a fixed element of a SR-group G. Then the mappings

$$r_a:\ x \to xa\,, \qquad l_a:\ x \to ax$$

of G onto G are homeomorphisms of G.

Proof. It is clear that r_a is a one-to-one and onto mapping. Since G is a SR-group, for any $\{u_n(x)\}$, $\{v_n(a)\}$ there exists a $\{w_n(xa)\}$ such that $u_n(x)v_n(a) \subseteq w_n(xa)$. Moreover $r_a(u_n(x)) = u_n(x)a \subseteq u_n(x)v_n(a) \subseteq w_n(xa) = w_n(r_a(x))$. Hence, r_a is continuous. By the same argument, $r_a^{-1}: x \to xa^{-1}$ is continuous.

Furthermore, $\{r_a(u_n(x))\}$ is a fundamental sequence of neighborhoods with respect to $xa \in G$. Therefore, r_a is a homeomorphism. The fact that l_a is a homeomorphism follows similarly. (Q. E. D.)

Definition 4. Translation. r_a and l_a are, respectively, called the right and left translation of G.

Corollary 1. Let O be an r-open,[7] F an r-closed,[8] and A any subset of a SR-group G and let $a \in G$. Then:

(i) Oa, aO, AO and OA are r-open.

(ii) Fa, aF are r-closed.

Proof. Since the mappings in Theorem 1 are homeomorphisms, (i) is obvious. By the same argument, Fa and aF are r-closed in (ii).

Since $AO = \bigcup_{a \in A} aO$, $OA = \bigcup_{a \in A} Oa$, and the union of r-open sets is r-open. (Q. E. D.)

Therefore,

Remark 4. r_a and l_a can be considered r-open and r-closed mappings.

Corollary 2. Let G be a SR-group. For $\forall x_1, x_2 \in G$, there exists a homeomorphism of G such that $f(x_1) = x_2$.

Namely, G is *homogeneous*[9]

Proof. Let $x_1^{-1}x_2 = a \in G$, and consider the mapping $f: x \to xa$. (Q. E. D.)

Theorem 2. If SR-group G satisfying F. Hausdorff's axiom (C)[10] is complete,[11] then G is of the second Category.

§ 3. The neighborhoods of identity of a SR-group.

Let G be a SR-group, and e be its identity. ε_n will denote the family of neighborhoods of e with rank n, and $\{U_n\}$, $\{V_n\}$, $\cdots$ fundamental sequences of neighborhoods with respect to e.

The system $\{\varepsilon_n\}$ possesses the following properties:

(A) for every V in ε, $e \in V$ (where $\varepsilon = \bigcup_{n=0}^{\infty} \varepsilon_n$.)

(B) for any U, V in ε, there is a W in ε such that $W \subseteq U \cap V$.

(α) for any V in ε and for any integer n, there is a m, $m \geqslant n$, and a U in ε_m such that $U \subseteq V$.

6) [5].
7), 8) [7], II, p. 788.
9) [14], p. 28.
10) F. Hausdorff: Grundzüge der Mengenlehre, 1914, p. 213.
11) [1], I, pp. 554–555.

(β) $G \in \varepsilon_0$.

These are obvious as the properties of neighborhoods in an R-space. This $\{\varepsilon_n\}$ has introduced in [5]. We shall call this system $\{\varepsilon_n\}$ a *fundamental system* of neighborhoods of e.

Furthermore, from (α), we get following properties:

(SR_1) For any $\{U_n\}$, $\{V_n\}$, there exists a $\{W_n\}$ such that $U_n V_n \subseteq W_n$.

(SR_2) For any $\{U_n\}$ and for any $x \in G$, there exists a $\{V_n\}$ such that $xU_n x^{-1} \subseteq V$.

($SR_3\, l$) (resp. ($SR_3\, r$)) Let x be any point of G. For any $\{U_n\}$ there exists a $\{v_n(x)\}$ such that $xU_n \subseteq v_n(x)$ (resp. $U_n x \subseteq v_n(x)$), and, conversely, for any $\{u_n(x)\}$, there exists a $\{V_n\}$ such that $u_n(x) \subseteq xV_n$ (resp. $u_n(x(\subseteq V_n x)$.

Proof. (SR_1) is immediate consequences of (α), putting $x=y=e$. We shall prove ($SR_3\, l$). Let $\{u_n(x)\}$ be some fundamental sequence of neighborhoods with respect to x. Because of (α), there is a $\{v_n(x)\}$ such that $u_n(x)\, U_n \subseteq v_n(x)$.

Since $x \in u_n(x)$, $xU_n \subseteq v_n(x)$. Conversely, taking some fundamental sequence of neighborhoods with respect to x^{-1}, say $\{v_n(x^{-1})\}$, and applying (α), there exists a $\{V_n\}$ such that $v_n(x^{-1})\, u_n(x) \subseteq V_n$. Since $x^{-1} \in v_n(x^{-1})$, $x^{-1}u_n(x) \subseteq V_n$, i.e. $u_n(x) \subseteq xV_n$. Similarly we can prove ($SR_3\, r$).

Next, we shall prove (SR_2). For any $\{U_n\}$ and for any $x \in G$, because of ($SR_3\, l$), we get a $\{v_n(x)\}$ such that $xU_n \subseteq v_n(x)$.

Then, from ($SR_3\, r$), there exists a $\{V_n\}$ such that $v_n(x) \subseteq V_n x$. Hence, $xU_n x^{-1} \subseteq V_n$.

Remark 5. (α) follows from (SR_1), (SR_2), ($SR_3\, l$), (or ($SR_3\, r$)). Therefore the three conditions above are not only necessary, but sufficient for a group G which is also an R-space to be a SR-group.

Proof. The proof is similar in [5]:

Take any $\{u_n(x)\}$, $\{v_n(y)\}$. From ($SR_3\, l$) and ($SR_3\, r$), there are $\{U_n\}$, $\{V_n\}$ such that $u_n(x) \subseteq xU_n$, $v_n(y) \subseteq V_n y$. Applying (SR_1), we get a $\{W_n\}$ such that $U_n V_n \subseteq W_n$, and moreover, by (SR_2), a $\{W_n'\}$ such that $xW_n x^{-1} \subseteq W_n'$. From ($SR\, r_3$) again, there is a $\{w_n(xy)\}$ such that $W_n' xy \subseteq w_n(xy)$. Then, $u_n(x)\, v_n(y) \subseteq xU_n V_n y \subseteq xW_n y \subseteq W_n' xy \subseteq w_n(xy)$.

Now, let G be a SR-group, where defined families of subsets, ε_n ($n=0, 1, 2, \cdots$), which satisfy axioms (A), (B), (α), (β), (SR_1), (SR_2), (SR_3). When we take the totality of xV for $V \in \varepsilon_n$ as $\varepsilon_n(x)$, ($SR_3\, l$) is obviously fullfilled, and G becomes a SR-group. Taking $\{V_x\,;\ V \in \varepsilon_n\}$ as $\varepsilon_n(x)$, we may obtain another SR-group. In any case convergence of sequences coincides.

§ 4. Sufficient conditions for (SR_1), (SR_2).

As sufficient conditions for (SR_1), (SR_2), respectively, we have

⟨1⟩ there exists a non-negative function $\phi(\lambda, \mu)$ defined for $\lambda \geqslant 0$, $\mu \geqslant 0$ such that $\lim_{\lambda, \mu \to \infty} \phi(\lambda, \mu) = \infty$, and the following hold; if $U \in \varepsilon_l$, $V \in \varepsilon_m$, $W \in \varepsilon_n$ and $UV \subseteq W$, then there exists a $n^* \geqslant \phi(l, m)$ and a W^* in $\varepsilon_n{}^*$ such that $UV \subseteq W^* \subseteq W$,

⟨2⟩ there exists a function $\psi(\lambda\,;\, x) \geqslant 0$ defined for $\lambda \geqslant 0$, $x \in G$ such that $\lim_{\lambda \to \infty} \psi(\lambda\,;\, x)$ for any fixed x, and the following holds; if $U \in \varepsilon_m$, $V \in \varepsilon_n$, $x \in G$, and $xUx^{-1} \subseteq V$, there exists a $n^* \geqslant \psi(m\,;\, x)$ and a V^* in $\varepsilon_n{}^*$ such that $xUx^{-1} \subseteq V^* \subseteq V$.

The proof is similar in [5].

When $\{\varepsilon_n\}$ satisfies the condition:

(***) if $U \in \varepsilon_l$, $V \in \varepsilon_m$, then $U \cap V \in \varepsilon_n$, where $n \geqslant \max(l, m)$.

⟨1⟩, ⟨2⟩ may be replaced by, respectively,

⟨1′⟩ there exists a function $\phi(\lambda, \mu)$ such as ϕ in ⟨1⟩, and the following hold; for any $U \in \varepsilon_l$, $V \in \varepsilon_m$, there exists a $n \geqslant \phi(l, m)$ and a W in ε_n such that $UV \subseteq W$.

⟨2′⟩ there exists a function $\psi(\lambda\,;\, x)$ such as ψ in ⟨2⟩, and the following holds; for any $U \in \varepsilon_n$ and for any $x \in G$, there exists a $n \geqslant \psi(m\,;\, x)$ and a V in ε_n such that $xUx^{-1} \subseteq V$.

§5. Subgroup, Normal subgroup, Quotient group.

In this section we will define several new notions, i. e. SR-subgroup, R-subgroup, SR-normal subgroup, R-normal subgroup, SR-quotient group, and R-quotient group.

Definition 5. SR-subgroup, R-subgroup.

(1°) Let G be a SR-group and H a subgroup of G. Then H, endowed with the rank induced[12] from G, is called a SR-subgroup.

(2°) Let G be an R-group and H a subgroup of G. Then H, endowed with the rank induced from G, is called an R-subgroup.

Definition 6. SR-normal subgroup, R-normal subgroup.

(i°) If G is a SR-group and if N is a normal subgroup of G, then N is called a SR-normal subgroup.

(ii°) If G is an R-group and if N is a normal subgroup of G, then N is called an R-normal subgroup.

Proposition 2. Every r-open subgroup H of a SR-group (hence of a R-group) G is r-closed.

Proof. For each $x \in G$, xH is r-open by Corollary 1.

Hence, $H = G - \cup xH$ is r-closed, because $\cup xH$ is r-open, where the union is taken over all pairwise disjoint cosets different from H. (Q. E. D.)

Proposition 3. Let U be a symmetric[13] neighborhood of e in an R-group G. Then $H = \bigcup_{n \geqq 1} U^n$ is an r-open and r-closed subgroup of G.

Proof. Let $x, y \in H$. Then there exist positive integers m, n such that $x \in U^m$, $y \in U^n$. Hence, $xy^{-1} \in U^m (U^n)^{-1} = U^m (U^{-1})^m = U^m U^n = U^{m+n} \subseteq H$. Thus, H is a subgroup of G. Now to show that H is r-open, we observe that for each $y \in H$, $yU \subseteq yH = H$. This proves that H is r-open and r-closed by Proposition 2. (Q. E. D.)

Proposition 4. If H is an r-closed R-subgroup of an R-group G, so is r-closure[14] $\bar{H}$. If H is an r-closed R-normal subgroup of G, so is $\bar{H}$.

Proof. By using $\bar{H} = H$, we get this Proposition.

Let G be a SR-group and H a subgroup of G. Let G/H denote the collection of all distinct cosets $\{xH\}$, $x \in G$. Let f be the canonical mapping of G into G/H (i. e. $f: x \to xH$). Then, for any fundamental sequence of neighborhoods of $x \in G$, we can consider $\{f(u_n(x))\}$ a fundamental sequence of neighborhoods with respect to $\dot{x} \in G/H$ ($\dot{x} \equiv xH$) (thus, we put $f(u_n(x)) \equiv \dot{u}_m(\dot{x})$). Therefore, G/H is an R-space (endowed with the rank induced from G). Thus,

Definition 7. SR-quotient space, R-quotient space.

(I°) Let G be a SR-group and H a subgroup of G. Then G/H, the collection of all distinct cosets $\{xH\}$, $x \in G$, is called a SR-quotient space.

(II°) If G is an R-group and if H is a subgroup of G, then G/H is called an R-quotient space.

Remark 6. f is an onto and r-continuous mapping.

Proposition 5. Let G be a SR-group and H a subgroup of G, then G/H is a homogeneous space.

Proof. Let $\dot{x}_1, \dot{x}_2 \in G/H$, then $\dot{x}_1 = x_1 H$ and $\dot{x}_2 = x_2 H$. Let a be in G such that $ax_1 = x_2$. Define the mapping $f_a: \dot{x} = xH \to (ax)H = a\dot{x}$ for $\forall \dot{x} \in G/H$. Then f_a is well-defined and is a one-to-one mapping of G/H onto itself. Also $f_a^{-1}: \dot{x} \to (a^{-1}x)H$. Obviously, f_a is bicontinuous. This f_a is a homeomorphism as is easy to check. Clearly, $f_a(\dot{x}_1) = a\dot{x}_1 = (ax_1)H = x_2 H = \dot{x}_2$ shows that G/H is a homogeneous space. (Q. E. D.)

Proposition 6. Let H be a subgroup of a SR-group G, and f the canonical mapping of G onto G/H. If $\{\varepsilon_n\}$ is a fundamental system of neighborhoods of $e \in G$, then $\{f(\varepsilon_n)\}$ is a fundamental system of neighborhoods of $\dot{e} \equiv f(e) \in G/H$.

Proof. For each ε_n, $f(\varepsilon_n)$ is regarded as a neighborhood of $\dot{e}$.

12) [2], II, pp. 549–550.

13) A subset U of a group G is said to be symmetric if $U = U^{-1}$.

14) [7], III, pp. 792–793.

Proposition 7. Let G be a SR-group (or R-group) and N a normal subgroup of G. Then

1) The canonical mapping $f: G \to G/N$ is an r-continuous and homomorphism.

2) G/N is a SR-group (or R-group).

Proof. These are obvious.

Definition 8. SR-quotient group, R-quotient group.

Let G be a SR-group (or R-group) and N a normal subgroup of G, then the group G/N is called a SR-quotient group (or R-quotient group).

Proposition 8. Let G be an R-group, N a normal subgroup of G, M any R-subgroup of G, and $f: G \to G/N$. Then $f(M)$ is an R-subgroup of G/N, and it is homeomorphic with MN/N.

Proof. By an isomorphism theorem of abstract groups.

Proposition 9. (The first law of isomorphism). Let N be a normal subgroup of an R-group G and M any R-subgroup of G. Let $f(\dot{m}) = m(M \cap N)$, $m \in M$. Then, f endows the rank of MN/N onto $M/M \cap N$.

Proof. By the above arguments.

Moreover,

Proposition 10. (The second law of isomorphism). Let G be an R-group, N and M two normal subgroups of G such that $N \subseteq M$. Then, G/M is homeomorphic with $(G/N)/(M/N)$.

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(To be continued)

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