On Generalized Continuous Groups I*

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Synopsis

First, in the ranked spaces, we will define the notion of the R-continuity. Next, using this notion we will define the notion of a generalized continuous group and call it the (Semi-) Ranked Group with indicator ω. After that, we will define some notions, namely, homeomorphism, homomorphism, isomorphism, subgroup, normal subgroup, quotient group and direct product group. Using these notions, we will attempt to construct a general theory of the Ranked Groups with indicator ω. Furtheremore, in this paper, we will also define some notions, namely, ranked ring, ranked field, linear ranked space, ortho-continuous group, para-continuous group, etc.

Introduction. Prof. K. Kunugi introduced, in 1954, the notion of a non-topological space, namely, the ***Ranked Space***,[1] as an extention of the *Metric space* and the *Normed Space*, and introduced further, in 1956, the notion of a generalized integral based on his theory of ranked spaces and called it the ***(E. R.) integral***.[2] After that, in 1968, M. Washihara [6] gave a special definition of the ***Ranked Group*** with indicator ω_0 (ω_0 is the first non-finite cardinal). But the ***general definition*** of the Ranked Group with *general indicator* ω has not been given yet. On the other hand, in the Note [12] H. R. Fischer introduced, in 1959, the notion of ***limit group*** as a generalization of the notion of the *topological groups.* In a sense, a ranked group is considered to be a limit group.[3] In this paper we will attempt to give a *general definition* of the ranked group with *general* indicator ω and construct its general theory.

§ *0.* ***Ranked Spaces.*** The purpose of this section is to explain some notions of ranked spaces. Let us introduce the notion of ranked spaces according to K. Kunugi [3] and S. Nakanishi [4].

Consider a non-empty set S (called a space) endowed with such a structure that each point p of S has a non-empty family $\{V(p)\}$ of subsets of $S(V(p),\ V(p)\subseteq S$, is called a neighbourhood of p) satisfying ***the axioms (A) and (B) of F. Hausdorff***.[4] Given a point p of S, we say that a monotone decreasing sequence of neighbourhoods $V_\alpha(p)$ is ***type*** γ, where γ is an ordinal number of ***Cantor***, if α runs over the set $0\leqslant\alpha<\gamma$ of all ordinal numbers and if $V_\alpha(p)\supseteq V_\beta(p)$ for all α, β with $0\leqslant\alpha<\beta<\gamma$:

$$(\gamma)\qquad V_0(p)\supseteq V_1(p)\supseteq V_2(p)\supseteq\cdots\supseteq V_\alpha(p)\supseteq\cdots\cdots,\ 0\leqslant\alpha<\gamma.$$

The sequence (γ) which has no neighbourhoods $V_\gamma(p), V_\gamma(p)\in\{V(p)\}$, such that $\bigcap_\alpha V_\alpha(p)\supseteq V_\gamma(p)\,(0\leqslant\alpha\leqslant\gamma)$ is said to be ***maximal***. We will denote by $\omega(p, S)$ the smallest ordinal number of types of maximal

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1) [1]
2) [2]
3) In the case of indicator ω, we get this statement by the method as is taken in the Note [6] (pp. 4-5).
4) [31], p. 213.

monotone decreasing sequences of neighbourhoods of p. Now, let us consider such a space that there is at least one point having a maximal monotone decreasing sequence of neighbourhoods. Then $\omega(S)$, $\omega(S) \equiv \inf_{p \in S} \omega(p, S) = \min_{p \in S} \omega(p, S)$, is called the ***depth of the space S***. $\omega(S)$ is an ***inaccessible*** ordinal number.[5]

Let us choose once for all an inaccessible ordinal number ω such that $\omega_0 \leqslant \omega \leqslant \omega(S)$. ω is called ***indicator of S***. Given an ordinal number α, which runs through the interval $0 \leqslant \alpha < \omega$, suppose that we have a set $\mathfrak{B}_\alpha$ of neighbourhoods, called ***neighbourhoods of rank*** α. Then S is said to be a ***ranked space*** if the sequence of sets $\mathfrak{B}_\alpha (0 \leqslant \alpha < \omega)$ satisfies the following axiom (***a***) of ***K. Kunugi***[6]:

(***a***) For every neighbourhood $V(p)$ of p ($p \in S$) and for every ordinal number α such that $0 \leqslant \alpha < \omega$, there exists an ordinal number $\beta = \beta(p, \alpha, V(p))$ and a neighbourhood $U(p)$ of p such that we have at the same time

$$\alpha \leqslant \beta < \omega,\ U(p) \subseteq V(p),\ U(p) \in \mathfrak{B}_\beta.$$

We will denote the ranked space by $\{S, \mathfrak{B}_\alpha\}$. A monotone decreasing sequence of neighbourhoods of points:

$$V_0(p_0) \supseteq V_1(p_1) \supseteq V_2(p_2) \supseteq \cdots \supseteq V_\alpha(p_\alpha) \supseteq \cdots,\ 0 \leqslant \alpha < \omega,$$

is said to be the ***fundamental sequence***, if there is an ordinal number $\gamma(\alpha)$ such that $V_\alpha(p_\alpha) \in \mathfrak{B}_{\gamma(\alpha)}$ for all α, $0 \leqslant \alpha < \omega$, and satisfies the following two conditions:

(i) $\gamma(0) \leqslant \gamma(1) \leqslant \gamma(2) \leqslant \cdots \leqslant \gamma(\alpha) \leqslant \cdots \quad (0 \leqslant \gamma(\alpha) < \omega)$, $\sup_\alpha \gamma(\alpha) = \omega$,

(ii) for each α, $0 \leqslant \alpha < \omega$, there is a number $\lambda = \lambda(\alpha)$ such that $\alpha \leqslant \lambda < \omega$, $p_\lambda = p_{\lambda+1}$ and $\gamma(\lambda) < \gamma(\lambda+1)$ (except the equality).

The ranked space S is said to be ***complete***, if, for every fundamental sequence $\{V_\alpha(p_\alpha);\ 0 \leqslant \alpha < \omega\}$ of neighbourhoods, we have $\bigcap_{\alpha=0}^{\omega} V_\alpha(p_\alpha) \neq \phi$.

Given a sequence $\{p_\alpha;\ 0 \leqslant \alpha < \omega\}$ of points of S and a point p of S, we say that the sequence $\{p_\alpha;\ 0 \leqslant \alpha < \omega\}$ ***ortho-converges*** to the point p, or that p is an ***ortho-limit*** of $\{p_\alpha;\ 0 \leqslant \alpha < \omega\}$, if there is a fundamental sequence $\{V_\alpha(p);\ 0 \leqslant \alpha < \omega\}$ consisting of neighbourhoods of p such that $V_\alpha(p) \ni p_\alpha$ for each α. In this case, we shall write

$$p \in \{\lim_\alpha p_\alpha\}.$$

$\{\lim_\alpha p_\alpha\}$ is not a set consisting of one point alone in general.

Let R, S be two ranked spaces with same indicator ω. Then we will say that the mapping $f\colon R \longrightarrow S$ is ***ortho-continuous*** at the point p in R iff

$$p \in \{\lim_\alpha p_\alpha\} \Rightarrow f(p) \in \{\lim_\alpha f(p_\alpha)\}.$$

The mapping f is said to be ***ortho-continuous*** if it is ortho-continuous at each point of R.

Let A be a subset of $\{S, \mathfrak{B}_\alpha\}$. For every point p of A, the neighbourhood of p in A is the set of points of A defined by the relation $V(p, A) = V(p) \cap A$, where $V(p)$ is a neighbourhood of p in S. We also define the family $\mathfrak{B}_\alpha(A)$ $(0 \leqslant \alpha < \omega)$ of neighbourhoods of rank α of points of A as follows:

$V(p, A) \in \mathfrak{B}_\alpha(A)$ iff $V(p) \in \mathfrak{B}_\alpha$, where $\mathfrak{B}_\alpha$ is a family of neighbourhoods of rank α in S.

Then, A is a ranked space with indicator ω. We call it a ***ranked space induced from*** $\{S, \mathfrak{B}_\alpha\}$.

Moreover, let us consider a ranked space A induced from S such that: for every $p \in A$ and for every fundamental sequence $\{V_\alpha(p, A);\ 0 \leqslant \alpha < \omega\}$ of neighbourhoods of p, there is a fundamental sequence $\{V_\alpha(p);\ 0 \leqslant \alpha < \omega\}$ of neighbourhoods of p in S for which we have $V_\alpha(p, A) = V_\alpha(p) \cap A$ for each α. We call this

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5) A limit number α is said to be ***inaccessible***, if, for every β with $\beta < \alpha$ and for every function $\alpha(\gamma)$ defined for γ with $0 \leqslant \gamma < \beta$, such that $0 \leqslant \alpha(\gamma) < \alpha$, we have always $\sup_\gamma \alpha(\gamma) < \alpha$.

6) [3], I, p. 319.

A a ***ranked subspace*** *of* $\{S, \mathfrak{V}_\alpha\}$. We will denote the subspace A of $\{S, \mathfrak{V}_\alpha\}$ by $\{A, \mathfrak{V}_\alpha(A)\}$. When $\{p_\alpha; 0 \leqslant \alpha < \omega\}$ is a sequence of points of A and p is a point of A, we have $p \in \{\lim_\alpha p_\alpha\}$ in A iff $p \in \{\lim_\alpha p_\alpha\}$ in S.[7]

As the trivial examles of ranked spaces, there are metric spaces,[8] *M. L.* Schwartz's distribution space (D),[9]..., etc.

§ *1*. ***Definition of Ranked Group.*** Throughout this paper we suppose that every ranked space $\{S, \mathfrak{V}_\alpha\}$ satisfies the following axiom (b):

(b) $$S \in \mathfrak{V}_0.$$

i) R-continuity.

Definition 1. A mapping f: $\{S, \mathfrak{V}_\alpha\} \ni \forall p \longrightarrow p' \in \{S', \mathfrak{V}'_\alpha\}$ is said to be ***R-continuous*** at the point p if the following condition is fulfilled:

For any fundamental sequence $\{V_\alpha(p); 0 \leqslant \alpha < \omega\}$ of any point p in S, there exists a fundamental sequence $\{V'_\alpha(p'); 0 \leqslant \alpha < \omega\}$ of the point $p' = f(p)$ in S' such that

$$f(V_\alpha(p)) \subseteq V'_\alpha(p') \quad (\forall_\alpha; 0 \leqslant \alpha < \omega).$$

The mapping f is said to be ***R-continuous*** if it is R-continuous at each point of S.

Proposition 1. Every R-continuous mapping is ortho-continuous. But thc converse is not true.

Definition 2. $f: \{S, \mathfrak{V}_\alpha\} \longrightarrow \{S', \mathfrak{V}'_\alpha\}$ is called a ***homeomorphism*** iff f is bijective and bi-R-continuous. In this case, the spaces $\{S, \mathfrak{V}_\alpha\}$ and $\{S', \mathfrak{V}'_\alpha\}$ are said to be ***homeomorphic*** to each other.

Proposition 2. Let f be a mapping of $\{S, \mathfrak{V}_\alpha\}$ into $\{S', \mathfrak{V}'_\alpha\}$ sucn that f: $S \ni \forall p \longrightarrow p' \in S'$. Then, for each α where $0 \leqslant \alpha < \omega$, we have the followings:

(1) if $f(\mathfrak{V}_\alpha(p)) \subseteq \mathfrak{V}'_\alpha(p')$ then f is R-continous at the point p,

thus,

(2) if f is bijective and if $f(\mathfrak{V}_\alpha(p)) \subseteq \mathfrak{V}'_\alpha(p')$ & $f^{-1}(\mathfrak{V}'_\alpha(p')) \subseteq \mathfrak{V}_\alpha(p)$, then f is a homeomorhism.

ii) Direct Product Ranked Space.

Let

$$\{S_\lambda, \mathfrak{V}_\alpha^{(\lambda)}(p_\lambda)\} \quad (p_\lambda \in S_\lambda; \lambda \in \Lambda, \Lambda \text{ is any indexing set})$$

be ranked spaces with same indicator ω. And put

$$\begin{cases} \bar{S} = \prod_{\lambda \in \Lambda} S_\lambda & \text{(direct product set of } S_\lambda) \\ p = (p_\lambda)_{\lambda \in \Lambda} \in \bar{S}, \quad p_\lambda \in S_\lambda \ (\lambda \in \Lambda) \\ \bar{\mathfrak{V}}_\alpha(p) = \{\prod_{\lambda \in \Lambda} V^{(\lambda)}(p_\lambda); V^{(\lambda)}(p_\lambda) \in \mathfrak{V}_{\alpha_\lambda}^{(\lambda)} (\alpha \leqslant \forall \alpha_\lambda < \omega) \ \& \ \mathrm{Min}\,(\alpha_\lambda; \lambda \in \Lambda) = \alpha\} & (\forall_\alpha; 0 \leqslant \alpha < \omega). \end{cases}$$

Then, $\bar{S}$ becomes a ranked space with indicator ω by $\bar{\mathfrak{V}}_\alpha$. (c. q. f. d.). We call it the direct product ranked space with indicator ω of $\{S_\lambda, \mathfrak{V}_\alpha^{(\lambda)}\} (\lambda \in \Lambda)$, and denote it by $\{\bar{S}, \bar{\mathfrak{V}}_\alpha\}$, $\{\prod_{\lambda \in \Lambda} S_\lambda, \prod_{\lambda \in \Lambda} \mathfrak{V}_\alpha^{(\lambda)}\}$, etc.

Now, for each $\lambda \in \Lambda$ and for any fundamental sequence $\{u_\alpha^{(\lambda)}(p_\lambda); 0 \leqslant \alpha < \omega\}$ of any point p_λ in S_λ, a sequence $\{U_\alpha(p); 0 \leqslant \alpha < \omega\}$ (such that $p = (p)_{\lambda \in \Lambda}$ and $U_\alpha(p) = (u_\alpha^{(\lambda)}(p_\lambda))_{\lambda \in \Lambda}$ $(\forall_\alpha; 0 \leqslant \alpha < \omega)$) is considered to be a fundamental sequence in $\bar{S}$.

Therefore, we define the fundamental sequence in $\bar{S}$ as follows:

7) [8], I, p. 619.

8) [30], [3]

9) [34], Chap. III, § 1; [3], II, p. 552.

Let $p=(p_\lambda)_{\lambda\in\Lambda}$ be any point of $\bar{G}=\prod_{\lambda\in\Lambda} G_\lambda$ and $U_\alpha(p)\,(0\leqslant\alpha<\omega)$ a system of elements of $\mathfrak{B}=\bigcup_{\alpha=0}^{\omega}\mathfrak{B}_\alpha(p)\,(0\leqslant\alpha<\omega)$. Then, the sequence $\{U_\alpha(p)\,;\,0\leqslant\alpha<\omega\}$ $(U_\alpha(p)\equiv(U_\alpha^{(\lambda)}(p))_{\lambda\in\Lambda})$ is said to be a ***fundamental sequence of p in*** $\bar{G}$ if, for each $\lambda\in\Lambda$, $\{u_\alpha^{(\lambda)}(p_\lambda)\,;\,0\leqslant\alpha<\omega\}$ is a fundamental sequence of p_λ in G_λ.

iii) Definition of Ranked Group with indicator ω.

Definition 3. (I) For $\{G,\mathfrak{B}_\alpha\}$ with indicator ω, if G is a group and the group operation

(i) $f_1:\{G\times G,\ \mathfrak{B}_\alpha\times\mathfrak{B}_\alpha\}\ni\forall(x,y)\longrightarrow xy\in\{G,\mathfrak{B}_\alpha\}$ is R-continuous,

then $\{G,\mathfrak{B}_\alpha\}$ is called the ***Semiranked Group*** *with indicator* ω. We denote this by $<G,\mathfrak{B}_\alpha>$.

(II) For $<G,\mathfrak{B}_\alpha>$, if the group operation

(ii) $f_2:\{G,\mathfrak{B}_\alpha\}\ni\forall x\longrightarrow x^{-1}\in\{G,\mathfrak{B}_\alpha\}$ is R-continuous,

then $<G,\mathfrak{B}_\alpha>$ is called the ***Ranked Group*** *with indicator* ω. We denote this by $(G,\mathfrak{B}_\alpha)$.

Proposition 3. For $(G,\mathfrak{B}_\alpha)$, the conditions (i) and (ii) are equivalent to the following condition (iii):

(iii) $f_3:\{G\times G,\ \mathfrak{B}_\alpha\times\mathfrak{B}_\alpha\}\ni\forall(x,y)\longrightarrow xy^{-1}\in\{G,\mathfrak{B}_\alpha\}$ is R-continuous.

Proof. $(\Rightarrow)$; Let $\{V_\alpha(p)\,;\,0\leqslant\alpha<\omega\}$ be any fundamental sequence of any point $p\in G\times G$ such that $p=(x,y)$ where $x,y\in G$. Then, there are two fundamental sequences $\{u_\alpha(x)\,;\,0\leqslant\alpha<\omega\}$ and $\{v_\alpha(y)\,;\,0\leqslant\alpha<\omega\}$ in G such that $V_\alpha(p)=(u_\alpha(x),\ v_\alpha(y))\,(0\leqslant\alpha<\omega)$. Since f_2 is R-continuous, for the fundamental sequence $\{v_\alpha(x)\,;\,0\leqslant\alpha<\omega\}$ in G, there is a fundamental sequence $\{v'_\alpha(y^{-1})\,;\,0\leqslant\alpha<\omega\}$ in G such that $v_\alpha(y)^{-1}\subseteq v'_\alpha(y^{-1})\,(0\leqslant\alpha<\omega)$. Moreover, since f_1 is R-continuous, for $\{u_\alpha(x)\,;\,0\leqslant\alpha<\omega\}$ and $\{v_\alpha'(y^{-1})\,;\,0\leqslant\alpha<\omega\}$, there is a fundamental sequence $\{w_\alpha(xy^{-1})\,;\,0\leqslant\alpha<\omega\}$ in G such that

$$u_\alpha(x)\cdot v'_\alpha(y^{-1})\subseteq w_\alpha(xy^{-1})\ (0\leqslant\alpha<\omega).$$

Hence, we have

f_3: $V_\alpha(p)=(u_\alpha(x),\ v_\alpha(y))\longrightarrow u_\alpha(x)\cdot v_\alpha(y)^{-1}\subseteq u_\alpha(x)\cdot v'_\alpha(y^{-1})\subseteq w_\alpha(xy^{-1})=w_\alpha(f_3(p))\ (0\leqslant\alpha<\omega)$. Namely, f_3 is R-continuous.

$(\Leftarrow)$; Let $\{u_\alpha(x)\,;\,0\leqslant\alpha<\omega\}$, $\{v_\alpha(y)\,;\,0\leqslant\alpha<\omega\}$ be any fundamental sequences of x,y in G. Since f_3 is R-continuous, for $\{v_\alpha(y)\,;\,0\leqslant\alpha<\omega\}$, there is a fundamental sequence $\{v'_\alpha(y^{-1})\,;\,0\leqslant\alpha<\omega\}$ in G such that

$$f_3:\ (u_\alpha(e),\ v_\alpha(y))\longrightarrow u_\alpha(e)\cdot v_\alpha(y)^{-1}\subseteq v'_\alpha(y^{-1})\ (0\leqslant v<\omega).$$

But, since $v_\alpha(y)^{-1}\subseteq u_\alpha(e)\cdot v_\alpha(y)^{-1}\subseteq v'_\alpha(y^{-1})$, f_2 is R-continuous. Hence, there is a fundamental sequence $\{v'_\alpha(y^{-1})\,;\,0\leqslant\alpha<\omega\}$ in G such that

$$u_\alpha(x)\cdot v_\alpha(y)=u_\alpha(x)\cdot(v_\alpha(y)^{-1})^{-1}\subseteq u_\alpha(x)\cdot(v'_\alpha(y^{-1})).^{-1}$$

Moreover, since f_3 is R-continuous, there is a fundamental sequence $\{w_\alpha(xy)\,;\,0\leqslant\alpha<\omega\}$ in G such that

$$u_\alpha(x)\cdot(v'_\alpha(y^{-1})^{-1})\subseteq w_\alpha(x(y^{-1})^{-1})=w_\alpha(xy).$$

Thus, we have

$$f_1:\ (u_\alpha(x),v_\alpha(y))\longrightarrow u_\alpha(x)\cdot v_\alpha(y)\subseteq w_\alpha(xy)\ (0\leqslant\alpha<\omega).$$

Namely, f_1 is R-continuous. (q. e. d.)

Proposition 4. For every ranked group G, the preceding conditions (i), (ii) and (iii) are respectively equivalent to the following conditions:

(i′) For $\forall x,\ y\in G$ and for any fundamental sequences $\{u_\alpha(x)\,;\,0\leqslant\alpha<\omega\}$, $\{v_\alpha(y)\,;\,0\leqslant\alpha<\omega\}$ of x,y, there is a fundamental sequence $\{w_\alpha(xy)\,;\,0\leqslant\alpha<\omega\}$ of xy in G such that

$$u_\alpha(x)\cdot v_\alpha(y)\subseteq w_\alpha(xy)\ (\forall\alpha\,;\,0\leqslant\alpha<\omega),$$

(ii′) For any fundamental sequence $\{u_\alpha(x)\,;\,0\leqslant\alpha<\omega\}$ of any point x in G, there is a fundamental sequence $\{v_\alpha(x^{-1})\,;\,0\leqslant\alpha<\omega\}$ of x^{-1} in G such that

$$u_\alpha(x)^{-1}\subseteq v_\alpha(x^{-1})\ (\forall\alpha\,;\,0\leqslant\alpha<\omega)$$

and

(iii′) For $\forall x, y \in G$ and for any fundamental sequences $\{u_\alpha(x);\ 0 \leqslant \alpha < \omega\}$, $\{v_\alpha(y);\ 0 \leqslant \alpha < \omega\}$ of x, y, there is a fundamental sequence $\{w_\alpha(xy^{-1});\ 0 \leqslant \alpha < \omega\}$ of xy^{-1} in G such that

$$u_\alpha(x) \cdot v_\alpha(y)^{-1} \subseteq w_\alpha(xy^{-1}) \ (\forall \alpha;\ 0 \leqslant \alpha < \omega).$$

In fact, for any $p \equiv (x, y) \in G \times G$ and for any $V_\alpha(p) \equiv (u_\alpha(x),\ v_\alpha(y)) \in \mathfrak{V}_\alpha \times \mathfrak{V}_\alpha\ (0 \leqslant \alpha < \omega)$ such that $\{u_\alpha(x)\}$, $\{v_\alpha(y)\}$ are fundamental sequences of x, y in G, the following statements are true:

(i′) $u_\alpha(x) \cdot v_\alpha(y) = f_1(V_\alpha(p)) \subseteq^{\mathfrak{A}} w_\alpha(f_1(p)) = w_\alpha(xy)$ for $\forall \alpha,\ 0 \leqslant \alpha < \omega$,

(ii′) $u_\alpha(x)^{-1} = f_2(u_\alpha(x)) \subseteq^{\mathfrak{A}} v_\alpha(f_2(x)) = v_\alpha(x^{-1})$ for $\forall \alpha,\ 0 \leqslant \alpha < \omega$,

and

(iii′) $u_\alpha(x) \cdot v_\alpha(y)^{-1} \subseteq f_3(V_\alpha(p)) \subseteq^{\mathfrak{A}} w_\alpha(f_3(p)) = w_\alpha(xy^{-1})$ for $\forall \alpha,\ 0 \leqslant \alpha < \omega$.

Therefore, this proposition is true.

Corollary. If the indicator of the ranked group is ω_0, then the notion of ranked group in our sense coincides with the notion of ranked group in the sense of M. Washihara [6].

By the definition of the ranked group, we have the following:

Proposition 5. For every $(G, \mathfrak{V}_\alpha)$, f_2: $x \longrightarrow x^{-1}$ is a homeomorphism:

Proposition 6. Let $a_1, a_2, \cdots\cdots, a_m$ be a finite system of elements of $(G, \mathfrak{V}_\alpha)$, let $a_1^{r_1} a_2^{r_2} \cdots a_m^{r_m} = c$ be a product of powers of the a's, where the powers may be positive or negative, and let $\{U_\alpha^{(1)}(a_1);\ 0 \leqslant \alpha < \omega\}$, $\{U_\alpha^{(2)}(a_2);\ 0 \leqslant \alpha < \omega\}, \cdots$, $\{U_\alpha^{(m)}(a_m);\ 0 \leqslant \alpha < \omega\}$ be arbitrary fundamental sequences of the elements $a_1, a_2, \cdots, a_m$. Then there is a fundamental sequence $\{W_\alpha(c);\ 0 \leqslant \alpha < \omega\}$ of the element c such that

$$U_\alpha^{(1)}(a_1)^{r_1} \cdot U_\alpha^{(2)}(a_2)^{r_2} \cdots U_\alpha^{(m)}(a_m)^{r_m} \subseteq W_\alpha(c) \ (0 \leqslant \alpha < \omega),$$

where $U_\alpha^{(i)}(a_i)$ is taken equal to $U_\alpha^{(j)}(a_j)$ if $a_i = a_j$, the same being true for a greater number of equal elements.

iV) Neighbourhood systems of the Identity.

(1°) Translations of the ranked group.

Proposition 7. Let a be a fixed element of $\langle G, \mathfrak{V}_\alpha \rangle$. Then the ***right*** and ***left translations*** of G, namely,

$$r_a : x \longrightarrow xa, \qquad l_a : x \longrightarrow ax$$

are homeomorphisms of G.

In fact, since $x, a \in \langle G, \mathfrak{V}_\alpha \rangle$, for any fundamental sequences $\{u_\alpha(x);\ 0 \leqslant \alpha < \omega\}$, $\{v_\alpha(a);\ 0 \leqslant \alpha < \omega\}$ of x, a in G, there is a fundamental sequence $\{w_\alpha(xa);\ 0 \leqslant \alpha < \omega\}$ of xa in G such that

$$u_\alpha(x) \cdot v_\alpha(a) \subseteq w_\alpha(xa) \ (0 \leqslant \alpha < \omega).$$

Hence, $r_a(u_\alpha(x)) = u_\alpha(x) \cdot a \subseteq u_\alpha(x) \cdot v_\alpha(a) \subseteq w_\alpha(xa) = w_\alpha(r_a(x))$. This shows that r_a is R-continuous. Moreover, r_α^{-1}: $x \longrightarrow a^{-1}x$ is R-continuous by the same argument as above. Hence, r_a is a homeomorphism. The fact that l_a is a homeomorphism follows similarly.

Proposition 8. Both $\langle G, \mathfrak{V}_\alpha \rangle$ and $(G, \mathfrak{V}_\alpha)$ are homogeneous. Namely, for $\forall\ p, q \in G$, there exists a homeomorphism f of G such that $f(p) = q$.

In fact, for $\forall\ p, q \in G$, consider the mapping f: $x \longrightarrow qp^{-1}x$. Then f is a homeomorphism by Proposition 7 and $f(p) = q$. Hence $\langle G, \mathfrak{V}_\alpha \rangle$ is a homogeneous space. Thus $(G, \mathfrak{V}_\alpha)$ is also a homogeneous space.

(2°) Neighbourhood systems of the identity.

Proposition 9. Let e be the identity of group G. And for every $(G, \mathfrak{V}_\alpha)$, put

$$\begin{cases} \mathfrak{V} \equiv \mathfrak{V}(e) = \bigcup_{\alpha=0}^{\omega} \mathfrak{V}_\alpha(e), \\ \text{for } \forall\ a \in G \text{ and for } \forall \alpha \text{ where } 0 \leqslant \alpha < \omega,\ \mathfrak{V}_\alpha(a) = \mathfrak{V}_\alpha(e) \cdot a = a \cdot \mathfrak{V}_\alpha(e). \end{cases}$$

Then, $\mathfrak{B}$ possesses the following properties:

1) for $\forall\ V\in\mathfrak{B}$, $e\in V$,

2) for $\forall\ U,\ V\in\mathfrak{B}$, there is a $W\in\mathfrak{B}$ such that $U\cap V\supseteq W$.

3) for $\forall\ V\in\mathfrak{B}$ and for any α, $0\leqslant\alpha<\omega$, there is a β, $\alpha\leqslant\beta<\omega$, and a U in $\mathfrak{B}_\beta$ such that $U\subseteq V$,

4) $G\in\mathfrak{B}_0$,

5) for $\forall\ U,\ V\in\mathfrak{B}$, there is a $W\in\mathfrak{B}$ such that $UV^{-1}\subseteq W$,

[or 5′) for $\forall\ U, V\in\mathfrak{B}$ there is a $W\in\mathfrak{B}$ such that $UV\subseteq W$, and for $\forall\ U\in\mathfrak{B}$ there is a $V\in\mathfrak{B}$ such that $U^{-1}\subseteq V$,]

and

6) for $\forall a\in G$ and for $\forall\ V\in\mathfrak{B}$, $aVa^{-1}\in\mathfrak{B}$.

Proposition 10. ***(Sufficient condition)***

Let G be an abstract group, $\mathfrak{B}_\alpha(e)$ $(0\leqslant\alpha<\omega)$ a system of families satisfying above conditions 1)—6) of its subsets including the identity e in G. If, for any $a\in G$ and for all α (where $0\leqslant\alpha<\omega$),

(H) $$\mathfrak{B}_\alpha(a)=\mathfrak{B}_\alpha(e)\cdot a \qquad [\text{or } \mathfrak{B}_\alpha(a)=a\cdot\mathfrak{B}_\alpha(e)]$$

then

(1) G becomes a ranked space with indicator ω

and

(2) $\{G,\mathfrak{B}_\alpha\}$ is a ranked group with indicator ω.

In fact, since every $\mathfrak{B}_\alpha(a)$ $(0\leqslant\alpha<\omega)$ satisfies the conditions 1)—4), G is a ranked space.

Moreover, for $\forall(x,y)\in G\times G$ and for $\forall(Ux,\ Vy)\in\mathfrak{B}(x)\times\mathfrak{B}(y)$ (where $U,V\in\mathfrak{B}(e)$), there exist V' and W in $\mathfrak{B}(e)$ such that

$$Ux\cdot(Vy)^{-1}=U\cdot xy^{-1}V^{-1}=U\cdot V'^{-1}xy^{-1}\subseteq Wxy^{-1}.$$

Therefore, $\{G,\mathfrak{B}_\alpha\}$ becomes a ranked group.

Proposition 11. If $\langle G,\mathfrak{B}_\alpha\rangle$ (or $(G,\mathfrak{B}_\alpha)$) satisfies the following condition

for $\forall\ a\in G$ and for $\forall\alpha$ where $0\leqslant\alpha<\omega$, $\mathfrak{B}_\alpha(a)=\mathfrak{B}_\alpha(e)\cdot a=a\cdot\mathfrak{B}_\alpha(e)$,

then, for $\forall p,q\in G$, there is a mapping f such that $f(\mathfrak{B}_\alpha(p))=\mathfrak{B}_\alpha(q)$ for each α where $0\leqslant\alpha<\omega$.

In fact, for $\forall x\in G$, consider the mapping $f:\ x\longrightarrow qp^{-1}x$.

§ 2. Subgroup, Normal Subgroup, Quotient Group.

i) Subgroup, Normal subgroup, Quotient group.

Definition 4. (I) Let G be a (Semi-) Ranked group with indicator ω. A subset H of G is called a ***subgroup*** of the (Semi-) Ranked group G if

⟨i⟩ H is a subgroup of the abstract group G,

⟨ii⟩ H is a ranked subspace of $\{G,\mathfrak{B}_\alpha\}$.

We denote this by $(\boldsymbol{H},\mathfrak{B}_\alpha(\boldsymbol{H}))$ (resp. $\langle\boldsymbol{H},\mathfrak{B}_\alpha(\boldsymbol{H})\rangle$).

(II) A subgroup N of a (Semi-) Ranked group G is called a ***normal subgroup*** of G if N is a normal subgroup of the abstract group G.

Proposition 12. $(H,\mathfrak{B}_\alpha(H))$ (resp. $\langle H,\mathfrak{B}_\alpha(H)\rangle$) becomes a (Semi-) Ranked group with indicator ω.

Proof. To prove this it is sufficient to show that the group operations in H are R-continuous in the ranked space H. Let a, b be two elements of the set H and let $\{u_\alpha(a)\cap H;\ 0\leqslant\alpha<\omega\}$, $\{v_\alpha(b)\cap H;\ 0\leqslant\alpha<\omega\}$ any fundamental sequences of a, b in H. Since $\{u_\alpha(a);\ 0\leqslant\alpha<\omega\}$, $\{v_\alpha(b);\ 0\leqslant\alpha<\omega\}$ are fundamental sequences of a, b in G, there exists a fundamental sequence $\{w_\alpha(ab^{-1});\ 0\leqslant\alpha<\omega\}$ of ab^{-1} in G such that

$$(u_\alpha(a)\cap H)\,(v_\alpha(b)\cap H)^{-1}\subseteq u_\alpha(a)\cdot v_\alpha(b)^{-1}\subseteq w_\alpha(ab^{-1})\ (0\leqslant\alpha<\omega).$$

On the other hand, since H is a subgroup of G we have the following:

$$(u_\alpha(a) \cap H)\ (v_\alpha(b) \cap H)^{-1} \subseteq HH^{-1} \subseteq H.$$

Hence, we have a fundamental sequence $\{w_\alpha(ab^{-1}) \cap H;\ 0 \leqslant \alpha < \omega\}$ of ab^{-1} in H such that

$$(u_\alpha(a) \cap H)\ (v_\alpha(b) \cap H)^{-1} \subseteq w_\alpha(ab^{-1}) \cap H\ (0 \leqslant \alpha < \omega).$$

Therefore, $\{H \times H,\ \mathfrak{B}_\alpha(H) \times \mathfrak{B}_\alpha(H)\} \ni \forall\ (a, b) \longrightarrow ab^{-1} \in \{H, \mathfrak{B}_\alpha(H)\}$ is R-continuous.

Remark Let $\{e\}$ be the identity subgroup of G. Since $V \cap \{e\} = \{e\}$ for $\forall V \in \mathfrak{B}(e) = \bigcup_{\alpha=0}^{\omega} \mathfrak{B}_\alpha(e)$, $\{e\}$ becomes a normal subgroup of $(G, \mathfrak{B}_\alpha)$.

Definition 5. Let H be a subgroup of $(G, \mathfrak{B}_\alpha)$ (or $\langle G, \mathfrak{B}_\alpha \rangle$) and let $\dot{G} \equiv G/H$ the totality of all left cosets of the subgroup in the group G. If we put

$$\begin{cases} \dot{V}(\dot{x}) \equiv V(x)/H = \{yH;\ y \in V(x) \text{ where } V(x) \in \mathfrak{B}_\alpha(x)\} \\ \text{and} \\ \dot{\mathfrak{B}}_\alpha(\dot{x}) \equiv \mathfrak{B}_\alpha(x)/H = \{\dot{V}(\dot{x});\ \dot{V}(\dot{x}) \equiv V(x)/H,\ V(x) \in \mathfrak{B}_\alpha(x)\}\ (0 \leqslant \alpha < \omega), \end{cases}$$

then $\dot{G}$ becomes a ranked space with indicator ω. The ranked space G/H thus obtained we shall call the ***space of left cosets*** of the subgroup H in the group G. Analogously we define the ***space of right cosets*** and use the symbol G/H for it also.

In the cases where there is no danger of ambiguity we shall make no distinction between the spaces of left and right cosets.

We denote the ***space of cosets*** by $\{\dot{G}, \dot{\mathfrak{B}}_\alpha\}$, $\{G/H, \mathfrak{B}_\alpha/H\}$, etc.

Proposition 13. $\{G/H, \mathfrak{B}_\alpha/H\}$ is a homogeneous space.

In fact, for $\forall\ \dot{a}, \dot{b} \in \dot{G} = G/H$, by the mapping $f\colon \dot{G} \ni \dot{x} \longrightarrow ba^{-1}\ \dot{x} \in \dot{G}$ we have $f(\dot{a}) = ba^{-1} \cdot \dot{a} = \dot{b}$.

Proposition 14. Let f be the ***canonical mapping*** of the space $\{G, \mathfrak{B}_\alpha\}$ on the space $\{G/H, \mathfrak{B}_\alpha/H\}$, i.e.,

$$f\colon \{G, \mathfrak{B}_\alpha\} \ni \forall x \longrightarrow \dot{x} = xH \in \{G/H,\ \mathfrak{B}_\alpha/H\}.$$

Then, we have $f(\mathfrak{B}_\alpha(x)) = \dot{\mathfrak{B}}_\alpha(\dot{x})$ for each α where $0 \leqslant \alpha < \omega$. Thus, f is an R-cotinuous mapping.

Proposition 15. If N is a normal subgroup of $(G, \mathfrak{B}_\alpha)$ (or $\langle G, \mathfrak{B}_\alpha \rangle$), then $\{G/N, \mathfrak{B}_\alpha/N\}$ becomes a (semi−) ranked group with indicator ω.

In fact, since G is a ranked group, for any fundamental sequences $\{\dot{u}_\alpha(\dot{a});\ 0 \leqslant \alpha < \omega\}$ $(\dot{u}_\alpha(\dot{a}) \equiv u_\alpha(a)N)$, $\{\dot{v}_\alpha(\dot{b});\ 0 \leqslant \alpha < \omega\}$ $(v_\alpha(\dot{b}) \equiv v_\alpha(b)N)$ of $\dot{a}$, $\dot{b}$ in G/N there exists a fundamental sequence $\{w_\alpha(ab^{-1});\ 0 \leqslant \alpha < \omega\}$ of ab^{-1} in G such that $u_\alpha(a) \cdot v_\alpha(b)^{-1} \subseteq w_\alpha(ab^{-1})$. Thus, we have $\dot{u}_\alpha(\dot{a}) \cdot \dot{v}_\alpha(\dot{b})^{-1} \subseteq u_\alpha(a) \cdot v_\alpha(b)^{-1}N \subseteq w_\alpha(ab^{-1})N \equiv \dot{w}_\alpha(\dot{a}\dot{b}^{-1})$. Hereupon the sequence $\{\dot{w}_\alpha(\dot{a}\dot{b}^{-1});\ 0 \leqslant \alpha < \omega\}$ is a fundamental sequence of $\dot{a}\dot{b}^{-1}$ in G/N.

Definition 6. Let N be a normal subgroup of $(G, \mathfrak{B}_\alpha)$ (resp. $\langle G, \mathfrak{B}_\alpha \rangle$). Then, the (semi−) ranked group $(G/N, \mathfrak{B}_\alpha/N)$ (resp. $\langle G/N, \mathfrak{B}_\alpha/N \rangle$) is called the ***quotient group*** with indicator ω of the (semi-) ranked group $(G, \mathfrak{B}_\alpha)$ (resp. $\langle G, \mathfrak{B}_\alpha \rangle$) by the normal subgroup N.

ii) Isomorphism, Automorphism, Homomorphism.

Let S, S' be two ranked spaces with indicator ω and let H a subset of S. We assume that the mapping $\varphi\colon \{S, \mathfrak{B}_\alpha\} \ni \forall p \xrightarrow{in} p' \in \{S', \mathfrak{B}'_\alpha\}$ is satisfying the following condition (*):

(*)

- (i) For any subset A of S, we have $\varphi(A \cap H) = \varphi(A) \cap \varphi(H)$,
- (ii) For any fundamental sequence $\{V_\alpha(p);\ 0 \leqslant \alpha < \omega\}$ of p in S, the sequence $\{V'_\alpha(p');\ 0 \leqslant \alpha < \omega\}$ $(V'_\alpha(p') \equiv \varphi(V_\alpha(p)))$ is a fundamental sequence of p' in S'

  and
- (iii) For any fundamental sequence $\{V'_\alpha(p', H');\ 0 \leqslant \alpha < \omega\}$ $(H' \equiv \varphi(H))$ of p' in H', there exists a fundamental sequence $\{V_\alpha(p, H);\ 0 \leqslant \alpha < \omega\}$ of p in H such that $\varphi(V_\alpha(p, H)) = V'_\alpha(p', H')$ for each α where $0 \leqslant \alpha < \omega$.

Then we have the following:

Proposition 16. $\varphi(\{H, \mathfrak{B}_\alpha(H)\}) = \{H', \mathfrak{B}'_\alpha(H')\}$.
(subspace) (subspace)

In fact, since φ is satisfying the condition (*), for any fundamental sequence $\{V'_\alpha(p', H'); 0 \leqslant \alpha < \omega\}$ of $p' = \varphi(p)$ in H', there is a $\{V_\alpha(p, H); 0 \leqslant \alpha < \omega\}$ such that $\varphi(V_\alpha(p, H)) = V'_\alpha(p', H')$ for each α where $0 \leqslant \alpha < \omega$. On the other hand, since H is a subspace of $\{S, \mathfrak{B}_\alpha\}$, for any fundamental sequence $\{V_\alpha(p, H); 0 \leqslant \alpha < \omega\}$ of p in H, there exists a fundamental sequence $\{V_\alpha(p); 0 \leqslant \alpha < \omega\}$ of p in S such that $V_\alpha(p, H) = V_\alpha(p) \cap H$ for each α where $0 \leqslant \alpha < \omega$. Therefore, we have the following:

$$V'_\alpha(p', H') = \varphi(V_\alpha(p, H)) = \varphi(V_\alpha(p) \cap H) = V'_\alpha(p') \cap H'.$$

And now, since $\{V'_\alpha(p'); 0 \leqslant \alpha < \omega\}$ is a fundamental sequence of p' in S', we have

$$\varphi(\{H, \mathfrak{B}_\alpha(H)\}) = \{H', \mathfrak{B}'_\alpha(H')\}.$$
(subspace) (subspace)

Definition 7. A mapping g of a (semi-) ranked group G into a (semi-) ranked group G' is called a homomorphism if

(i) g is a homomorphism of the abstract group G into the abstract group G',

(ii) g satisfies the condition (*).

For instance, the canonical mapping: $G \ni \forall x \longrightarrow xN \in G/N$ is a homomorphism.

Proposition 17. If $(G, \mathfrak{B}_\alpha) \overset{g}{\sim} (G', \mathfrak{B}'_\alpha)$ and H is a subgroup of $(G, \mathfrak{B}_\alpha)$, then $g(H)$ is a subgroup of $(G', \mathfrak{B}'_\alpha)$.

Definition 8. A mapping f of a (semi-) ranked group G on a (semi-) ranked group G' is called an ***isomorphim*** if

(i) f is an isomorphism of the abstract group G on the abstract group G',

(ii) f is a homeomorphism of the ranked space G on the ranked space G'.

Two (semi-) ranked groups are called ***isomorphic*** if there exists an isomorphism of one group on the other. In particular, an isomorphism of a (semi-) ranked group G into itself is called an ***automorphism*** of the group G.

Remark Let H be a subgroup of $(G, \mathfrak{B}_\alpha)$ and let $(G, \mathfrak{B}_\alpha) \overset{f}{\cong} (G', \mathfrak{B}'_\alpha)$. In general, $f(H)$ does not form a subspace of $(G', \mathfrak{B}'_\alpha)$. But if f satisfies the condition (ii) in (*), then $f(H)$ becomes a subspace of $\{G', \mathfrak{B}'_\alpha\}$ Thus, in this case, $f(H)$ is a subgroup of $(G', \mathfrak{B}'_\alpha)$.

Proposition 18. If $(G, \mathfrak{B}_\alpha) \overset{g}{\sim} (G', \mathfrak{B}'_\alpha)$ and if $N \equiv g^{-1}(e')$ is a subspace of $(G, \mathfrak{B}_\alpha)$, then

(1) N is normal in $(G, \mathfrak{B}_\alpha)$,

(2) $(G/N, \mathfrak{B}_\alpha/N) \cong (G', \mathfrak{B}'_\alpha)$,

(3) This isomorphism satisfies the condition (ii) in (*). Thus, this isomophism satisfies the condition (*).

Proof. As is known from group theory, we have $G/N \overset{f}{\cong} G'$. This algebraic homomorphism f is $G/N \ni aN \longrightarrow g(a) \in G'$. Now, for any fundamental sequence $\{V_\alpha(a)N; 0 \leqslant \alpha < \omega\}$ of $\dot{a}$ in G/N, we have $f(V_\alpha(a)N) = g(V_\alpha(a)) \equiv V'_\alpha(a')$. And the sequence $\{V'_\alpha(a'); 0 \leqslant \alpha < \omega\}$ is a fundamental sequence of a' in G'. Hence f is R-continuous.

Now, let $\{V'_\alpha(a'); 0 \leqslant \alpha < \omega\}$ be any fundamental sequence of a' in G', and put $H' = G'$. Then, by the condition (iii), there exists a fundamental sequence $\{V_\alpha(a); 0 \leqslant \alpha < \omega\}$ of a in G such that $g(V_\alpha(a)) = V'_\alpha(a')$ for each α where $0 \leqslant \alpha < \omega$. Thus, we have the following:

$$f^{-1}(V'_\alpha(a')) = f^{-1}(g(V_\alpha(a))) = V_\alpha(a)N \quad (0 \leqslant \alpha < \omega).$$

Since $\{V_\alpha(a)N; 0 \leqslant \alpha < \omega\}$ becomes a fundamental sequence of $\dot{a}$ in G/N, f^{-1} is R-continuous. Therefore, f is an isomorphism.

Proposition 19. Let N be a normal subgroup of $(G, \mathfrak{B}_\alpha)$, M any subgroup of $(G, \mathfrak{B}_\alpha)$, and

$(G,\mathfrak{B}_\alpha)\overset{g}{\sim}(G/N,\mathfrak{B}_\alpha/N)$. Then $g(M)$ is a subgroup of $(G/N,\mathfrak{B}_\alpha/N)$. And if MN is a subspace of $\{G,\mathfrak{B}_\alpha\}$ and N a subspace of MN then MN/N is a subgroup of the ranked group G/N, and $g(M)$ is isomorphic with HN/N.

In fact, this isomorphism is given by the mapping f: $mN\longrightarrow g(m)$ $(m\in M)$.

Remark. Assume that $(G,\mathfrak{B}_\alpha)\overset{g}{\sim}(G',\mathfrak{B}'_\alpha)$. And let $N\equiv g^{-1}(e')$ be a subspace of $(G,\mathfrak{B}_\alpha)$, A a subgroup including N of $(G,\mathfrak{B}_\alpha)$. And suppose that AN becomes a subspace of G. Then since the restriction $g|A$ of g on A is a homomorphism, we have $AN/N\cong g(A)$ in the sense of the ranked group. In particular, if g is the canonical mapping of G *onto* G/N, then A/N which is a quotient group of the ranked group A is isomorphic with A/N which is a subgroup of the ranked group G/N.

Proposition 20. If H is a normal subgroup of $(G,\mathfrak{B}_\alpha)$ and a subgroup N of H is normal in $(G,\mathfrak{B}_\alpha)$ and H forms a subspace of G, then, in the sense of the ranked group, we have $(G/N)/(H/N)\cong G/H$.

In fact, as is known from group theory, we have $G/H\overset{f}{\cong}(G/N)/(H/N)$. This algebraic isomorphism f is given by the composition mapping $h\cdot g$ of two canonical mappings such that $G\overset{g}{\longrightarrow}G/N\overset{h}{\longrightarrow}(G/N)/(H/N)$. Since $h\cdot g$ is canonical, f is an isomorphism in the sense of the ranked group.

Remark. Let M,N be two subgroups of $(G,\mathfrak{B}_\alpha)$. Both MN and $M\cap N$ are not always subspaces of $\{G,\mathfrak{B}_\alpha\}$. Thus, both MN and $M\cap N$ are not always subgroups of $(G,\mathfrak{B}_\alpha)$. Therefore, the *Second Isomorphism Theorem* is not always true.

§ 3. *Direct Product (semi–) Ranked Group.*

i) Direct product (semi–) ranked group.

Proposition 21. Let Λ be any indexing set. For each $\lambda\in\Lambda$, let $(G_\lambda,\mathfrak{B}^{(\lambda)}_\alpha)$(resp. $\langle G_\lambda,\mathfrak{B}^{(\lambda)}_\alpha\rangle$) be a (semi–) ranked group with indicator ω. Then, the direct product ranked space $\{\bar{G},\bar{\mathfrak{B}}_\alpha\}$, i.e., $\{\prod_{\lambda\in\Lambda}G_\lambda,\prod_{\lambda\in\Lambda}\mathfrak{B}^{(\lambda)}_\alpha\}$ is a (semi–) ranked group with indicator ω.

Proof. We have only to show that the mapping: $(x,y)\longrightarrow xy^{-1}$ of $\bar{G}\times\bar{G}$ onto $\bar{G}$ is R-continuous. Let x,y be two elements of $\bar{G}$, and $\{U_\alpha(x);\ 0\leqslant\alpha<\omega\}$, $\{V_\alpha(y);\ 0\leqslant\alpha<\omega\}$ two fundamental sequences of x,y in $\bar{G}$ such that, for each $\lambda\in\Lambda$, $U^{(\lambda)}_\alpha(x)\equiv(u^{(\lambda)}_\alpha(x_\lambda))_{\lambda\in\Lambda}$ and $V_\alpha(y)\equiv(v^{(\lambda)}_\alpha(y_\lambda))_{\lambda\in\Lambda}$. By the definition of the fundamental sequences in the direct product ranked space $\bar{G}$, for each $\lambda\in\Lambda$, $\{u^{(\lambda)}_\alpha(x_\lambda);\ 0\leqslant\alpha<\omega\}$ and $\{v^{(\lambda)}_\alpha(y_\lambda);\ 0\leqslant\alpha<\omega\}$ are respectively fundamental sequences of x_λ and y_λ in $\{G_\lambda,\mathfrak{B}^{(\lambda)}_\alpha\}$. Since each G_λ is a ranked group with indicator ω, for each $\lambda\in\Lambda$, there exists a fundamental sequence $\{w^{(\lambda)}_\alpha(x_\lambda y_\lambda^{-1});\ 0\leqslant\alpha<\omega\}$ of $x_\lambda y_\lambda^{-1}$ in G_λ such that $u^{(\lambda)}_\alpha(x_\lambda)\cdot v^{(\lambda)}_\alpha(y_\lambda)^{-1}\subseteq w^{(\lambda)}_\alpha(x_\lambda y_\lambda^{-1})$. Hence, the sequence $\{W_\alpha(xy^{-1});\ 0\leqslant\alpha<\omega\}$ such that $W_\alpha(xy^{-1})\equiv(w^{(\lambda)}_\alpha(x_\lambda y_\lambda^{-1}))_{\lambda\in\Lambda}$ becomes a fundamental sequence of xy^{-1} in $\bar{G}$. Thus, for each $\lambda\in\Lambda$, we have the following:

$$U_\alpha(x)\cdot V_\alpha(y)^{-1}=(u^{(\lambda)}_\alpha(x_\lambda)\cdot v^{(\lambda)}_\alpha(y_\lambda)^{-1})_{\lambda\in\Lambda}\subseteq(w^{(\lambda)}_\alpha(x_\lambda y_\lambda^{-1}))_{\lambda\in\Lambda}=W_\alpha(xy^{-1}).$$

Therefore $\{\bar{G},\bar{\mathfrak{B}}_\alpha\}$ becomes a ranked group with indicator ω.

Definition 9. The (semi–) ranked group $\{\bar{G},\bar{\mathfrak{B}}_\alpha\}$ is called the ***direct product (semi–) ranked group*** with indicator ω of the ranked groups $(G_\lambda,\mathfrak{B}^{(\lambda)}_\alpha)$ (resp. $\langle G_\lambda,\mathfrak{B}^{(\lambda)}_\alpha\rangle$) (where $\lambda\in\Lambda$). We denote this by $(\bar{G},\bar{\mathfrak{B}}_\alpha)$ (resp. $\langle\bar{G},\bar{\mathfrak{B}}_\alpha\rangle$) and $(\prod_{\lambda\in\Lambda}G_\lambda,\prod_{\lambda\in\Lambda}\mathfrak{B}^{(\lambda)}_\alpha)$ (resp. $\langle\prod_{\lambda\in\Lambda}G_\lambda,\prod_{\lambda\in\Lambda}\mathfrak{B}^{(\lambda)}_\alpha\rangle$),etc.

Proposition 22. Let Λ be any indexing set, $\bar{G}=\prod_{\lambda\in\Lambda}G_\lambda$ the direct product (semi–) ranked group of the (semi–) ranked groups $G_\lambda\,(\lambda\in\Lambda)$, and $\{V_\alpha(p);\ 0\leqslant\alpha<\omega\}$ $(p\equiv(p_\lambda)_{\lambda\in\Lambda},\ v_\alpha(p)\equiv(v^{(\lambda)}_\alpha(p_\lambda))_{\lambda\in\Lambda})$ any fundamental sequence of p in $\bar{G}$. Then, by the λ-th projection mapping pr_λ: $\prod_{\lambda\in\Lambda}G\ni{}^V(a_\lambda)_{\lambda\in\Lambda}\rightarrow a_\lambda\in G_\lambda$, $pr_\lambda(\{V_\alpha(p);\ 0\leqslant\alpha<\omega\})=\{v^{(\lambda)}_\alpha(p_\lambda);\ 0\leqslant\alpha<\omega\}$ becomes a fundamental sequence of p_λ in G_λ. Thus, for each $\lambda\in\Lambda$, pr_λ is R-continuous. Moreover, $pr_\lambda\,(\lambda\in\Lambda)$ is a homomorphism of $\bar{G}$ onto $\bar{G}_\lambda$.

Proposition 23. For each i $(i=1,2,\ldots,m)$, let N_i be a normal subgroup with indicator ω of the (semi-) ranked group G_i with indicator ω. Then, $N_1\times\cdots\times N_m$ is a normal subgroup with indicator ω of the direct product (semi-) ranked group $G_1\times\cdots\times G_m$ with indicator ω, and, in the sense of the ranked group, we have

$$G_1\times\cdots\times G_m/N_1\times\cdots\times N_m\cong G_1/N_1\times\cdots\times G_m/N_m.$$

Proposition 24. Let $\bar{G}=\prod_{\lambda\in\Lambda}G_\lambda$ be a direct product (semi-) ranked group of some (semi-) ranked groups $G_\lambda\,(\lambda\in\Lambda)$. Then, $\bar{G}$ is complete iff G_λ is complete for each $\lambda\in\Lambda$.

ii) Direct product decomposition.

Proposition 25. Let $N_1,\ldots,N_m$ be a system of ranked groups with same indicator ω, e_i the identity of $N_i\,(i=1,2,\ldots,m)$, G' the direct product (semi-) ranked group of $N_1,\ldots,N_m$. And let $\varphi_i:\ N_i\ni x_i\longrightarrow(e_1,\ldots,e_{i-1},\ x_i,\ e_{i+1},\ldots,e_m)\in G'$ for each $i=1,2,\ldots,m$. Then we have the following statements: For each $i=1,2,\ldots,m$,

(1) φ_i becomes a homeomorphism of the ranked group N_i onto an induced space $N'_i(=\varphi_i(N_i))$ in G' and N'_i is normal in G'.

(2) In the sense of ranked groups, we have $N_i\overset{\varphi_i}{\cong}N'_i$, $N_1\times\cdots\times N_m\cong N'_1\times\cdots\times N'_m$.

(3) In algebraic sense, G' decomposes into the direct product of its subgroups $N'_1,\ldots,N'_m$.

(4) For any $\{V^{(i)}_\alpha(x'_i)\}$ (F. S. of x'_i in N'_i, $1\leqslant i\leqslant m$), there is a $\{V_\alpha(x')\}$ (F. S. of x' in G') such that $V^{(1)}_\alpha(x'_1)\cdots V^{(m)}_\alpha(x'_m)\subseteq V_\alpha(x')$ $(x'=x'_1\cdots x'_m)$.

Definition 10. Let G be a (semi-) ranked group with indicator ω, and $N_1,\ldots,N_m$ a system of normal subgroups of the (semi-) ranked group G. We say that the (semi-) ranked group G ***decomposes*** into the direct product of its subgroups $N_1,\ldots,N_m$ if the following conditions are fulfilled:

(1°) In algebraic sense, the group G can be decomposed into the direct product of its subgroups $N_1,\ldots,N_m$,

(2°) For any fundamental sequence $\{V_\alpha(x);\ 0\leqslant\alpha<\omega\}$ of x in G, and for each $i=1,2,\ldots,m$, there exists a fundamental sequence $\{V^{(i)}_\alpha(x_i);\ 0\leqslant\alpha<\omega\}$ of x_i in N_i such that $V_\alpha(x)=V^{(1)}_\alpha(x_1)\cdots V^{(m)}_\alpha(x_m)$ $(0\leqslant\alpha<\omega)$ $(x=x_1\cdots x_m,\ x_i\in N_i)$,

(3°) For each $i=1,2,\ldots,m$, and for any fundamental sequence $\{U^{(i)}_\alpha(x_i);\ 0\leqslant\alpha<\omega\}$ of x_i in N_i, there exists a fundamental sequence $\{U_\alpha(x);\ 0\leqslant\alpha<\omega\}$ of $x=x_1\cdots x_m$ in G such that $U^{(1)}_\alpha(x_1)\cdots U^{(m)}_\alpha(x_m)\subseteq U_\alpha(x)$ $(0\leqslant\alpha<\omega)$.

From this definition we have the following statement:

Proposition 26. Suppose that the ranked group G can be decomposed into the direct product of its ranked subgroups $N_1,\ldots,N_m$. And let G' be the direct product (semi-) ranked group of the ranked groups $N_1,\ldots,N_m$. Then we have the following statements:

(1) $\varphi:\ G'\ni\forall x=(x_1,\ldots,x_m)\longrightarrow x_1\cdots x_m\in G$ becomes an isomorphism of the ranked group G' to the ranked group G,

(2) φ satisfies the condition (ii) in (*),

(3) $\varphi\cdot\varphi_i$ becomes an R-continuous identity mapping of N_i onto itself for each $i=1,2,\ldots,m$.

Proposition 27. Let G be a ranked group, and $N_1,\ldots,N_m$ a system of normal subgroup of G. And suppose that G is decomposed into the direct product of $N_1,\ldots,N_m$. Then the projection mapping $pr_i:\ G\ni(a_i)_{1\leqslant i\leqslant m}\longrightarrow a_i\in N_i$ is R-continuous for every $i=1,2,\ldots,m$.

Moreover, as is known from the abstract group theory, a ranked group $(G,\mathfrak{V}_\alpha(G))$ is isomorphic to the direct product of two subgroups $(A,\mathfrak{V}_\alpha(A))$ and $(B,\mathfrak{V}_\alpha(B))$ if $(A,\mathfrak{V}_\alpha(A))$ and $(B,\mathfrak{V}_\alpha(B))$ are normal subgroups such that $A\cap B=e$, $A\cup B=G$, $\mathfrak{V}_\alpha(A)\times\mathfrak{V}_\alpha(B)=\mathfrak{V}_\alpha(A\times B)$ $(0\leqslant\alpha<\omega)$. In general, we have the following:

Proposition 28. A ranked group $(G,\mathfrak{V}_\alpha(G))$ is isomorphic to the direct product of its subgroups $(G_i,\mathfrak{V}^{(i)}(G_i))$ $(i=1,2,\ldots,m)$ if

(1) every G_i is an abstract normal subgroup of G,

(2) $G_i\cap(\bigcup_{j\neq i}G_j)=e$ for every $j=1,2,\ldots,m$,

(3) $G=\bigcup_{i=1}^{m} G_i$ and $\mathfrak{B}_\alpha(G)=\prod_{i=1}^{m}\mathfrak{B}_\alpha^{(i)}(G_i)$ for every α, $0\leqslant\alpha<\omega$.

iii) Embeddings of any group in a product group.

If, for each $\lambda\in\Lambda$ ($\lambda\in\Lambda$, any indexing set), $G_\lambda=G$, the product $\prod_{\lambda\in\Lambda}G_\lambda$ is denoted simply by G^Λ. G^Λ is the set of all mappings of Λ into G, and if G is a group so is G^Λ. Now if $\Lambda=G$, which is a group, then G^G is also a group.

By the mapping η_r: $G\ni a\longrightarrow r_a\in G^G$ (or η_l: $a\longrightarrow l_a$), G is mapped onto $\eta_r(G)\subseteq G^G$ (or $\eta_l(G)\subseteq G^G$). Therfore, we have the following:

Proposition 29. Any abstract group G can be embedded into G^G, i.e., there exists a one-to-one mapping of G onto a subset of G^G.

Notations. The mappings η_r: $a\longrightarrow r_a$ and η_l: $a\longrightarrow l_a$ of G into G^G will be called the ***right*** and ***left canonical embeddings*** of G into G^G respectively. In case G is an abelian group, $r_a=l_a$ and thus $\eta_r=\eta_l$.

Proposition 30. Let G be a (semi-) ranked group. Then G is isomorphic to $\eta_r(G)\subseteq G^G$.

In fact, for every $(G,\mathfrak{B}_\alpha)$ we have the following:

$$a(\in G)\overset{\eta_r}{\longleftrightarrow}r_a,\quad U(\in\mathfrak{B})\overset{\eta_r}{\longleftrightarrow}r_U\equiv\{r_a\,|\,a\in U\},\quad \mathfrak{B}_\alpha\overset{\eta_r}{\longleftrightarrow}r_{\mathfrak{B}_\alpha}\equiv\{r_U\,|\,U\in\mathfrak{B}_\alpha\}\ (0\leqslant\alpha<\omega).$$

§ 4. *Other ranked algebraic systems.*

i) Ranked Rings.

Definition 11. An abstract ring R is called a ***ranked ring*** if the set R is a ranked space and if the following conditions are fulfilled:

(1) The mapping $(x,y)\longrightarrow x-y$ of $R\times R$ into R is R-continuous,

(2) The mapping $(x,y)\longrightarrow xy$ of $R\times R$ into R is R-continuous.

These conditions (1) and (2) are respectively equivalent to the following conditions:

(1′) For any fundamental sequences $\{u_\alpha(x);\ 0\leqslant\alpha<\omega\}$, $\{v_\alpha(y);\ 0\leqslant\alpha<\omega\}$ of any points x,y in R, there is a fundamental sequence $\{w_\alpha(x-y);\ 0\leqslant\alpha<\omega\}$ of $x-y$ in R such that

$$u_\alpha(x)-v_\alpha(y)\subseteq w_\alpha(x-y)\ (\forall\alpha;\ 0\leqslant\alpha<\omega),$$

(2′) For any fundamental sequence $\{u_\alpha(x);\ 0\leqslant\alpha<\omega\}$, $\{v_\alpha(y);\ 0\leqslant\alpha<\omega\}$ of any points x,y in R, there is a fundamental sequence $\{w_\alpha(xy);\ 0\leqslant\alpha<\omega\}$ of xy in R such that

$$u_\alpha(x)\cdot v_\alpha(y)\subseteq w_\alpha(xy)\ (\forall\alpha;\ 0\leqslant\alpha<\omega).$$

Moreover the condition (1) is equivalent to the following condition (1″) and the condition (1‴):

(1″) The mappings $R\times R\ni(x,y)\longrightarrow x+y\in R$ and $R\ni x\longrightarrow -x\in R$ are R-continuous.

(1‴)

① For any fundamental sequence $\{u_\alpha(x);\ 0\leqslant\alpha<\omega\}$, $\{v_\alpha(y);\ 0\leqslant\alpha<\omega\}$ of any points x,y in R, there is a fundamental sequence $\{w_\alpha(x+y);\ 0\leqslant\alpha<\omega\}$ of $x+y$ in R such that

$$u_\alpha(x)+v_\alpha(y)\subseteq w_\alpha(x+y)\ (\forall\alpha;\ 0\leqslant\alpha<\omega)$$

and

② For any fundamental sequence $\{u_\alpha(x);\ 0\leqslant\alpha<\omega\}$ of any point x in R, there is a fundamental sequence $\{v_\alpha(-x);\ 0\leqslant\alpha<\omega\}$ of $-x$ in R such that

$$-u_\alpha(x)\subseteq v_\alpha(-x)\ (\forall\alpha;\ 0\leqslant\alpha<\omega).$$

Definition 12. (I) An abstract subring H of the ranked ring R is called a ***subring*** of the ranked ring R if the set H is a subspace of the ranked space R.

(II) A subset I of the ranked ring R is called an ***ideal*** of the ranked ring R if the set I that is an abstract ideal of the abstract ring R is also a subspace of the ranked space R,

(III) R/I is called a ***quotient ring.*** (R/I becomes a ranked ring.)

(IV) Let $(R)_{\lambda\in\Lambda}$ be a family of ranked rings with same indicator. Then the direct product $\bar{R}=\prod_{\lambda\in\Lambda}\bar{R}_\lambda$ becomes a ranked ring. This ranked ring $\bar{R}$ is called the ***product ring*** of the ranked rings (R_λ).

In a ranked ring R every right translation r_a (resp. every left translation l_a) is R-continuous (and is a homeomorphism if a^{-1} exists in R).

Let S be a ranked space, and let f and g be two mappings of S into a ranked ring R. If f and g are R-continuous at a point $p\in S$, then $f+g$, $-f$ and fg are R-continuous at this point. (Attend to $f+g$: $x\rightarrow f(x)+g(x)$, $-f$: $x\rightarrow -f(x)$, fg: $x\longrightarrow f(x)g(x)$.) Thus we have the following:

Proposition 31. The R-continuous mappings of the ranked space S into the ranked ring R form a subring of the ring R^S of all mappings of S into R.

Definition 13. (I) A mapping g of a ranked ring R into a ranked ring R' is called a ***homomorphism*** if

(i) g is a homomorphism of the abstract ring R into the abstract ring G',

(ii) g satisfies the condition (*).

(II) A mapping f of a ranked ring R onto a ranked ring R' is called an ***isomorphism*** if

(i) f is an isomorphism of the abstract ring R onto the abstract ring R',

(ii) f is a homeomorphism of the ranked space R onto the ranked space R'.

Two ranked rings are called ***isomorphic*** if there exists an isomorphism of one ring onto the other. In particular, an isomorphism of a ranked ring R into itself is called an ***automorphism*** of the ring R.

Proposition 32. Let R and R' be two ranked rings with same indicator ω and H a subring of the ranked ring R. If g is a homomorphism of R onto R', then $g(H)$ is a subring of the ranked ring R'. If f is an isomorphism of R onto R' and if f satisfies the condition (ii) in (*) then $f(H)$ is a subring of the ranked ring R'.

ii) Ranked Modules.

Definition 14. Given a ranked ring A with an identity element, a set E is called a ***ranked left A-module*** if the following conditions are fulfilled:

(1) E is an abstract left A-module

(2) E is a ranked additive group,

(3) The mapping $(\lambda, x)\longrightarrow\lambda x$ of $A\times E$ into E is R-continuous.

We define similarly the notion of a ***ranked right A-module.*** Since every right A-module can be considered as a left A°-module, where A° is the opposite ring of A, and since the ranked structure of A is compatible with the ring structure of A°, there is no need to distinguish between ranked right A-modules and ranked left A°-modules.

Let $(E_\nu)_{\nu\in\Lambda}$ be an arbitrary family of ranked A-modules, and let $\bar{E}=\prod_{\nu\in\Lambda}E_\nu$ be the A-module which is the product of the E_ν. For each $\mu\in\Lambda$ the mapping $(\lambda, x)\longrightarrow(\lambda, pr_\mu x)$ satisfies the condition (ii) in (*), and the mapping $(\lambda, x_\mu)\longrightarrow\lambda x_\mu$ is R-continuous. Since the mapping $(\lambda, x)\longrightarrow\lambda\cdot pr_\mu x$ is the composition of $(\lambda, x_\mu)\longrightarrow\lambda x_\mu$ and $(\lambda, x)\longrightarrow(\lambda, pr_\mu x)$ for each $\mu\in\Lambda$, the mapping $(\lambda, x)\longrightarrow\lambda\cdot pr_\mu x$ is an R-continuous mapping of $A\times E$ into E_μ. Therefore $\bar{E}$ becomes a ranked A-module.

iii) Ranked Fields

Definition 15. A set K is called a ***ranked field*** if the following conditions are fulfilled:

(1) K is an abstract field,

(2) K is a ranked space,

(3) The algebraic operations operating in K are R-continuous in the ranked space K.

If K is an abstract devision ring we shall denote by K^* the ***multiplicative group*** of non-zero elements of K.

If K is a ranked field then the mapping $x \longrightarrow x^{-1}$ of K^* into K^* is R-continuous.
The condition (3) is equivalent to the following condition (3′):

(3′)
① For any fundamental sequences $\{u_\alpha(x); 0 \leqslant \alpha < \omega\}$, $\{v_\alpha(y); 0 \leqslant \alpha < \omega\}$ of any points x, y in K, there are two fundamental sequences $\{w_\alpha(x+y); 0 \leqslant \alpha < \omega\}$, $\{w_\alpha'(xy); 0 \leqslant \alpha < \omega\}$ of $x+y$, xy in K such that

$$u_\alpha(x) - v_\alpha(y) \subseteq w_\alpha(x-y), \quad u_\alpha(w) \cdot v_\alpha(y) \subseteq w_\alpha(xy) \quad (\forall_\alpha; 0 \leqslant \alpha < \omega),$$

② For any point $x \neq 0\,(x \in K)$ and for any fundamental sequence $\{u_\alpha(x); 0 \leqslant \alpha < \omega\}$, there is a fundamental sequence $\{v_\alpha(x^{-1}); 0 \leqslant \alpha < \omega\}$ of x^{-1} in K such that

$$u_\alpha(x)^{-1} \subseteq v_\alpha(x^{-1}) \quad (\forall\,; 0 \leqslant \alpha < \omega).$$

If $a \neq 0$, the translations ℓ_a and r_a are homeomorphisms of K onto itself. Thus, the mapping $\ell : x \longrightarrow ax+b$ is a one-to-one R-continuous mapping for all $a, b \in K$. In fact, for any fundamental sequence $\{u_\alpha(x); 0 \leqslant \alpha < \omega\}$ of x in K, there are respectively four fundamental sequences $\{u_\alpha'(a); 0 \leqslant \alpha < \omega\}$, $\{u_\alpha''(b); 0 \leqslant \alpha < \omega\}$, $\{u_\alpha'''(ax); 0 \leqslant \alpha < \omega\}$ and $\{w_\alpha(ax+b); 0 \leqslant \alpha < \omega\}$ of a, b, ax and $ax+b$ in K such that

$$\ell(u_\alpha(x)) = a \cdot u_\alpha(x) + b \subseteq u_\alpha'(a) \cdot u_\alpha(x) + u_\alpha''(b) \subseteq u_\alpha'''(ax) + u_\alpha''(b) \subseteq w_\alpha(ax+b) = w_\alpha(\ell(x))$$

for all α where $0 \leqslant \alpha < \omega$. Hence ℓ is R-continuous. Therefore if $a \neq 0$, ℓ is a homeomorphism. Note that the translations ℓ_a and r_a are ***automorphisms*** of the (ranked) ***additive group*** of K if $a \neq 0$. If ℓ_a (resp. r_a) satisfies the condition (ii) in (*), and if $V \in \mathfrak{V}(0)$, then $aV \in \mathfrak{V}(0)$ (resp. $Va \in \mathfrak{V}(0)$) for $a \neq 0$.

Proposition 33. In the ranked field K if $\{\lim_\alpha a_\alpha\} \ni a$ and $\{\lim_\alpha b_\alpha\} \ni b$ then we have the following properties:

(1) $\{\lim_\alpha (a_\alpha + b_\alpha)\} \ni a+b$, (2) $\{\lim_\alpha (-a_\alpha)\} \ni -a$,

(3) $\{\lim_\alpha a_\alpha b_\alpha\} \ni ab$, (4) if $a \neq 0$ and $a_\alpha \neq 0$, $\{\lim_\alpha a_\alpha^{-1}\} \ni a^{-1}$.

In fact, since the algebraic operations operating in K are R-cotinuous we have this proposition.

iv) Linear Ranked Spaces.

Definition 16. Given a ranked field K with indicator ω, a set L is called a ***left-linear ranked space*** (resp. ***right-linear ranked space***) with indicator ω over K if the following conditions are fulfilled:

(1) L is an abstract left linear space (resp. right linear space),

(2) L is a ranked space with indicator ω,

(3) The mapping $(\lambda, x) \longrightarrow \lambda x$ of $K \times L$ into L is R-continuous.

Proposition 34. In view of (H) above condition (3) is equivalent to the conjunction of the following three conditions:

(3′) For each $x_0 \in L$, the mapping $\lambda \longrightarrow \lambda x_0$ is R-continuous at the point $\lambda = 0$,

(3″) For each $\lambda_0 \in K$, the mapping $x \longrightarrow \lambda_0 x$ is R-continuous at the point $x = 0$,

(3‴) The mapping $(\lambda, x) \longrightarrow \lambda x$ is R-continuous at the point (0,0).

Proposition 35. For every $\alpha \in K$ and every $b \in L$, the mapping $x \longrightarrow \alpha x + b$ of L into itself is R-continuous. And if $\alpha \neq 0$ this mapping becomes a homeomorphism of L onto itself.

An algebraic isomorphism f (resp. algebraic homomorphism g) of a linear ranked space S to a linear ranked space T is called an ***isomorphism*** (resp. ***homomorphism***) of S to T if f is a one-to-one bi-R-continuous mapping (resp. G satisfies the condition (*)).

EXAMPLE Let E be a ranked space with indicator ω_0, and also a linear space over the real or complex field C. In C we define S_n and $\mathfrak{V}_n$ $(0 \leqslant n < \omega_0)$ as follows:

$$\left.\begin{array}{l} S_n(\lambda) \equiv \{\mu; 0 \leqslant |\mu - \lambda| < 1/n,\ \mu \in C\} \text{ for each } \lambda \in C, \\ \mathfrak{V}_n \ni S_n(\lambda) \text{ for all } n = 0, 1, 2, \dots. \end{array}\right] .$$

Then C becomes a ranked space with indicator ω_0 by $\mathfrak{V}_n$. Moreover, if $\{\lim_{n\to\omega_0} \lambda_n\} \ni \lambda$ in C then we have $\lim_{n\to\infty} \lambda_n = \lambda$ in the topological sense. Therefore this linear ranked space $\{E, \mathfrak{V}_n\}$ becomes a linear ranked space in the sence of M. Washihara [5; II].

§ 5. ***Ortho-continuous group, Para-continuous group.*** In this section we will give other definitions of ranked groups. They are the ortho-continuous groups and the para-continuous groups.

Given a sequence $\{p_\alpha; 0\leqslant\alpha<\omega\}$ of points of $\{S, \mathfrak{V}_\alpha\}$ and a point p of $\{S, \mathfrak{V}_\alpha\}$. We say that the sequence $\{p_\alpha; 0\leqslant\alpha<\omega\}$ ***para-converges***[10] to the point p, or that p is a ***para-limit*** of $\{p_\alpha; 0\leqslant\alpha<\omega\}$, if there is a monotone decreasing sequence $\{V_\alpha(p_\alpha); 0\leqslant\alpha<\omega\}$ consisting of neighbourhoods of p_α and if $V_\alpha(p_\alpha)$ satisfies the following conditions:

1) $V_\alpha(p_\alpha) \in \mathfrak{V}_{\gamma(\alpha)}$,
2) $\gamma(0)\leqslant\gamma(1)\leqslant\gamma(2)\leqslant\cdots\cdots\leqslant\gamma(\alpha)\leqslant\cdots\cdots (0\leqslant\gamma(\alpha)<\omega)$, $\sup_\alpha \gamma(\alpha)=\omega$,
3) $p\in V_\alpha(p_\alpha)$ for all α, $0\leqslant\alpha<\omega$.

In this case, we shall write

$$p\in\{para\text{-}\lim_\alpha p_\alpha\}.$$

Definition 17. Let R, S be two ranked spaces with same indicator ω. Then we will say that the mapping $f: R\longrightarrow S$ is ***para-continuous*** at the point p in R iff

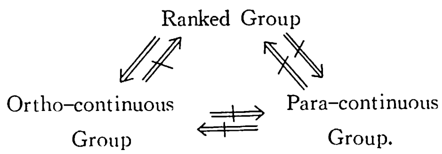
$$p\in\{para\text{-}\lim_\alpha p_\alpha\} \Rightarrow f(p)\in\{para\text{-}\lim_\alpha f(p_\alpha)\}.$$

The mapping f is said to be ***para-continuous*** if it is para-continuous at each point of R.

Definition 18. For $\{G, \mathfrak{V}_\alpha\}$ with indicator ω, if G is an abstract group and the group operations $\{G\times G, \mathfrak{V}_\alpha\times\mathfrak{V}_\alpha\}\ni(x,y)\longrightarrow xy\in\{G, \mathfrak{V}_\alpha\}$, $\{G, \mathfrak{V}_\alpha\}\ni x\longrightarrow x^{-1}\in\{G, \mathfrak{V}_\alpha\}$ are ortho-continuous (resp. para-continuous), then $\{G, \mathfrak{V}_\alpha\}$ is called the ***ortho-continuous group*** (resp. ***para-continuous group***) with indicator ω.

Since two notions of ortho-convergence and para-convergence do not necessarily coincide with each other,[11] two notions of ortho-continuity and para-continuity do not necessarily coincide with each other.[12] Therefore an ortho-continuous group is not always a para-continuous group and conversely a para-continuous group is not always an ortho-continuous group. On the other hand, from ***Proposition 1*** we have the following statement:

Proposition 36 . If G is a ranked group in the sense of Definition 3, G becomes an ortho-continuous group. But the converse is not true. Thus we have



Remark. If G is a linear ranked space in the sense of M. Yamaguchi [9; II], then G is not always an ortho-continuous additive group. But G becomes a ranked additive group with indicator ω_0 in the sense of Definition 3. (See [5; II])

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10) [17], pp. 23-24.

11) Ibid., pp. 24-25, Propositions 2 and 2′.

12) For the sequence $\{p_\alpha; 0\leqslant\alpha<\omega\}$ $(p_\alpha\equiv p, 0\leqslant\alpha<\omega)$, these two notions of continuity coincide with each other.

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(To be continued.)

References

[1] K. Kunugi : Sur les Espaces Complets et Régulièrement Complets, I–III. Proc. Japan Acad., 30 (1954) 553–556, 912–916; 31 (1955) 49–53.

[2] ———— : Application de la méthode des espaces rangés à la theorie de l' integration, I. Proc. Japan Acad., 32 (1956) 215–220.

[3] ———— : Sur la méthode des espaces rangés, I, II. Proc. Japan Acad., 42 (1966) 318–322, 549–554.

[4] S. Nakanishi : On Generalized Integrals, I–III. Proc. Japan Acad., 44 (1968) 133–138, 225–230, 904–909.

[5] M. Washihara : On Ranked Spaces and Linearity, I, II. Proc. Japan Acad., 43 (1967) 584–589; 45 (1969) 238–242.

[6] ———— : On the Ranked Group. Proc. Japan Acad., 44 (1968) 246–250.

[7] Y. Yoshida : Sur le produit des espaces rangés, I,II. 42 (1966) 477–481, 611–615.

[8] ———— : Sur les structures des espaces rangés, I,II. Proc. Japan Acad., 42 (1966) 166–619, 1144–1148

[9] M. Yamaguchi : Calculus in Ranked Vector Spaces, I–VI. Proc. Japan Acad., 44 (1968) 207–212, 213–218, 307–311, 312–317, 424–433.

[10] ———— : On a Ranked Vector Space. Proc. Japan Acad., 46 (1970) 422–426.

[11] H. Okano : On Closed Subspaces of the Complete Ranked Spaces. Proc. Japan Acad., 33 (1957) 336–337.

[12] H. R. Fischer : Limesräume. Math. Ann., (1959) 269–303.

[13] J. W. Tukey : Convergence and uniformity in topology. Princeton (1940).

[14] D. Montgomery : Continuity in topological groups. Bull. Amer. Math. Soc., 42 (1936) 879–882.

[15] Ta–Sun. Wu : Continuity in topological groups. Proc. Amer. Math. Soc., 13 (1962) 452–453.

[16] T. Husain : Semitopological groups and linear spaces. Math. Ann., 160 (1965) 146–160.

[17] H. Nagashima, K. Yajima and Y. Sakamoto : On an equivalence of convergences in ranked spaces. Proc. Japan Acad., 43 (1967) 23–27.

[18] ———— : On the Sets of Points in the Ranked Space. I–III. Proc. Japan Acad., 43 (1967) 941–945; 44 (1968) 788–791, 792–795.

[19] T. Mori : 位相線型空間 (Lebesgue 積分論と Banach 空間論の発展として). 数学, 12 (1961) 210–225.

[20] Л. С. Понтрягин : Непрерывные Группы, ГостехиздАт (1954).

[21] L. S. Pontorjagin : Topologische Gruppen, B. G. Teubner, Leipzig (1957–1959).

[22] ———————— : Topological groups, Princeton University Press, Princeton, N. J. (1958).

[23] D. Montgomery and L. Zippin : Topological transformation groups, Interscience Publishers, Inc., New York, (1955).

[24] A. Weil : L'integration dans la Groupes Topologiques et ses Applications, Hermann Cie., Paris (1940).

[25] T. Tannaka : 位相群論, 岩波 (1949).

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***) Tokyo University of Sciences.

[26] N. Bourbaki : Espaces Vectoriels Topologiques. Hermann, Paris (1966), Chap. I–II.
[27] ———— : Topologie Generale. Livre III. Hermann, Paris (1960), Chap. I–III.
[28] ———— : General Topology. Part 1, 2. Addison–Wesley Pub. Company, London (1966),
[29] J. L. Kelley : General topology. D. Van Nostrand Company, New York. (1965).
[30] E. T. Copson : Metric Spaces. Cambridge University Press (1968).
[31] F. Hausdorff : Grundzüge der Mengenlehre. Leipzig (1914).
[32] ———— : Set Theory. Chelsea Pub. Company, New York (1962).
[33] K. Yoshida : Functional Analysis. Second Edition. Springer–Verlag, Berlin (1968).
[34] L. Schwartz : Theorie des distributions, Hermann Cie, Paris (1966).
[35] ———— : Functional Analysis, New York University (1964).
[36] I. M. Gel'fand and G. E. Shilov : Generalized Functions, Vol. 2, Academic Press, New York (1968).
[37] A. Г. Курош : Теория Групп. Москва (1967).
[38] M. Hall : The Theory of Groups. The Macmillan Company, New York (1965).
[39] H. J. Zassenhaus : The Theory of Groups. Chelsea Pub. Company, New York (1958).
[40] T. Inagaki : 点集合論，岩波 (1968).
[41] G. Köthe : Topologische lineare Räume, I. Springer–Verlag, Berlin (1960).
[42] L. Garsoux : Espaces vectoriels topologiques et distributions. Dunod, Paris (1963).
[43] L. Nachbin : The Haar integral. D. Van Nostrand Company, Princeton, N. J. (1965).
[44] J. L. Kelley and I. Namioka : Linear Topological Spaces. D. Van Nostrand Co., Inc., Princeton, N. J. (1970).
[45] T. Shintani : On the Ranked Groups. 日本数学会秋季全国講演会講演 (1970).