Some considerations in the Ranked Spaces

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Synopsis

The purpose of this paper is to study some properties of sets of points in the ranked spaces.

We shall use the same terminology that is introduced in [1] and [2]. And throughout this paper we shall always treat ranked spaces with general indicator ω.

1. Derived sets, Adherences.

Definition 1. Let R be a ranked space and let E any subset of R. A point p in R is called an ***accumulation point***[1] of E iff there exists a fundamental sequence $\{V_\alpha(p);\ 0\leqslant\alpha<\omega\}$ of p in R such that $V_\alpha(p)\cap(E-p)\neq\phi$ for all α, $0\leqslant\alpha<\omega$. The set of all accumulation points of E is called the ***derived set*** of E. We denote this by $E.^d$ $E^a\equiv\{p\,|\,p\in\{\lim_\alpha p_\alpha\},\ p_\alpha\in E\,(0\leqslant\alpha<\omega)\}$ is called the ***adherence*** of E and each point of E^a is called an ***adherent point*** of E.

Proposition 1. $E\subseteq E^a$, $E\subseteq F\Rightarrow E^a\subseteq F^a$, $\phi^a=\phi$, $R^a=R$, $E^a\subseteq E^{aa}$.

Proposition 2. $p\in E^d\Leftrightarrow p\in(E-p)^a$.

Proof. If p is a point of E^d then there is a fundamental sequence $\{V_\alpha(p);\ 0\leqslant\alpha<\omega\}$ of p in R such that $V_\alpha(p)\cap(E-p)\neq\phi$ for all α, $0\leqslant\alpha<\omega$. Thus there is a sequence $\{p_\alpha;\ 0\leqslant\alpha<\omega\}$ such that $p_\alpha\in V_\alpha(p)\cap(E-p)$ for all α, $0\leqslant\alpha<\omega$. Thus we have $p\in\{\lim_\alpha p_\alpha\}$ $(p_\alpha\in E-p,(0\leqslant\alpha<\omega))$. Therefore we have $p\in(E-p)^a$. Conversely, if $p\in(E-p)^a$ then there is a sequence $\{p_\alpha;\ 0\leqslant\alpha<\omega\}$ such that $p\in\{\lim_\alpha p_\alpha\}$ $(p_\alpha\in E-p,(0\leqslant\alpha<\omega))$. Thus there is a fundamental sequence $\{V_\alpha(p);\ 0\leqslant\alpha<\omega\}$ of p such that $p_\alpha\in V_\alpha(p)\cap(E-p)$ for all α, $0\leqslant\alpha<\omega$. Hence we have $V_\alpha(p)\cap(E-p)\neq\phi$ for all α, $0\leqslant\alpha<\omega$. This shows $p\in E^d$.

Proposition 3. $E^a=E\cup E^d$, $(E\cup F)^d\subseteq E^d\cup F^d$, $(E\cup F)^a=E^a\cup F^a$.

Proof. From $p\in E^d\Leftrightarrow p\in(E-p)^a\subseteq E^a$ we get $E^d\subseteq E^a$. Hence $E\cup E^d\subseteq E^a$. Conversely, let p be an element of E^a. In the case of $p\in E$ we get $E^a\subseteq E\subseteq E\cup E^d$. In the case of $p\notin E$, if $p\notin E^d$ then we have $E^a\supseteq E\cup E^d\not\ni p$, i. e., $p\notin E^a$. This is a contradiction. Hence we have $p\in E^d$. Thus we have $E^a\subseteq E^d\subseteq E\cup E^d$. Therefore we have $E^a=E\cup E^d$.

Proposition 4. For any subset E of the ranked space R, the following two conditions are equivalent:

(1) $p\in E^a$.

(2) There is a fundamental sequence $\{V_\alpha(p);\ 0\leqslant\alpha<\omega\}$ of p in R such that $V_\alpha(p)\cap E\neq\phi$ for all α, $0\leqslant\alpha<\omega$.

Proof. Let p be a point of E^a. Then we have $p\in E$ or $p\in E^d$. And there is a fundamental sequence $\{V_\alpha(p);\ 0\leqslant\alpha<\omega\}$ of p in R. In this time, if $p\in E$ then $V_\alpha(p)\cap E\neq\phi$ for all α, $0\leqslant\alpha<\omega$, and if $p\in E^d$ then $V_\alpha(p)\cap E\supseteq V_\alpha(p)\cap(E-p)\neq\phi$ for all α, $0\leqslant\alpha<\omega$. Hence if $p\in E^a$ there is a fundamental sequence

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1) [4], II, p. 788.

$\{V_\alpha(p);\ 0\leqslant\alpha<\omega\}$ of p in R such that $V_\alpha(p)\cap E\neq\phi$ for all $\alpha, 0\leqslant\alpha<\omega$. Conversely, suppose that there is a fundamental sequence $\{V_\alpha(p);\ 0\leqslant\alpha<\omega\}$ of a point p in R such that $V_\alpha(p)\cap E\neq\phi$ for all $\alpha, 0\leqslant\alpha<\omega$. Then if $p\in E$ we have $p\in E^a$ and if $p\notin E$, since $E-p=E$, we have $V_\alpha(p)\cap(E-p)=V_\alpha(p)\cap E\neq\phi$ for all $\alpha, 0\leqslant\alpha<\omega$. Thus we get $p\in E^d\subseteq E^a$, i. e., $p\in E^a$.

Since $\phi^d=\phi$ & $E\subseteq F\Rightarrow E^d\subseteq F^d$ & $p\in E^d\Rightarrow p\in(E-p)^d$ are true, we have the following statement:

Proposition 5. The ranked space R becomes a ***space*** $(\boldsymbol{V})$ in the sense of M. Fréchet.[2)]

2. Open sets, Closed sets.

Definition 2. Let $E^c\equiv R-E$ be the complementary set of a subset E of the ranked space R. Then $E^i\equiv E-(E^c)^d$, each point of E^i, $E^e\equiv(E^c)^i$, each point of E^e, $E^f\equiv R-(E^i\cup E^e)$ and each point of E^f are respectively called the ***interior*** of E, an ***inner point*** of E, the ***exterior*** of E, an ***outer point*** of E, the ***frontier*** of E and a ***boundary point*** of E.

From these definitions we have the following statement:

Proposition 6. $E^i\subseteq E\subseteq E^a$, $\phi^i=\phi$, $E^i\cap E^e=\phi$, $E^i=E-E^{ca}=E^{cac}$, $E^f=E^{cf}$, $E^{ac}=E^{ci}$, $E^a=E^{cic}$, $E^{ic}=E^{ca}$, $R=E^i\cup E^e\cup E^f$ (direct sum)$=E^a\cup E^e$ (direct sum)$=E^a\cup E^{ci}$ (direct sum).

Definition 3. Let E be a subset of the ranked space R. If $E^d\subseteq E$, i. e., $E^a=E$ then E is called a ***closed set*** in R. And if E^c is a closed set in R then E is called an ***open set*** in R. (These definitions coincide with the definitions in the Note [3]).

Proposition 7. A subset E of the ranked space R is an open set in R iff $E^i=E$.

Proof. If E is open in R then $E^{ca}=E^c$. From $E^i=E-E^{ca}$ we get $E=E^i$. Conversely, if $E=E^i$ then $E\cap(E^c)^d=E^i\cap(E^c)^d=(E-(E^c)^d)\cap(E^c)^d=\phi$. Hence we have $(E^c)^d\subseteq E^c$. Thus we get $E^c=E^c\cup(E^c)^d=E^{ca}$. Therefore E is open in R.

Proposition 8. For any subset E of the ranked space R, the following two conditions are equivalent:

(1) $p\in E^i$,

(2) For any fundamental sequence $\{V_\alpha(p);\ 0\leqslant\alpha<\omega\}$ $(F.\ S.$ in $R)$ of p *in* E, there is an $\alpha', 0\leqslant\alpha'<\omega$, such that $E\supseteq V_{\alpha'}(p)$.

Proof. If $p\in E^i$ then we have $p\in E-(E^c)^d$, i. e., $p\in E$ & $p\notin(E^c)^d$. Thus, for any fundamental sequence $\{V_\alpha(p);\ 0\leqslant\alpha<\omega\}$ of p in E there is an $\alpha', 0\leqslant\alpha'<\omega$, such that $V_{\alpha'}(p)\cap(E^c-p)=\phi$. From $E^c-p=E^c$ we have $V_{\alpha'}(p)\cap E^c=\phi$. Hence, we have $E\supseteq V_{\alpha'}(p)$. Conversely, if, for any fundamental sequence $\{V_\alpha(p);\ 0\leqslant\alpha<\omega\}$ of p in E, there is an $\alpha', 0\leqslant\alpha'<\omega$, such that $E\supseteq V_{\alpha'}(p)$ then we have $p\in E$ & $V_{\alpha'}(p)\cap E^c=\phi$. Thus we have $V_{\alpha'}(p)\cap(E^c-p)=\phi$, i. e., $p\notin(E^c)^d$. Hence we have $p\in E-(E^c)^d=E^i$.

Corollary 1. $E\subseteq F\Rightarrow E^i\subseteq F^i$.

Corollary 2. A subset E of the ranked space R is open in R iff, for $\forall\ p\in E$ and for $\forall\ \{V_\alpha(p);\ 0\leqslant\alpha<\omega\}$ $(F.\ S.$ of p in $R)$, there is an $\alpha', 0\leqslant\alpha'<\omega$, such that $E\supseteq V_{\alpha'}(p)$. Therefore, our notion of open sets coincides with the notion of open sets in the sense of the Note [4].

In fact, if E is open in R we get $(\Rightarrow)$ by Proposition 8. Conversely, if, for $\forall p\in E$ and for $\forall\{V_\alpha(p);\ 0\leqslant\alpha<\omega\}$ $(F.\ S.$ of p in $R)$, there is an $\alpha', 0\leqslant\alpha'<\omega$, such that $E\supseteq V_{\alpha'}(p)$ then we have $p\in E^i$, i. e., $E\subseteq E^i$. Thus we have $E=E^i$.

From Corollary 2 we have the followings:

Proposition 9. If both E and F are open (resp. closed) in the ranked space R, then $E\cup F$ (resp. $E\cap F$) is open (resp. closed) in R, but $E\cap F$ (resp. $E\cup F$) is not always open (resp. closed) in R.

In fact, let E and F be two closed sets in R. From $(E\cap F)^a=(E\cap F)^{cic}=(E^c\cup F^c)^{ic}=(E^c\cup F^c)^c=E\cap F$ it follows that $E\cap F$ is closed in R.

Corollary. Any union (resp. intersection) of open sets (resp. closed sets) in R is open (resp. closed in R.

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2) T. Inagaki: Point sets theory (In Japanese) (1957), p. 17.

3. Continuous mappings.

Prof. K. Kunugi introduced the notion of ***ortho-continuity*** in the Note [2]. We introduced another notion of continuity ([7]).

Definition 4. Let R, S be two ranked spaces with same indicator ω. Then we will say that the mapping $f: R \longrightarrow S$ is ***R-continuous*** at the point p in R if the following condition is fulfilled:

For any fundamental sequence $\{V_\alpha(p);\ 0\leqslant\alpha<\omega\}$ of any point p in R, there is a fundamental sequence $\{U_\alpha(q);\ 0\leqslant\alpha<\omega\}$ of the point $q=f(p)$ in S such that $f(V_\alpha(p))\subseteq U_\alpha(q)$ for all $\alpha, 0\leqslant\alpha<\omega$.

The mapping f is said to be R-***continuous*** if it is R-continuous at each point of R.

From the definition of R-continuity we get the following statement:

Proposition 10. Every R-continuous mapping is ortho-continuous, but the converse is not always true.

Proposition 11. Let R, S be two ranked spaces with same indicator and let $f: R\longrightarrow S$ a mapping of R into S. Then the following three conditions are equivalent.

(1) f is R-continuous at a point p in R.

(2) Let E be a subset of R and let $p\in E^a$, then we have $f(p)\in f(E)^a$, i. e., $f(E^a)\subseteq f(E)^a$.

(3) Let F be a subset of S and let $p\in f^{-1}(F)^a$, then we have $f(p)\in F^a$.

Proof. (1)$\Rightarrow$(2); Let f be an R-continuous mapping at the point p, then for any fundamental sequence $\{V_\alpha(p);\ 0\leqslant\alpha<\omega\}$ of p in R there is a fundamental sequence $\{U_\alpha(f(p));\ 0\leqslant\alpha<\omega\}$ of $f(p)$ in S such that $f(V_\alpha(p))\subseteq U_\alpha(f(p))$ $(\forall\alpha;\ 0\leqslant\alpha<\omega)$.

Since p is belonging to E^a there is a fundamental sequence $\{V_\alpha(p);\ 0\leqslant\alpha<\omega\}$ of p in R such that $V_\alpha(p)\cap E\neq\phi$ $(\forall\alpha;\ 0\leqslant\alpha<\omega)$. Thus, we have $f(V_\alpha(p))\cap f(E)\neq\phi$ $(\forall\alpha;\ 0\leqslant\alpha<\omega)$. Therefore, we get $f(E)\cap U_\alpha(f(p))\neq\phi$ $(\forall\alpha;\ 0\leqslant\alpha<\omega)$, i. e., $f(p)\in f(E)^a$.

(2)$\Rightarrow$(1); Now we suppose that the mapping f is not R-continuous at the point p in R. Then, for any fundamental sequence $\{U_\alpha(f(p));0\leqslant\alpha<\omega\}$ of $f(p)$ in S there is an α', $0\leqslant\alpha'<\omega$, such that $f(V_{\alpha'}(p))\not\subseteq U_{\alpha'}(f(p))$. Thus, we have the following fact:

$$\exists p_{\alpha'}\in V_{\alpha'}(p)\ \&\ f(p_{\alpha'})\notin U_{\alpha'}(f(p)).$$

Let E be the set of all points $p_{\alpha'}$ satisfying the above condition. Then we have $p\in E^a$ & $f(E)\cap U_{\alpha'}(f(p))=\phi$, i. e., $f(p)\notin f(E)^a$. Therefore, if $f(p)\in f(E)^a$ then f is R-continuous at the point p in R.

(2)$\Rightarrow$(3); Put $E=f^{-1}(F)$. From $f(E)=ff^{-1}(F)$, using the condition (2), it follows that $p\in E^a=f^{-1}(F)^a$ $\Rightarrow f(p)\in f(E)^a=f(f^{-1}(F))^a\subseteq F^a$.

(3)$\Rightarrow$(2); Put $F=f(E)$ for the set E such that $p\in E^a$. From $E\subseteq f^{-1}\ f(E)=f^{-1}(F)$ it follows that $E^a\subseteq f^{-1}(F)^a$. Thus, we have $p\in f^{-1}(F)^a$. Therefore, by the condition (3), we have $f(p)\in F^a=f(E)^a$.

Proposition 12. Let R, S and T be three ranked spaces with same indicator and let $R\xrightarrow{f}S\xrightarrow{g}T$. If f is R-continuous at the point $p\in R$ and if g is R-continuous at the point $q=f(p)\in S$ then the composed mapping $g\cdot f$ is R-continuous at the point p in R.

In fact, we have the following fact:

$$g(f(U_\alpha(p)))\subseteq g(\exists V_\alpha(f(p))\subseteq\exists W_\alpha(g(f(p)))\ (\forall\alpha;\ 0\leqslant\alpha<\omega).$$

Let $\mathfrak{S}$ be the set of all ranked spaces with same indicator ω and let $M(R, S)$ be the set of all R-continuous mappings of $R(\in\mathfrak{S})$ into $S(\in\mathfrak{S})$. Furtheremore we shall denote by the form $V_\alpha(p)\xrightarrow{f}V_\alpha'(p')$ the fact that for any fundamental sequence $\{V_\alpha(p);\ 0\leqslant\alpha<\omega\}$ of p in $R\in\mathfrak{S}$ there exists a fundamental sequence $\{V'_\alpha(p');\ 0\leqslant\alpha<\omega\}$ of $p'=f(p)$ in $R'\in\mathfrak{S}$.

Now we have the following three statements.

(i) For $\forall f \in M(R, S)$, $\forall g \in M(S, T)$ and $\forall h \in M(T, U)$, $h\cdot(g\cdot f)=(h\cdot g)\cdot f$ is satisfied,

(ii) For each $R \in \mathfrak{C}$, there exists the identity morphism $1_R \in M(R, R)$ and $f\cdot 1_R=f$ is satisfied for $\forall f \in M(R, S)$

and

(iii) $1_R\cdot g=g$ is satisfied for $\forall g \in M(S, R)$.

In fact, from the form $V_\alpha(p) \xrightarrow{f} V'_\alpha(p') \xrightarrow{g} V''_\alpha(p'') \xrightarrow{h} V'''_\alpha(p''')$ it follows that the form $V_\alpha(p) \xrightarrow{h\cdot(g\cdot f)} V'''_\alpha(p''')$. On the other hand we have the form $V_\alpha(p) \xrightarrow{f} V'_\alpha(p') \xrightarrow{h\cdot g} V'''_\alpha(p''')$. Therefore $h\cdot(g\cdot f)=(h\cdot g)\cdot f$ is true. Moreover the conditions (ii) and (iii) are clearly true. Therefore we get following statement:

Proposition 13. We have ***the category*** $\mathfrak{C}$ ***of ranked spaces.*** Objects, all ranked spaces with same indicator ω; morphisms, all R-continuous mappings $f: R \longrightarrow S$ of one space $R \in \mathfrak{C}$ into a second one $S \in \mathfrak{C}$.

4. Homeomorphisms.

Let R, S be two ranked spaces with same indicator and let $f: R \longrightarrow S$ be a one-to-one R-continuous mapping of R *onto* S.

Proposition 14. Let E be a subset of the ranked space R. If $p \in (E-p)^a$ then we have $f(p) \in (f(E)-f(p))^a$.

Proof. Since f is one-to-one we have $f(E-p)=f(E)-f(p)$. And since $f(p)$ is R-continuous, if $p \in (E-p)^a$ then we have $f(p) \in f(E-p)^a$. On the other hand we have $f(E-p)^a=(f(E)-f(p))^a$. Thus we have $f(p) \in (f(E)-f(p))^a$.

Definition 5. If $f: R \longrightarrow S$ is bijective and bi-R-continuous then it is called the ***homeoniorhism.***

If f is a homeomorphism then its inverse mapping f^{-1} is also a homeomorphism.

Proposition 15. Let $f: R \longrightarrow S$ be a homeomorphism and E a subset of R. Then we have the following statement:

$$p \in (E-p)^a \Leftrightarrow f(p) \in (f(E)-f(p))^a.$$

Proof. Put $q=f(p)$. And let F be a subset of S. Since $f^{-1}(q)$ becomes a homeomorphism we have the following fact:

$$q \in (F-q)^a \Rightarrow f^{-1}(q) \in (f^{-1}(F)-f^{-1}(q))^a.$$

Now put $f^{-1}(F)=E$. Then we have the following statement: $f(p) \in (f(E)-f(p))^a \Rightarrow p \in (E-p)^a$.

Proposition 16. Let $f: R \longrightarrow S$ be a bijection. Then the following three conditions are equivalent:

(1) f is a homeomorphism.

(2) $f(E^a)=f(E)^a$ is satisfied for any subset E of R.

(3) $f^{-1}(F^a)=f^{-1}(F)^a$ is satisfied for any subset F of S.

Proof. (1)$\Rightarrow$(2); By Proposition 15 we get $p \in (E-p)^a \Leftrightarrow f(p) \in f(E-p)^a$. Hereupon replace $E-$p by E. Then we get $f(E^a)=f(E)^a$.

(2)$\Rightarrow$(1); Since $f(E^a) \subseteq f(E)^a$ is satisfied for any subset E of R, f is R-continuous on R and $f^{-1}(f(E)^a) \subseteq E^a$ is satisfied for any subset E of R. Put $f(E)=F$. Then we have $E=f^{-1}(F)$. Thus, we have $f^{-1}(F^a) \subseteq f^{-1}(F)^a$. Therefore, f^{-1} is R-continuous on S.

(2)$\Rightarrow$(3); Put $f^{-1}(F)=E$. Then we have $F=f(E)$. From $f(E^a)=f(E)^a$ we have $f(f^{-1}(F)^a)=F^a$. Thus we have $f^{-1}(F)^a=f^{-1}(F^a)$.

(3)$\Rightarrow$(2); This is clear.

Proposition 17. Let $f: R \longrightarrow S$ be a homeomorphism. Then we have $f(E^i)=f(E)^i$ for any subset E of R.

Proof. If we replace E by E^c in the equality $f(E^a)=f(E)^a$ then we have $f((E^c)^a)=f(E^c)^a$. On the other hand we have $f(E^c)^a=f(E^{ic})=(f(E^i))^c$ & $f(E^c)^a=(f(E)^c)^a=f(E)^{ic}$. Hence we have $f(E^i)=f(E)^i$.

Let R, S be two ranked spaces with same indicator. Let E be a subset of R and F a subset of S. If there is a homeomorphism f such that $F=f(E)$ then the set F is said to be ***homeomorphic*** with the set E. This homeomorphic relation is denoted by $E\sim F$.

Proposition 18. $E\sim E$, $E\sim F \Rightarrow F\sim E$, $E\sim F$ & $F\sim G \Rightarrow E\sim G$.

5. Open mappings, closed mappings.

Definition 6. A mapping f on a ranked space with indicator ω to another ranked space with same indicator ω is ***open*** (resp. ***closed***) iff the image of each open set (resp. closed set) is open (resp. closed).

Proposition 19. Let $f: R \longrightarrow S$ be a bijection of the ranked space R to the ranked space S. Then the following conditions are coincide with each other.

(1) f is open. (2) f is closed.

Moreover we get the following propositions.

Proposition 20. Let R, S, T be three ranked spaces with same indicator and let $R \xrightarrow{f} S \xrightarrow{g} T$. If both f and g are open (resp. closed) then the composed mapping $g\cdot f$ is open (resp. closed).

Proposition 21. $f: R \longrightarrow S$ is a homeomorphism iff f is a one-to-one R-continuous open (or closed) mapping.

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