On Generalized Continuous Groups II

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Synopsis.
In this paper we report of some results on the ranked groups and on the linear ranked spaces.

§ 1. Direct Product Decomposition**.

We shall give a better definition of Direct Product Decomposition than proceeding one [14].

Let $(N_i, \mathfrak{B}_\alpha^{(i)})$ $(1 \leqslant i \leqslant m)$ be ranked groups with same indicator ω and $\{G', \mathfrak{B}_\alpha'\}$ a direct product space of the ranked spaces $\{N_i, \mathfrak{B}_\alpha^{(i)}\}$ $(1 \leqslant i \leqslant m)$ defined as follows:

$$G' = N_1 \otimes \cdots \otimes N_m \quad \text{(direct product group)};$$

$$\mathfrak{B}_\alpha'(x') = \{\prod_{i=1}^{m} \otimes V^{(i)}(x_i)\,;\ V^{(i)}(x_i) \in \mathfrak{B}_{\alpha_i}^{(i)}(x_i)\ \ (\alpha \leqslant \forall \alpha_i < \omega)\ \&\ \mathrm{Min}\,(\alpha_1, \dots, \alpha_m) = \alpha\}$$

for any $x' = (x_1, \dots, x_m)$ $(x_i \in N_i)$ and any α such that $0 \leqslant \alpha < \omega$.

And we define the fundamental sequence in $\{G', \mathfrak{B}_\alpha'\}$ as follows:

Let $x' = (x_1, \dots, x_m)$ be any point of $\{G', \mathfrak{B}_\alpha'\}$ and $V_\alpha'(x')$ $(0 \leqslant \alpha < \omega)$ an element of $\mathfrak{B}' = \bigcup_{\alpha=0}^{\omega} \mathfrak{B}'_\alpha(x')$. Then the sequence $\{V_\alpha'(x');\ 0 \leqslant \alpha < \omega\}$ such that $V_\alpha'(x') \equiv (V_\alpha^{(i)}(x_i))_{1 \leqslant i \leqslant m}$ is said to be a fundamental sequence of x' in G' if, for each i, $1 \leqslant i \leqslant m$, $\{V_\alpha^{(i)}(x_i);\ 0 \leqslant \alpha < \omega\}$ is a fundamental sequence of x_i in G_i.

Difinition. Let $(N_i, \mathfrak{B}_\alpha^{(i)})$ $(1 \leqslant i \leqslant m)$ be normal in $(G, \mathfrak{B}_\alpha)$. We say that $(G, \mathfrak{B}_\alpha)$ *decomposes* into the direct product of its subgroups $(N_1, \mathfrak{B}_\alpha^{(1)}), \dots, (N_m, \mathfrak{B}_\alpha^{(m)})$ if the following two conditions are fulfilled:

(i) G can be decomposed into the algebraically direct product of its subgroups $N_1, \dots, N_m$;

(ii) For any fundamental sequence $\{V_\alpha(x)\}$ of x in G and for each $i = 1, 2, \dots, m$, there exists a fundamental sequence $\{V_\alpha^{(i)}(x_i)\}_{0 \leqslant \alpha \leqslant \omega}$ of x_i in N_i such that $V_\alpha(x) = V_\alpha^{(1)}(x_1) \cdots V_\alpha^{(m)}(x_m)$ $(x = x_1 \cdots x_m)$.

Theorem A. We have the followings.

(1) *There is a mapping* φ_i *of* $\overset{RG^{1)}}{N_i}$ *into* $\overset{RG}{G'}$ *and* $\overset{IS^{2)}}{N'_i}(=\varphi_i(N_i))$ *becomes a normal subgroup of* $\overset{RG}{G'}$. *And in the sense of the ranked groups, we have*

$$N_i \overset{\varphi_i}{\cong} N_i', \quad \prod_{i=1}^{m} \otimes N_i \cong \prod_{i=1}^{m} \otimes N_i'.$$

(2) $\overset{RG}{G'}$ *can be decomposed into the algebraically direct product of its subgroups* $N'_1, \dots, N'_m$.

(3) *For each* $i = 1, 2, \dots, m$, *and any fundamental sequence* $\{V_\alpha^{(i)}(x'_i)\}_{0 \leqslant \alpha < \omega}$ *of* x'_i *in* N'_i, *there exists a fundamental sequence* $\{V_\alpha(x')\}$ *of* $x' = x'_1 \cdots x'_m$ *in* G' *such that* $V_\alpha^{(1)}(x'_1) \cdots V_\alpha^{(m)}(x'_m) \subseteq V_\alpha(x')$ *for all* α, $0 \leqslant \alpha < \omega$.

Proof.

(1) ; Put $\varphi_i : \overset{RG}{N_i} \ni x_i \overset{in}{\longmapsto} (e_1, \dots, e_{i-1}, x_i, e_{i+1}, \dots, e_m) \in \overset{RG}{G'}$ and $N'_i \equiv \varphi_i(N_i)$ $(1 \leqslant i \leqslant m)$.

Then N'_i is a normal subgroups of the abstract group G' and each φ_i is an algebraic isomorphism.

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1) RG = ranked group.

2) IS = induced ranked space.

By $\mathfrak{B}_\alpha'(N_i') \equiv \mathfrak{B}_\alpha' \cap N_i'$ $(0 \leqslant \alpha < \omega)$ N_i' becomes an induced space of $\overset{RG}{G'}$. We will show that N_i' becomes a subspace of G'. For any $V'(x_i', N_i') \in \mathfrak{B}_\alpha'(N_i')$ we have

$$V'(x_i', N_i') = N_i' \cap (V^{(1)}(e_1), \dots, V^{(i-1)}(e_{i-1}), V^{(i)}(x_i), V^{(i+1)}(e_{i+1}), \dots, V^{(m)}(e_m))$$
$$= (e_1, \dots, e_{i-1}, N_i, e_{i+1}, \dots, e_m) \cap (V^{(1)}(e_1), \dots, V^{(i-1)}(e_{i-1}), V^{(i)}(x_i), V^{(i+1)}(e_{i+1}), \dots, V^{(m)}(e_m))$$
$$= (e_1 \cap V^{(1)}(e_1), \dots, e_{i-1} \cap V^{(i-1)}(e_{i-1}), N_i \cap V^{(i)}(x_i), e_{i+1} \cap V^{(i+1)}(e_{i+1}), \dots, e_m \cap V^{(m)}(e_m))$$
$$\equiv V'(x'_i) \in \mathfrak{B}_\alpha'(x'_i).$$

Therefore, for every fundamental sequence $\{V'_\gamma(x'_i, N'_i)\}_{0 \leqslant \gamma < \omega}$ of x'_i in N'_i, $\{V'_\gamma(x'_i)\}_{0 \leqslant \gamma < \omega}$ becomes a fundamental sequence of x'_i in G'.

Now, since $V'_\gamma(x'_i, N'_i) \subseteq N'_i$, for any fundamental sequence $\{V'_\gamma(x'_i, N'_i)\}_{0 \leqslant \gamma < \omega}$, there exists a fundamental sequence $\{V'_\gamma(x'_i)\}_{0 \leqslant \gamma < \omega}$ of x'_i in G' such that $V'_\gamma(x'_i, N'_i) = N'_i \cap V'_\gamma(x'_i, N'_i) = N'_i \cap V'_\gamma(x'_i)$.

Thus N'_i becomes a ranked subspace of $\overset{RG}{G'}$. Hence N'_i is normal in G'.

Next we will show that $N_i \overset{\varphi_i}{\cong} N'_i$ and $\prod_{i=1}^{m} \otimes N_i \cong \prod_{i=1}^{m} \otimes N'_i$. It is clear that φ_i is a bijection. Let $V(x_i)$ be any element of $\mathfrak{B}_\alpha^{(i)}(x_i)$. Since $V(x_i) \subseteq N_i$ we have $N_i \cap V(x_i) = V(x_i)$.

Therefore we get

$$\varphi_i(N_i \cap V(x_i)) = (e_1, \dots, e_{i-1}, N_i \cap V(x_i), e_{i+1}, \dots, e_m) \in \mathfrak{B}'_\alpha \cap N'_i \equiv \mathfrak{B}'_\alpha(N'_i)$$

and

$$\varphi_i(N_i \cap V(x_i)) = (e_1, \dots, e_{i-1}, V(x_i), e_{i+1}, \dots, e_m) \in \mathfrak{B}'_\alpha(x') \text{ when } x' = (e_1, \dots, e_{i-1}, x_i, e_{i+1}, \dots, e_m).$$

Thus, for any fundamental sequence $\{V_\gamma(x_i)\}_{0 \leqslant \gamma < \omega}$ of x_i in N_i, $\{\varphi_i(V_\gamma(x_i))\}_{0 \leqslant \gamma < \omega} \equiv \{(e_1, \dots, e_{i-1}, V_\gamma(x_i), e_{i+1}, \dots, e_m)\}_{0 \leqslant \gamma < \omega}$ becomes a fundamental sequence of x'_i in N'_i. Thus φ_i is a rank preserving R–continuous mapping of N_i onto N'_i.

Now, let $\{V'_\alpha(x'_i, N'_i)\}_{0 \leqslant \alpha < \omega}$ be a fundamental sequence of $x'_i = (e_1, \dots, e_{i-1}, x_i, e_{i+1}, \dots, e_m)$ in N'_i. Since N'_i is a subspace of G' there is a fundamental sequence of x'_i in G', $\{V'_\alpha(x'_i)\}_{0 \leqslant \alpha < \omega}$ such that $V'_\alpha(x'_i, N'_i) = N'_i \cap V'_\alpha(x'_i) \in \mathfrak{B}'_{\gamma(\alpha)}(N'_i)$ for some $\gamma(\alpha)$, $0 \leqslant \gamma(\alpha) < \omega$.

From $V'_\alpha(x'_i) = (V_\alpha^{(1)}(e_1), \dots, V_\alpha^{(i-1)}(e_{i-1}), V_\alpha^{(i)}(x_i), V_\alpha^{(i+1)}(e_{i+1}), \dots, V_\alpha^{(m)}(e_m))$ and $V_\alpha^{(i)}(x_i) \subseteq N_i$ we have

$$V'_\alpha(x'_i, N'_i) = (e_1, \dots, e_{i-1}, N_i, e_{i+1}, \dots, e_m) \cap (V_\alpha^{(1)}(e_1), \dots, V_\alpha^{(i-1)}(e_{i-1}), V_\alpha^{(i)}(x_i), V_\alpha^{(i+1)}(e_{i+1}), \dots, V_\alpha^{(m)}(e_m))$$
$$= (e_1, \dots, e_{i-1}, N_i \cap V_\alpha^{(i)}(x_i), e_{i+1}, \dots, e_m)$$
$$= (e_1, \dots, e_{i-1}, V_\alpha^{(i)}(x_i), e_{i+1}, \dots, e_m).$$

Thus we get $\varphi^{-1}(V'_\alpha(x'_i, N'_i)) = V_\alpha^{(i)}(x_i)$.

Hence if $\{V'_\alpha(x'_i, N'_i)\}_{0 \leqslant \alpha < \omega}$ is a fundamental sequence of x'_i in N'_i then $\{\varphi_i^{-1}(V'_\alpha(x'_i, N'_i))\}_{0 \leqslant \alpha < \omega}$, i.e., $\{V_\alpha^{(i)}(x_i)\}_{0 \leqslant \alpha < \omega}$ becomes a fundamental sequence of $x_i = \varphi_i^{-1}(x'_i)$ in N_i.

Therefore φ_i^{-1} is an R–continuous mapping. Thus φ_i is an isomorphism in the sense of ranked groups.

Thus we have $N_i \overset{\varphi_i}{\cong} N'_i$.

$$\text{Put} \qquad \varphi: \quad \begin{array}{ccccc} N_1 & \ni & x_1 \overset{\varphi_1}{\longmapsto} x'_1 & \in & N'_1 \\ \otimes & & & & \otimes \\ \vdots & & \vdots \quad \vdots \quad \vdots & & \vdots \\ \otimes & & & & \otimes \\ N_i & \ni & x_i \overset{\varphi_i}{\longmapsto} x'_i & \in & N'_i \\ \otimes & & & & \otimes \\ \vdots & & \vdots \quad \vdots \quad \vdots & & \vdots \\ \otimes & & & & \otimes \\ N_m & \ni & x_m \overset{\varphi_m}{\longmapsto} x'_m & \in & N'_m . \end{array}$$

Then there exists a fundamental sequence of x'_i in N'_i, $\{V_\alpha^{(i)\prime}(x'_i)\}_{0 \leqslant \alpha < \omega}$ such that $\varphi_i(V_\alpha^{(i)}(x_i)) \subseteq V_\alpha^{(i)\prime}(x'_i)$ $(x'_i = \varphi_i(x_i)$, $0 \leqslant \alpha < \omega)$ for each $\{V_\alpha^{(i)}(x_i)\}_{0 \leqslant \alpha < \omega}$ (i. e., F. S. of x_i in N_i, $1 \leqslant i \leqslant m$). Thus there is a fundamental sequence of $(x'_1, \dots, x'_m)$ in $N'_1 \otimes \cdots \otimes N'_m$, $\{(V_\alpha^{(1)\prime}(x'_1), \dots, V_\alpha^{(m)\prime}(x'_m))\}_{0 \leqslant \alpha < \omega}$ such that $\varphi((V_\alpha^{(1)}(x_1), \dots, V_\alpha^{(m)}(x_m))) \subseteq (V_\alpha^{(1)\prime}(x'_1), \dots, V_\alpha^{(m)\prime}(x'_m))$ $(0 \leqslant \alpha < \omega)$ for all $\{(V_\alpha^{(1)}(x_1), \dots, V_\alpha^{(m)}(x_m))\}_{0 \leqslant \alpha < \omega}$ (i.e., F. S. of $(x_1, \dots, x_m)$ in $N_1 \otimes \cdots \otimes N_m$). Namely φ is R–continuous.

Analogously φ^{-1} becomes an R-continuous mapping. Thus φ is an isomorphism in the sense of ranked groups. Therefore we have $\prod_{i=1}^{m}\otimes N_i \overset{\varphi}{\cong} \prod_{i=1}^{m}\otimes N'_i$.

(2) ; From abstract group theory we get this statement.

(3) ; Since there exists $\{V_\alpha^{(i)}(x'_i)\}_{0\leqslant\alpha<\omega}$ (i.e., F.S. of x'_i in G') such that $V_\alpha^{(i)\prime}(x'_i) = V_\alpha^{(i)}(x'_i) \cap N_i$ for each $\{V_\alpha^{(i)\prime}(x_i')\}_{0\leqslant\alpha<\omega}$, there exists $\{V_\alpha'(x')\}$ (i.e., F.S. of $x'=x_i'\cdots x_m'$ in G') such that $V_\alpha^{(1)\prime}(x_1')\cdots V_\alpha^{(m)\prime}(x_m')$ $= (V_\alpha^{(1)}(x_1') \cap N_1)\cdots(V_\alpha^{(m)}(x'_m) \cap N_m) \subseteq V_\alpha^{(1)}(x_1')\cdots V_\alpha^{(m)}(x'_m) \subseteq V_\alpha'(x'_1\cdots x'_m)$ in G'. (*Q.E.D.*)

Theorem B. *Let $\overset{RG}{G}$ be the direct product decomposition of $N_1,\dots,N_m$ and G' the direct product group of $\overset{RG}{N_1},\dots,N_m$. Then there exists an isomorphism of the ranked group G' onto the ranked group G. And there exists an R-continuous identity mapping of the ranked group N_i onto itself for each $i=1,2,\dots,m$.*

Proof. Let $\{V_\alpha'(x')\} \equiv \{(V_\alpha^{(1)}(x_1),\dots, V_\alpha^{(m)}(x_m))\}$ ($V'_\alpha(x') \in \mathfrak{V}'_\gamma(x')$ for some γ, $0\leqslant\gamma<\omega$) be any fundamental sequence of x' in G'.

Since G' is a direct product ranked group we have $V_\alpha^{(i)}(x_i) \in \mathfrak{V}^{(i)}_{\gamma(\alpha)}(x^i)$ for some $\gamma(\alpha)$, $0\leqslant\gamma(\alpha)<\omega$, and $\{V_\alpha^{(i)}(x_i)\}_{0\leqslant\alpha<\omega}$ becomes a fundamental sequence of x_i in N_i.

Since $\overset{RG}{G}$ is the direct product decomposition of $N_1,\dots,N_m$, for every fundamental sequences $\{V_\alpha^{(1)}(x_1)\},\dots,$ $\{V_\alpha^{(m)}(x_m)\}$ in the groups $N_1,\dots,N_m$, there exists a fundamental sequence of x' in G, $\{U_\alpha(x')\}$, such that $V_\alpha^{(1)}(x_1)\cdots V_\alpha^{(m)}(x_m) \subseteq U_\alpha(x')$, $x'=x_1\cdots x_m\,(0\leqslant\alpha<\omega)$. Thus we have $\varphi(V_\alpha'(x')) = V_\alpha^{(1)}(x_1)\cdots V_\alpha^{(m)}(x_m)$ $\subseteq U_\alpha(x')$. Therefore φ is R-continuous.

Conversely, let $\{V_\alpha(x)\}$ be any fundamental sequence of $x=x_1\cdots x_m\,(x_i\in N_i)$ in G.
Since $\overset{RG}{G}$ is decomposed into $\overset{RG}{N_1},\dots,\overset{RG}{N_m}$, there is a fundamental sequence of x_i in N_i, $\{V_\alpha^{(i)}(x_i)\}_{0\leqslant\alpha<\omega}(1\leqslant i\leqslant m)$, such that

$$V_\alpha(x) = V_\alpha^{(1)}(x_1)\cdots V_\alpha^{(m)}(x_m) = V_\alpha^{(1)}(x_1)\otimes\cdots\otimes V_\alpha^{(m)}(x_m).$$

Thus we have

$$\varphi^{-1}(V_\alpha(x)) = (V_\alpha^{(1)}(x_1),\dots, V_\alpha^{(m)}(x_m)) \equiv V_\alpha'(x'),\ x'=(x_1,\dots,x_m).$$

Since $\{V'_\alpha(x')\}$ becomes a fundamental sequence of x' in G', φ^{-1} is R-continuous.

Next, it is clear that $\varphi\circ\varphi_i$ is the identity mapping of $\overset{RG}{N_i}$ onto itself.

Moreover we have

$$\varphi\circ\varphi_i(V_\alpha^{(i)}(x_i)) = \varphi((e_1,\dots,e_{i-1}, V_\alpha^{(i)}(x_i), e_{i+1},\dots,e_m)) = V_\alpha^{(i)}(x_i).$$

Thus $\varphi\circ\varphi_i$ is R-continuous. (*Q.E.D.*)

Theorem C. *If G is the direct product decomposition of its subgroups $(N_1,\mathfrak{V}_\alpha^{(1)})$ and $(N_2,\mathfrak{V}_\alpha^{(2)})$ then we have $(G/N_1,\mathfrak{V}_\alpha/N_1) \cong (N_2,\mathfrak{V}_\alpha^{(2)})$.*

Proof. Let $\forall(x_1,x_2) \in G$ and $\varphi : N_2 \ni x_2 \longmapsto x_2 N_1 \in G/N_1$. It is clear that φ becomes an algebraic isomorphism of N_2 onto G/N_1.

Next, for $\forall\ V^{(2)}(x_2) \in \mathfrak{V}_\beta^{(2)}$, we have

$V^{(2)}(x_2)\cdot N_1 = V^{(2)}(x_2)\cdot V^{(1)}(x_1)N_1$ (for some $V^{(1)}(x_1) \in \mathfrak{V}_\alpha^{(1)})$),

$= V^{(1)}(x_1)\cdot V^{(2)}(x_2)\cdot N_1 \in \mathfrak{V}_\gamma/N_1$ (for $\gamma=\mathrm{Min}\ (\alpha,\beta)$).

Because abstract group G is the direct product of N_1 and N_2.

Thus $\{\varphi(V_\alpha^{(2)}(x_2))\}$ becomes a fundamental sequence of $\varphi(x_2) = x_2N_1$ in G/N_1 for any fundamental sequence $\{V_\alpha^{(2)}(x_2)\}$ of x_2 in N_2.

Therefore φ is R-continuous.

Conversely, we have

$$\varphi^{-1}(V_\alpha^{(1)}(x_1)\cdot V_\alpha^{(2)}(x_2)\cdot N_1) = \varphi^{-1}(V^{(2)}(x_2)\cdot N_1) = V_\alpha^{(2)}(x_2)\quad(0\leqslant\alpha<\omega)$$

for any fundamental sequence $\{V_\alpha(x)\cdot N_1\} \equiv \{V_\alpha^{(1)}(x_1)\cdot V_\alpha^{(2)}(x_2)\cdot N_1\}$ of $x=x_1x_2$ in G/N_1.

Namely, φ^{-1} is R-continuous.

Therefore φ becomes an isomorphism in the sense of ranked groups. (*Q.E.D.*)

§ 2. Convergences in the Ranked Group.

Let us consider two convergences, i. e., *ortho-convergence* and *para-convergence* in the ranked group.

Theorem. *Let* $(G, \mathfrak{B}_\alpha)$ *be a ranked group with indicator* $\omega \geqslant \omega_0$. *Suppose that*

$$\mathfrak{B}_\alpha(a) = a \cdot \mathfrak{B}_\alpha(e) = \mathfrak{B}_\alpha(e) \cdot a \quad (0 \leqslant \forall \alpha < \omega, \ \forall a \in G)$$

and $\{p_\alpha V_\alpha(e)\}$ $(p_\alpha \in G)$ is monotone decreasing iff $\{V_\alpha(e)\}$ is monotone decreasing.

Then we have

$$\{ortho\text{-}\lim_\alpha p_\alpha\} \ni p \text{ in } G \Leftrightarrow \{para\text{-}\lim_\alpha p_\alpha\} \ni p \text{ in } G$$

for any sequence $\{p_\alpha\}_{0 \leqslant \alpha < \omega}$ *in* $(G, \mathfrak{B}_\alpha)$.

Remark. If $(G, \mathfrak{B}_\alpha)$ is commutative we have always $a \cdot \mathfrak{B}_\alpha(e) = \mathfrak{B}_\alpha(e) \cdot a$ for each α, $0 \leqslant \alpha < \omega$, and any $a \in G$.

Proof of the theorem. Since $\{ortho\text{-}\lim_\alpha p_\alpha\} \ni p$ there exists a fundamental sequence $\{V_\alpha(p)\}$ such that $V_\alpha(p) \ni p_\alpha$ for each α, $0 \leqslant \alpha < \omega$. As $(G, \mathfrak{B}_\alpha)$ is a ranked group there exists a fundamental sequence $\{V_\alpha'(p^{-1})\}$ such that $p_\alpha^{-1} \in V_\alpha(p)^{-1} \subseteq V_\alpha'(p^{-1})$ for each α, $0 \leqslant \alpha < \omega$.

On the other hand, there exist a fundamental sequence $\{U_\alpha(e)\}$ and a monotone decreasing sequence $\{U_\alpha'(p_\alpha^{-1})\}$ such that

$e \in p_\alpha^{-1} \cdot V_\alpha(p) = p_\alpha^{-1} \cdot U_\alpha(e)\, p$ (from $V_\alpha(p) \ni p_\alpha$ and $\mathfrak{B}_\alpha(p) = p \cdot \mathfrak{B}_\alpha(e)$)

$= U_\alpha'(p_\alpha^{-1}) \cdot p$ $(U_\alpha'(p_\alpha^{-1}) \in \mathfrak{B}_{\gamma(\alpha)}(p_\alpha^{-1}),\ \gamma(\alpha) \uparrow \omega$ as $\alpha \uparrow \omega)$ (from $\mathfrak{B}_\alpha(p) = p \cdot \mathfrak{B}_\alpha(e)$).

Thus we get $U_\alpha'(p_\alpha^{-1}) \ni p^{-1}$ for each α, $0 \leqslant \alpha < \omega$.

Therefore we have

$$p = (p^{-1})^{-1} \in U_\alpha'(p_\alpha^{-1})^{-1} \subseteq U_\alpha''(p_\alpha) \quad (0 \leqslant \alpha < \omega)$$

for some monotone decreasing sequence $\{U_\alpha''(p_\alpha)\}$ such that $U_\alpha''(p_\alpha) \in \mathfrak{B}_{\delta(\alpha)}(p_\alpha)$, $\delta(\alpha) \uparrow \omega$ as $\alpha \uparrow \omega$.

Namely, we have $\{para\text{-}\lim_\alpha p_\alpha\} \ni p$.

Conversely suppose that $\{para\text{-}\lim_\alpha p_\alpha\} \ni p$. Since there exists a monotone decreasing sequence $\{V_\alpha(p_\alpha)\}$ such that $V_\alpha(p_\alpha) \ni p$ & $V_\alpha(p_\alpha) \in \mathfrak{B}_{\varepsilon(\alpha)}(p_\alpha)$ for each α, $0 \leqslant \alpha < \omega$, and $\varepsilon(\alpha) \uparrow \omega$ as $\alpha \uparrow \omega$, we get

$$p \in V_\alpha(p_\alpha) = p_\alpha \cdot V_\alpha'(e) \quad (0 \leqslant \alpha < \omega)$$

for some fundamental sequence $\{V_\alpha'(e)\}$ in $(G, \mathfrak{B}_\alpha)$. Thus there is a point $p_\alpha' \in V_\alpha'(e)$ such that $p_\alpha \cdot p_\alpha' = p$ for each α, $0 \leqslant \alpha < \omega$.

Thus there exist two fundamental sequences $\{U_\alpha'(e)\}$ and $\{U_\alpha(p)\}$ such that

$$p_\alpha = p \cdot p'^{-1}_\alpha \in p \cdot V_\alpha'(e)^{-1} \subseteq p \cdot U_\alpha'(e) = U_\alpha(p)$$

for each α, $0 \leqslant \alpha < \omega$. Therefore we get $\{ortho\text{-}\lim_\alpha p_\alpha\} \ni p$.

This completes the proof.

§ 3. Linear Ranked Spaces.

We shall introduce linear ranked spaces as certain generalized normed linear spaces.

Let E be a linear space over real or complex field K and also a ranked space with indicator ω_0.

We now introduce following notations:

$E \equiv \{E, \mathfrak{B}_n\}$ (i. e., a ranked space);

$\mathfrak{B} \equiv \bigcup_{n=0}^{\omega_0} \mathfrak{B}_n$;

$\| x \|_V \equiv$ the rank of $V(x) \in \mathfrak{B}$;

$\{u_n(x)\} \equiv$ a fundamental sequence of x in E;

$\mathfrak{F}(x) \equiv$ all of fundamental sequences with respect to x in E.

Suppose that E satisfies the following condition (I) or (II) :

(I) (i) For $\forall \lambda \in K$ and $\forall V \in \mathfrak{B}(x)$, there is a $W \in \mathfrak{B}(\lambda x)$ such that

$$\lambda V \subseteq W \ \& \ \| \lambda x \|_W = \left[\frac{\| x \|_V}{|\lambda|} \right]$$

(above [ ] is the Gaussian symbol);

(ii) For $\forall U \in \mathfrak{B}(x)$ and $\forall V \in \mathfrak{B}(y)$, there is a $W \in \mathfrak{B}(x+y)$ such that

$$U+V \subseteq W \ \& \ \|x+y\|_W \leqslant Min. \{[\frac{\|x\|_U}{2}], [\frac{\|y\|_V}{2}]\}$$

(thus $\|x+y\|_W \leqslant Min. \{\|x\|_U, \|y\|_V\}$).

(II) (1°) For $\forall \lambda \in K$ and $\forall \{u_n(x)\} \in \mathfrak{F}(x)$, there is a $\{w_n(\lambda x)\} \in \mathfrak{F}(\lambda x)$ such that

$$\lambda \cdot u_n(x) \subseteq w_n(\lambda x) \ \& \ \|\lambda x\|_{w_n} = [\frac{\|x\|_{u_n}}{|\lambda|}]$$

for all n, $0 \leqslant n < \omega_0$;

(2°) For $\forall \{u_n(x)\} \in \mathfrak{F}(x)$ and $\forall \{v_n(y)\} \in \mathfrak{F}(y)$, there is a $\{w_n(x+y)\} \in \mathfrak{F}(x+y)$ such that

$$u_n(x) + v_n(y) \subseteq w_n(x+y) \ \& \ \|x+y\|_{w_n} \leqslant Min. \{[\frac{\|x\|_{u_n}}{2}], [\frac{\|y\|_{v_n}}{2}]\}$$

for all n, $0 \leqslant n < \omega_0$.

Definition. We call above E a ***linear ranked space*** over K.

Remark 1. Above E becomes a *linear ranked space* in the sense of [14, p. 58].

Remark 2. From the axiom (a) we have following statements:

(1) For $\forall U \in \mathfrak{B}(x)$ there exists $V \in \mathfrak{B}(x)$ such that $U \supseteq V$ and $\|x\|_U \leqslant \|x\|_V < \omega_0$.

(2) For $\forall \{u_n(x)\} \in \mathfrak{F}(x)$ there exists $\{v_n(x)\} \in \mathfrak{F}(x)$ such that

$u_n(x) \supseteq v_n(x)$ & $\|x\|_{u_n} \leqslant \|x\|_{v_n} < \omega_0$ for all n, $0 \leqslant n < \omega_0$.

Remark 3. $\|x\|_V \geqslant 0$ for all $V \in \mathfrak{B}(x)$. $\|x\|_{V=\{x\}(one\ point\ set)} \geqslant \omega_0$. $[\frac{1}{|\lambda|}] = \omega_0$ for $\lambda = 0$.

Examples of above spaces.

Type (I); (1) (Semi-) Normed space $(E, \|\cdot\|)$.

Let $v(n; 0) = \{x \in E; \|x\| < \frac{1}{n}\}$, $\mathfrak{B}_n(0) = \{v(n; 0)\}$ (one set family) and let $\mathfrak{B}_0 = \{E\}$.

(2) Countably normed space $(\Phi, \{\|\cdot\|_p\}_{p=1,2,\dots})$.

Let $v(n; 0) = \{\varphi \in \Phi; \|\varphi\|_{\forall p} < \frac{1}{n}\}$, $\mathfrak{B}_n(0) = \{v(n; 0)\}$ (one set family) and let $\mathfrak{B}_0 = \{\Phi\}$.

(3) Countably normed space as linear Metric Space.

I. M. Gel'fand [8, p. 21] introduced a metric in a countably normed space. *A* metric space is considered as a ranked space with depth ω_0.

(4) Perfect space. (See Gel'fand [8, p. 54]).

(5) Dual space of countably normed space $(\Phi^*, \{\|\cdot\|^*_p\}_{p=1,2,\dots})$.

Let $v(n, p; 0) = \{f \in \Phi_p^*; \|f\|^*_p < \frac{1}{n}\}$, $\mathfrak{B}_n(0) = \{v(n, p; 0); p = 1, 2, \cdots\}$ and let $\mathfrak{B}_0 = \{\Phi^*\}$.

(6) L. Schwartz's distribution space D.

Let $v(n, k; 0) = \{\varphi \in D; \text{Car.}\, \varphi \subseteq [-k, k], \max_{0 \leqslant l < n} \sup_x |\overset{(l)}{\varphi}(x)| < \frac{1}{n}\}$ $(k > 0)$ and let $\mathfrak{B}_n(0) = \{v(n, k; 0); \forall k > 0\}$.

(7) Dual space D' of space D.

Let $v(n; 0) \equiv v(n; 0, \{m_1, \dots, m_n\}) = \bigcap_{j=1}^{n} U_{nj}$ where arbitary integers $m_1 \leqslant m_2 \leqslant \cdots \leqslant m_n$ and

$U_{nj} = \{\tau \in D'; \sup_{\varphi \in v(m_j; 0, 1, K_j)} |\tau(\varphi)| \leqslant \frac{1}{n}\}$ and let $\mathfrak{B}_n(0) =$ all of above $v(n; 0)$.

Type (II); (i) All examples in type (I).

(ii) Union space of Countably normed spaces. $\Phi^{(\infty)}$.

Let $\Phi^{(\infty)} = \bigcup_{m=1}^{\infty} \Phi^{(m)}$ be the union space of countably normed spaces $\Phi^{(m)}$ $(m = 1, 2, \cdots)$ where $\Phi^{(1)} \subset \Phi^{(2)} \subset \cdots \subset \Phi^{(m)} \subset \cdots$ & the systems $\{\|\cdot\|_p^{(m)}\}_{p=1,2,\dots}$ and $\{\|\cdot\|_p^{(m+1)}\}_{p=1,2,\dots}$ are equivalent in $\Phi^{(m)}$.

And put $v(n, m; 0) = \{\varphi \in \Phi^{(m)}; \|\varphi\|_n^{(m)} < \frac{1}{n}\}$, $\mathfrak{B}_n(0) = \{v(n, m; 0); m = 1, 2, \cdots\}$ for $n \geqslant 1$, and $\mathfrak{B}_0 = \{\Phi^{(\infty)}\}$,

(iii) Conjugate space Φ'.

Let Φ' be the conjugate space to a countably normed space Φ. Then we have $\Phi'=\bigcup_{p=1}^{\infty}\Phi'_{(p)}$ by Gel'fand [8, p. 36].

(iv) Nuclear Space in the sense of Y. Nagakura. See [7 ; II].

§ 4. A theorem on locally convex linear topological spaces.

1. We consider again linear ranked space in the sense of [14, p. 58].

Theorem. *Locally convex linear topological space S becomes a linear ranked space.*

***Corollary* 1.** *Conjugate space S' to a locally convex linear topological space S is a linear ranked space.*

***Corollary* 2.** *Let $S_1\subset S_2\subset\cdots\subset S_k\subset\cdots$ be an increasing sequence of locally convex linear topological spaces $S_k\,(k=1,2,\cdots)$. Then the **inductive limit space**[3] $S=\bigcup_{k=1}^{\infty}S_k$ becomes a linear ranked space. (Thus the conjugate space to S is so.)*

Proof. Suppose that the locally convex topology on above linear space S is defined by a system of semi-norms $\{p_\alpha(x)\}_{\alpha\in A}$ on S. Let

$$v(n, B\,;0)=\{x\,;\,\alpha\in B\Rightarrow p_\alpha(x)<\frac{1}{n}\}\ \text{for any finite subset } B \text{ of } A$$

and let

$$\mathfrak{B}_n(0)=\{v(n, B\,;0)\ ;\ \forall B\subseteq A\}.$$

Then we get above theorem.

Since S' is a locally convex space, we get corollary 1.

Finally, we shall prove corollary 2. Let $p_\alpha(x)$ be a semi-norm on S such that, by the topology on S_k, $p_\alpha(x)$ is continuous on S_k for each $k=1,2,\cdots$. Since all of above semi-norms $\{p_\alpha(x)\}_{\alpha\in A}$ defines the locally convex topology on S, S is a locally convex linear topological space. Thus we get corollary 2. (*Q.E.D.*)

2. ***Examples of such linear ranked spaces.***

(1) D and its Fourier transformation $\hat{D}$.

(2) Fréchet spaces (thus Banach spaces).

(3) LF-spaces (i. e., the inductive limit space of Fréchet spaces).

(4) Bornological spaces and the inductive limit space of Bornological spaces.

(5) Barrelled spaces and the inductive limit space of Barrelled spaces.

(6) Montel spaces and the inductive limit space of Montel spaces.

(7) The conjugate spaces to above spaces.

(8) L. Hörmander's space $\mathcal{F}(\Omega)=\bigcap_{i\in I}\mathcal{B}^{loc}_{p_i,k_i}(\Omega)$.[4]

§ 5. Linear Forms and Extension Theorem.

Let R and S be two linear spaces over the same field Φ (of real or complex numbers). The mapping f of R into S is called ***linear*** if

$$f(x+y)=f(x)+f(y),\ f(\lambda x)=\lambda f(x),$$

for all $x\in R$, $y\in R$ and $\lambda\in\Phi$. The linear mapping f is one-to-one iff $f^{-1}(0)=\{0\}$; in general $f^{-1}(0)$ is a linear subspace of R. Moreover in the set L of all linear mappings of R into S, addition and multiplication by scalars can be defined by

$$(f+g)\,(x)=f(x)+g(x),\ \ (\lambda f)\,(x)=\lambda(f(x))\ ;$$

then L becomes a linear space over Φ.

***Proposition* 1.** *When R and S are both linear ranked spaces, the R-continuous linear mappings*

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3) [11], p. 27.

4) [12], p. 77.

of R into S form a linear subspace of L.

Because the R-continuity of f and g implies the R-continuity of $f+g$ and λf.

***Proposition* 2.** *If R and S are (homegeneous) linear ranked spaces and f is a linear mapping of R into S, then f is R-continuous on R iff f is R-continuous at the origin.*

Proof. If f is R-continuous at 0, and $\{U_n(0)\}$ is any fundamental sequence of 0 in R, there is a fundamental sequence $\{V_n(0)\}$ of 0 in S such that $f(U_n(0))\subseteq V_n(0)$ for every n, $0\leqslant n<\omega_0$. Then for each point a of R, $f(a+U_n(0))=f(a)+f(U_n(0))\subseteq f(a)+V_n(0)$, and so f is R-continuous at a.

Definition. If R is a linear space over Φ, a linear mapping of R into the scalar field Φ itself is called a ***linear form*** (or ***linear functional***) on R. A linear form f on a linear ranked space R is called ***continuous*** at $x\in R$ if

$$\{\lim_{\alpha} x_\alpha\}\ni x \text{ in } R\Rightarrow\lim_{\alpha} f(x_\alpha)=x \text{ in } \Phi.$$

Remark. If R is a normed linear space then we have

$$\{\lim_{\alpha} x_\alpha\}\ni x \text{ in } R\Leftrightarrow\lim_{\alpha} \| x_\alpha-x \| =0 \text{ in } R.$$

***Proposition* 3.** *If a linear form f on a (homogeneous) linear ranked space R is continuous at 0, then f is continuous on the whole of R.*

Proof. If $\{\lim_{n} x_n\}\ni x$ then there is a fundamental sequence $\{v_n(0)+x\}$ of x such that $v_n(0)+x\ni x_n$ for each n, $0\leqslant n<\omega_0$. Then $\{v_n(0)\}$ becomes a fundamental sequence of 0 and we have $v_n(0)\ni x_n-x$ for each n, $0\leqslant n<\omega_0$. Thus

$$f(x_n)-f(x)=f(x_n-x)\longrightarrow 0.$$

***Proposition* 4.** *Let f and g be two continuous linear forms on R and X an r-dense subset of R. When $f(x)=g(x)$ for any $x\in X$, we have $f=g$ on R.*

Proof. For any $x\in R$ there is a sequence $\{x_n\}$ such that $\{\lim_{n} x_n\}\ni x$ and $x_n\in X$.
Since f and g are continuous we get $f(x)=g(x)$ from $f(x_n)=g(x_n)$.
From linear topological space theory[5], we have the **Hahn-Banach theorem**, i. e.,

Theorem (Hahn-Banach extension theorem). *Suppose that $p(x)$ is a positive homogeneous subadditive function on a real linear space R. If a linear form $q(z)$, defined on a linear subspace X, satisfies*

$$q(z)\leqslant p(z) \quad \text{for } z\in X$$

then $q(z)$ can be extended to a linear form ℓ, defined on the whole of R, which satisfies

$$\ell x\leqslant p(x) \quad \text{for } x\in R.$$

If R is a linear ranked space and $p(x)$ is continuous at 0, then ℓ is also continuous.

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5) [9], p. 190

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