Some structures on the Ranked Spaces

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Synopsis

The aim of this paper is to study the notion of sheaves treated by the method of ranked spaces.

Throughout this paper we shall treat only ranked spaces with same indicator $\omega \geqslant \omega_0$. And let these ranked spaces be always satisfying axioms (A), (B), (a) and (b)[1].

§ 1. Sheaves.

1°. Sheaf of sets.

Let $\boldsymbol{F}$ and M be two ranked spaces and $\pi : \boldsymbol{F} \longrightarrow M$ a unique mapping of $\boldsymbol{F}$ into M.

Definition 1. The mapping π is called ***local R-homeomorphism*** iff any $f \in \boldsymbol{F}$ has an r-open neighborhood of f in $\boldsymbol{F}$, $\widetilde{U}(f)$, such that $\pi : \widetilde{U} \longrightarrow \pi(\widetilde{U})$ is an R-homeomorphism and $\pi(\widetilde{U})$ becomes an r-open neighborhood of $\pi(f)$ in M.

Definition 2. A ***sheaf*** (of sets) on M is a triple $(\boldsymbol{F}, \pi, M)$ where

(i) $\boldsymbol{F}$ is a ranked space.

(ii) $\pi : \boldsymbol{F} \longrightarrow M$ is a local R-homeomorphism onto M.

This π is called the ***projection*** of $(\boldsymbol{F}, \pi, M)$.

Definition 3. The ***stalk*** over $x \in M$ is the subset $\boldsymbol{F}_x = \pi^{-1}(x)$ of $\boldsymbol{F}$.

Definition 4. A ***section*** s of $\boldsymbol{F}$ over a subset X of M is an R-continuous mapping from X into $\boldsymbol{F}$ such that $\pi \circ s$ is the identity 1.

For any $y \in \boldsymbol{F}$, there exists a section over some $V \subset M$ passing through y. Take V to be a homeomorph under π of some neighborhood W of y and let $s = (\pi \mid W)^{-1}$.

The set of all sections over X is denoted by $\Gamma(X, \boldsymbol{F})$ and $\boldsymbol{F}(X)$. Especially, for every point $x \in M$, $\boldsymbol{F}_x = \Gamma(x, \boldsymbol{F})$ holds.

For two subsets X and Y (of M), $X \subset Y$ decides the mapping $\rho_X{}^Y$ such that

$$\rho_X{}^Y : \Gamma(Y, \boldsymbol{F}) \longrightarrow \Gamma(X, \boldsymbol{F}), \rho_X{}^Y(s) = s \circ i \text{ where } i : X \longrightarrow Y \text{ is an inclusion map.}$$

This $\rho_X{}^Y$ is called a ***restriction*** of the section s.

Proposition 1. *For a sheaf $(\boldsymbol{F}, \pi, M)$, we have*

(i) *The stalk over $x \in M$ is discrete.*

(ii) *The projection π is an r-open mapping.*

(iii) *Every section of $\boldsymbol{F}$ over $X \subset M$ is an r-open mapping.*

(iv) *If any two sections agree at a point $x_0 \in M$ then they agree in an r-open neighborhood of x_0.*

Proof. (i) and (ii) are clear. (iii); Let $s : V \longrightarrow \boldsymbol{F}$ be a section over an r-open set V in M. For any point $x \in V$ there is an r-open neighborhood $\widetilde{U}$ of $s(x) \in \boldsymbol{F}_x$. Then the set $W = V \cap \pi(\widetilde{U})$ becomes

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1) [8], I, pp 46–47.

an r–open neighborhood of x. Since the projection $\pi: \widetilde{U} \longrightarrow \pi(\widetilde{U})$ is an R–homeomorphism and $\pi \circ s(W) = W$ holds for W, the set $s(W)$ is an r–open neighborhood of $s(x)$. That is, s is an r–open mapping. (iv); Let $s: U \longrightarrow \boldsymbol{F}$ and $t: V \longrightarrow \boldsymbol{F}$ be two sections over r–open sets (of M) U and V respectively. And suppose that $s(x_0) = t(x_0)$ at a point $x_0 \in U \cap V$. From $W = \pi(s(U) \cap t(V))$ follows $s(x) = t(x)$, $x \in W$. Moreover s, t and π are r–open mappings. Thus W is an r–open neighborhood of x_0. (*Q. E. D.*)

2°. Sheaf of groups.

Definition 5. A ***sheaf*** of (resp. abelian) groups on M is a triple $(\boldsymbol{F}, \pi, M)$ where

(i) $(\boldsymbol{F}, \pi, M)$ is a sheaf of sets on M.

(ii) The stalk $\boldsymbol{F}_x = \pi^{-1}(x)$ of $\boldsymbol{F}$ at any point $x \in M$ is a (resp. an abelian) group.

(iii) The group operations are R-continuous.

Similarly, a sheaf of A–modules is defined.

We shall now list elementary consequences of above definition.

Proposition 2. *If $(\boldsymbol{F}, \pi, M)$ is a sheaf of groups on M then the set $\Gamma(X, \boldsymbol{F})$ forms a group and its group operations are given by following relations:*

$$(st)(x) = s(x) \cdot t(x), \quad s^{-1}(x) = s(x)^{-1}, \quad s, t \in \Gamma(X, \boldsymbol{F}), \ x \in X.$$

For each point $x \in M$, let e_x be the unit of group $\boldsymbol{F}_x$. Then e_x assigns a section over M such that $e: M \longrightarrow \boldsymbol{F}$, $e(x) = e_x$, $x \in M$.

Proposition 3. *If $(\boldsymbol{F}, \pi, M)$ is a sheaf of groups we have the followings:*

(a) *The restrictions of its section*

$$\rho_X^Y: \Gamma(Y, \boldsymbol{F}) \longrightarrow \Gamma(X, \boldsymbol{F}), \ \rho_x^X: \Gamma(X, \boldsymbol{F}) \longrightarrow \boldsymbol{F}_x, \ X \subset Y, \ x \in X$$

become group homomorhisms.

(b) $\rho_X^X = 1$.

(c) $\rho_X^Y \circ \rho_Y^Z = \rho_X^Z$ *when* $X \subset Y \subset Z$.

Example: Constant sheaf. Let F be a set (or an A–module) and let F discrete. Make the direct product $\boldsymbol{F} = M \times F$ for a ranked space M. Then $\boldsymbol{F}$ becomes a sheaf of sets on M by the projection π such that $\pi: \boldsymbol{F} \longrightarrow M$, $\pi(x, a) = x$, $x \in M$, $a \in F$. $\boldsymbol{F}$ is called the constant sheaf on M.

3°. Presheaf.

Definition 6. Let M be a ranked space. *A* ***presheaf*** on M is a system $\boldsymbol{F}$ where

(i) Each r–open set U in M assigns an A–module $\boldsymbol{F}(U)$.

(ii) For any two r–open sets U and V such that $U \subset V$, there is a group homomorphism $\rho_U^V: \boldsymbol{F}(V) \longrightarrow \boldsymbol{F}(U)$ such that

(a) $\rho_V^V = 1$ and (b) $\rho_U^V \circ \rho_V^W = \rho_U^W$ when $U \subset V \subset W$.

§ 2. Homomorphisms, subsheaves direct sum of sheaves and quotient sheaves.

Definition 7. (a) Let $(\boldsymbol{F}, \pi, M)$ and $(\boldsymbol{F}', \pi', M)$ be two sheaves of sets on space M. A ***homomorphism*** of sheaves (of sets) $h: \boldsymbol{F} \longrightarrow \boldsymbol{F}'$ is an R-continuous mapping such that $\pi = \pi' \circ h$, i. e., $h(\boldsymbol{F}_x) \subset \boldsymbol{F}_x'$ for all $x \in M$.

(b) Let $(\boldsymbol{F}, \pi, M)$ and $(\boldsymbol{F}', \pi', M)$ be two sheaves of groups (resp. A–modules) on M. A ***homomorphism*** of sheaves of groups (resp. A–modules), $h: \boldsymbol{F} \longrightarrow \boldsymbol{F}'$, is an R–continuous mapping such that $\pi = \pi' \circ h$ and the restriction $h_x: \boldsymbol{F}_x \longrightarrow \boldsymbol{F}_x'$ of h to stalk is a group (resp. A–module) homomorphism for all x.

h becomes an r–open mapping and a local R–homeomorphism.

Let $\boldsymbol{F}_i = (\boldsymbol{F}_i, \pi_i, M)$ $(i = 1, 2)$ be two sheaves of sets. Suppose that

$\boldsymbol{F}_1 \oplus \boldsymbol{F}_2 = \{(f_1, f_2); f_i \in \boldsymbol{F}_i, \pi_1(f_1) = \pi_2(f_2)\}$ is the direct product ranked space of $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$, and let $\tilde{\pi}(f_1, f_2) = \pi_1(f_1) = \pi_2(f_2) \in M$. Then $(\boldsymbol{F}_1 \oplus \boldsymbol{F}_2, \tilde{\pi}, M)$ is a sheaf (of sets) on M.

Definition 8. $(F_1 \dotplus F_2, \tilde{\pi}, M)$ is called the ***direct sum*** of F_1 and F_2.

Definition 9. A ***subsheaf H*** of a sheaf of sets (resp. groups, A-modules), $F=(F, \pi, M)$, is a sheaf of sets (resp. groups, A-modules), $H=(H, \pi, M)$, such that $H \subset F$ and the identity mapping $1: H \longrightarrow F$ is a homomorphism of sheaves.

Let E be a ranked space, R an equivalence relation on E, and let $\dot{E} \underset{\leftarrow}{\equiv} E/R$ the quotient set. Let $\dot{V}(\dot{x})$ be a neighborhood of $\dot{x} \in \dot{E}$ and let $V(x)$ a neighborhood of some $x \in \dot{x}$ such that $\dot{V}(\dot{x}) = V(x)/R$. By letting the rank of $\dot{V}(\dot{x}) \equiv$ the rank of $V(x)$, the quotient set $\dot{E}$ becomes a ranked space. This $\dot{E} \underset{\leftarrow}{\equiv} E/R$ is called a ***ranked quotient space.***

Definition 10. Let H be a subsheaf of F. Then a ranked quotient space $\bigcup_{x \in M} F_x/H_x$ becomes a sheaf on M. This sheaf is called a ***quotient sheaf*** and denoted by F/H.

If a homomorphism of sheaves (of groups), $h: F \longrightarrow G$ is bijective then h is called an ***isomorphism*** of sheaves (of groups) and denoted by $h: F \cong G$. In this case h is an R-homeomorphism.

From topological sheaf theory, we get following proposition.

Proposition 4. *If $\varphi: F \longrightarrow G$ is a homomorphism of sheaves of groups and if φ satisfies the condition* (*) *in* [8 ; I, *p.* 51] *then $G' \underset{\leftarrow}{\equiv} \varphi(F)$* is a subsheaf *of G and the kernel of φ, $F' \underset{\leftarrow}{\equiv} \varphi^{-1}(0)$, is a subsheaf of F. Moreover we have $F/F' \cong G'$.*

References

[1] K. Kunugi : Sur la méthode des espaces rangés. I, II. Proc. Japan Acad., 42 (1966), 318–322, 549–544.

[2] F. Hirzebruch : Neue topologische Methode in der algebraische Geometrie, Springer–Verlag, Berlin (1956).

[3] R. Godement : Topologie Algébrique et Théorie des Faisceax, Hermann, Paris (1958).

[4] R. G. Swan : The Theory of Sheaves, The University of Chicago Press, Chicago (1968).

[5] G. E. Bredon : Sheaf Theory, McGraw–Hill Book Company, New York (1967).

[6] S. Hitotsumatsu : Theory of analytic functions of several complex variables, Baihūkan Book Company, Japan (1960).

[7] H. Komatsu : Sato's Hyperfunctions and Partial Differential Equations with Constant Coefficients, Univ. Tokyo Seminar Notes No. 22, 1968.

[8] T. Shintani : On Generalized Continuous Groups I, Mem. Tomakomai Tech. College, 6 (1971), 45–60 : Some considerations in the Ranked Spaces, Ibid., 6 (1971), 61–65.