On Some Structures of the Ranked Spaces

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(Received January 10, 1973)

Synopsis.

In this paper we construct the completion of the ranked spaces. This problem has not been solved yet. The completion is essential and very important notion in the theory of functional analysis. Moreover we give a few important examples of ranked spaces.

1. **Introduction.**** K. Kunugi ([5]) studied the Baire's category theorem in a ranked space. He showed then that in a ropological space which is at the same time a complete[1)] ranked space following generalized Baire's theorem holds.

Theorem (K. Kunugi ([5])). If a topological space R is a complete ranked space with indicator $\omega \geqslant \omega_0$, then, for any well-ordered sequence

$$G_0,\ G_1, \dots, G_\alpha, \dots\ ;\ 0 \leqslant \alpha < \omega$$

of topologically open and topologically everywhere dense subsets in R, $\bigcap_\alpha G_\alpha$ is also topologically everywhere dense in R.

This theorem is a generalization of Baire's theorem which states that every complete metric space or every locally compact regular space is a Baire space.

After that, K. Kunugi ([5]) and H. Okano ([12]) studied a completion of a ranked space on a topological space. Their theory has been applying to extend the Lebesgue integrals or the Denjoy integrals by K. Kunugi ([6]), H. Okano ([13]), S. Nakanishi ([11]) and many mathematicians. Their generalized integrals are called the ***(E. R.) integrals.***

Recently, the notion of the ranked space is being studied as a generalization of the topological space by using ***new general topological methods*** ([7]). But, by such standpoint, the general construction of the completion of the ranked spaces has not been given yet.

We shall grasp the notion of the ranked spaces as a generalization of the notion of the metric spaces or the uniform spaces ([18]) or the extended uniform spaces ([2]).
From this stand point, in 3 and 4, we shall construct the completion of the ranked spaces ***without assuming the uniformity property*** for the ranked spaces. And such construction is purely done by the new methods of ranked spaces.

In 5, we shall give a simple example of the ranked space with non-indicator ω_0. In 6, we shall show that every $\sum$-space has the structure as a ranked space with indicator ω_0.

2. **Preliminaries.**

Throughout this paper we assume that the ranked spaces $\{R, \mathfrak{V}_\alpha\}$ satisfy the axioms (A) and (B) of

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** Throughout this paper we shall use the same terminology that is introduced in [7] and [14].

1) See p. 84.

F. Hausdorff, the axioms (a) and (b) of K. Kunugi ([7]) and that they have the indicator ω, $\omega_0 \leqslant \omega \leqslant \omega(R)$ ($\omega(R)$ is the depth of R and ω is an inaccessible ordinal number). New, let us recall some basic concepts in the general ranked spaces.

A monotone decreasing sequence of neighborhoods of points in R :

$$V_0(p_0) \supseteq V_1(p_1) \supseteq V_2(p_2) \supseteq \cdots \supseteq V_\alpha(p_\alpha) \supseteq \cdots, \quad 0 \leqslant \alpha < \omega,$$

is said to be a ***fundamental sequence***, if there is an ordinal number $\gamma(\alpha)$ such that $V_\alpha(p_\alpha) \in \mathfrak{V}_{\gamma(\alpha)}$ for all α, $0 \leqslant \alpha < \omega$, and satisfies the following two conditions :

(i) $\gamma(0) \leqslant \gamma(1) \leqslant \gamma(2) \leqslant \cdots \leqslant \gamma(\alpha) \leqslant \cdots \quad (0 \leqslant \gamma(\alpha) < \omega)$, $\sup_\alpha \gamma(\alpha) = \omega$,

(ii) for each α, $0 \leqslant \alpha < \omega$, there is a number $\lambda = \lambda(\alpha)$ such that $\alpha \leqslant \lambda < \omega$, $p_\lambda = p_{\lambda-1}$ and $\gamma(\lambda) < \gamma(\lambda+1)$ (except the equality).

The ranked space R is said to be ***complete***, if, for every fundamental sequence $\{V_\alpha(p_\alpha)\ ;\ 0 \leqslant \alpha < \omega\}$ of neighborhoods, we have $\bigcap_{\alpha=0}^{\omega} V_\alpha(p_\alpha) \neq \phi$.

Given a sequence $\{p_\alpha\ ;\ 0 \leqslant \alpha < \omega\}$ of points of R and a point p of R, we say that the sequence $\{p_\alpha\ ;\ 0 \leqslant \alpha < \omega\}$ ***r-converges or ortho-converges*** to the point p, or that p is an ***r-limit ar an ortho-limit*** of $\{p_\alpha\ ;\ 0 \leqslant \alpha < \omega\}$, if there exists a fundamental sequence $\{V_\alpha(p)\ ;\ 0 \leqslant \alpha < \omega\}$ consisting of neighborhoods of p such that $V_\alpha(p) \ni p_\alpha$ for each α. In this case, we write $\{\lim_\alpha p_\alpha\} \ni p$. $\{\lim_\alpha p_\alpha\}$ is not a set consisting of one point alone in general.

By $\mathrm{cl}_r(E)$, we show the set of all r-limit points of a subset E of the ranked space R. we say that a set E is ***r-dense*** in the ranked space R if $\mathrm{cl}_r(E) = R$. A set $E(\subseteq R)$ is called ***r-cosed*** if $\mathrm{cl}_r(E) = E$.

Let R, S be two ranked spaces with same indicator ω. We say that a mapping $f: R \longrightarrow S$ is r-continuous at a point p in R if $\{\lim_\alpha p_\alpha\} \ni p \Rightarrow \{\lim_\alpha f(p_\alpha)\} \ni f(p)$. A mapping f is said to be r-continuous if it is r-continuous at each point in R.

A mapping $f: \{R, \mathfrak{V}_\alpha\} \longrightarrow \{R', \mathfrak{V}_\alpha'\}$ is called an ***r-homeomorphism*** if f is bijective and *bi-r*-continuous. In this case, the spaces R and R' are said to be ***r-homeomorphic*** or ***r-equivalent***.

Let A be a subset of $\{R, \mathfrak{V}_\alpha\}$. Put $V(p, A) = V(p) \cap A$ for each $V(p) \in \mathfrak{V}_\alpha(p)$ $(0 \leqslant \alpha < \omega)$ and $\forall p \in A$. By the relation $\mathfrak{V}_\alpha(A) \equiv \mathfrak{V}_\alpha(p) \cap A$, A becomes a ranked space with indicator ω. Then A is called the ranked space induced from R or the r-subspace of R.

Given a sequence $\{p_\alpha\ ;\ 0 \leqslant \alpha < \omega\}$ of points of $\{R, \mathfrak{V}_\alpha\}$ and a point p of $\{R, \mathfrak{V}_\alpha\}$, we say that the sequence $\{p_\alpha\ ;\ 0 \leqslant \alpha < \omega\}$ ***para-converges*** to the point p, or that p is a ***para-limit*** of $\{p_\alpha\ ;\ 0 \leqslant \alpha < \omega\}$, if there is a monotone decreasing sequence $\{V_\alpha(p_\alpha)\ ;\ 0 \leqslant \alpha < \omega\}$ consisting of neigborhoohds of p_α $(0 \leqslant \alpha < \omega)$ and if $V_\alpha(p_\alpha)$ satisfies the following three conditions :

1) $V_\alpha(p_\alpha) \in \mathfrak{V}_{\gamma(\alpha)}$ $(\exists \gamma(\alpha),\ 0 \leqslant \gamma(\alpha) < \omega)$,

2) $\gamma(0) \leqslant \gamma(1) \leqslant \gamma(2) \leqslant \cdots \leqslant \gamma(\alpha) \leqslant \cdots$, $\sup_\alpha \gamma(\alpha) = \omega$,

3) $p \in V_\alpha(p_\alpha)$ for all α, $0 \leqslant \alpha < \omega$.

In this case, we write $\{para\text{-}\lim_\alpha p_\alpha\} \ni p$.

Proposition. *Every sequence of points* $\{p_\alpha\ ;\ 0 \leqslant \alpha < \omega\}$ *in* R *is* r *and para-convergence at a point* $p \in R$ *if and only if there exist two monotone decreasing sequences of neighborhoods* $\{U_\alpha(p_\alpha)\ ;\ 0 \leqslant \alpha < \omega\}$ *and* $\{V_\alpha(p)\ ;\ 0 \leqslant \alpha < \omega\}$ *such that*

(1) $U_\alpha(p_\alpha) \in \mathfrak{V}_{\gamma(\alpha)}$, $V_\alpha(p) \in \mathfrak{V}_{\delta(\alpha)}$ *and* $\gamma(\alpha), \delta(\alpha) \uparrow \omega$ *as* $\alpha \uparrow \omega$

and

(2) $U_\alpha(p_\alpha) \cap V_\alpha(p) \supseteq \{p_\alpha, p\}$ *for all* α, $0 \leqslant \alpha < \omega$.

3. Completion of Ranked Spaces.***

We will consider the problem of completion : construction of a complete ranked space containing

*** 1971年10月，日本数学会秋季総合分科会（於京都大学）にて一部講演.

a given ranked space as an r-dense subspace.

Let $\{R, \mathfrak{B}_\alpha\}$ be a ranked space, $\omega(R) \geqslant \omega_0$ the depth of R, $\omega(\omega_0 \leqslant \omega \leqslant \omega(R))$ the indicator of R and let $\mathcal{F}$ the set of all fundamental sequences of neighborhoods in R.

In general, we have not always $\bigcap_{\alpha=0}^{\omega} V_\alpha(p_\alpha) \neq \phi$ for $\forall \{V_\alpha(p_\alpha)\,;\ 0 \leqslant \alpha < \omega\} \in \mathcal{F}$.

We shall, hereafter, assume the following postulates for $\{R, \mathfrak{B}_\alpha\}$:

(T_0*) For a point p and any point q such that $p \neq q$, there exist $\{u_\alpha(p)\,;\ 0 \leqslant \alpha < \omega\} \in \mathcal{F}$ and μ, $0 \leqslant \mu < \omega$, such that $u_\mu(p) \not\ni q$.

(C*) If $\{V_\alpha(p_\alpha)\}$, $\{U_\alpha(q_\alpha)\} \in \mathcal{F}$, then, for each index α, $0 \leqslant \alpha < \omega$, there are $\lambda \equiv \lambda(\alpha) = \lambda(\alpha, V_\alpha(p_\alpha), U_\alpha(q_\alpha)) \geqslant \alpha$ and $\delta \equiv \delta(\alpha) = \delta(q_\alpha, \lambda)$ $(0 \leqslant \lambda, \delta < \omega)$ such that if

$$U_{\delta(\alpha)}(q_{\delta(\alpha)}) \cap V_{\lambda(\alpha)}(p_{\lambda(\alpha)}) \neq \phi$$

then

$$U_{\delta(\alpha)}(q_{\delta(\alpha)}) \subseteq V_\alpha(p_\alpha).$$

(r) If $U(p) \supseteq V(q)$, $U(p) \in \mathfrak{B}_{\gamma'}$, $V(q) \in \mathfrak{B}_{\gamma''}$, and $\gamma' < \gamma''$, then for any γ, $\gamma' < \gamma \leqslant \gamma''$, there exist a point $r \in R$ and a neighborhood $W(r) \in \mathfrak{B}_\gamma$ such that $U(p) \supseteq W(r) \supseteq V(q)$.

(r′) Under the assumption in (r), there exist a rank γ and a neighorhood $W(p) \in \mathfrak{B}_\gamma$ such that $\gamma' < \gamma \leqslant \gamma''$ and $U(p) \supseteq W(p) \supseteq V(q)$.[2)]

These postulates hold always for every metric space. And these postulates are weaker than the assumptions in the Notes [5] and [12].

Then we will prove that the following theorem holds :

Completion Theorem (T. Shintani ([15])). *Let R be a given ranked space of depth $\omega(R) \geqslant \omega_0$ and let ω, $\omega_0 \leqslant \omega \leqslant \omega(R)$, be any indicator of the space R. If R satisfies the postulates (T_0*), (C*), (r) and (r'), then there exists the completion $\hat{R}$ of R such that $\hat{R}$ is a ranked space with indicator ω and it is uniquely determined by R. Namely, for a given ranked space R with indicator ω, we can always construct the ranked space $\hat{R}$ with indicator ω. Moreover $\hat{R}$ fulfills the following properties :*

(*i*) *In $\hat{R}$, for any fundamental sequence $\{\hat{V}_\alpha(P_\alpha)\,;\ 0 \leqslant \alpha < \omega\}$, $\bigcap_{\alpha=0}^{\omega} \hat{V}_\alpha(P_\alpha) \neq \phi$ holds always. Namely R is complete.*

(*ii*) *There exists an r-homeomorphism of R onto an r-subspace $R' \equiv f(R)$ of $\hat{R}$.*

(*iii*) *$\hat{R} = cl_r(f(R))$, i. e., $f(R)$ is r-dense in $\hat{R}$.*

***Corollary* 1.** *For any two indicators ω_μ and ω_ν $(\mu < \nu)$, if $\hat{R}$ is ω_ν-completion of R then $\hat{R}$ is ω_μ-completion of R.*

***Corollary* 2.** *If R satisfies the postulates (T_0*) and (C*), then, essentially, there exists the completion $\hat{R}$ of R. Thus, for each fundamental sequence $\{\hat{V}_\alpha(P_\alpha)\,;\ 0 \leqslant \alpha < \omega\}$ in $\hat{R}$, the sequence $\{P_\alpha\,;\ 0 \leqslant \alpha < \omega\}$ of points in $\hat{R}$ is always para-convergent in $\hat{R}$.*

Remark. The completion in the ordinary sense of the metric space R is a completion of the ranked space R with indicator ω_0 in above sence.

4. Proof of Completion Theorem.

Definition. We will call two members $\Pi' = \{V_\alpha'(p_\alpha')\}$ and $\Pi'' = \{V_\alpha''(p_\alpha'')\}$ of $\mathcal{F}$ equivalent and write $\Pi' \sim \Pi''$ if for every α, $0 \leqslant \alpha < \omega$, there are $p_\alpha \in R$ and a member $\{V_\alpha(p_\alpha)\} \in \mathcal{F}$ which have the following property : for every β, $0 \leqslant \beta < \omega$, there exists ϑ, $0 \leqslant \vartheta < \omega$, such that

$$V_\vartheta'(p_\vartheta') \cup V_\vartheta''(p_\vartheta'') \subseteq V_\beta(p_\beta).$$

Using the axioms (b), (r) and the transfinite induction, we get the following lemma.

2) [12], p. 338.

***Lemma* 1.** *For each* $\Pi=\{V_\alpha(p_\alpha)\}\in\mathfrak{F}$ *there are a sequence of points* $\{q_\alpha;\ 0\leqslant\alpha<\omega\}$ *and a fundamental sequence* $\Pi^*=\{U_\gamma(q_\gamma);\ 0\leqslant\alpha<\omega\}$ *such that*

1) $\{p_\alpha;\ 0\leqslant\alpha<\omega\}$ *is a subsequence of* $\{q_\alpha;\ 0\leqslant\alpha<\omega\}$,

2) Π *is a subsequence of* Π^*,

3) $U_\gamma(q_\gamma)\in\mathfrak{B}_\gamma\ (\forall\gamma,\ 0\leqslant\gamma<\omega)$

and

4) $\Pi^*\sim\Pi$.

***Lemma* 2.** *If* $\Pi'=\{V_\nu'(p_\nu')\}$ *and* $\Pi''=\{V_\nu''(p_\nu'')\}$ *are such that for every* θ, $0\leqslant\theta<\omega$, $V_\theta'(p_\theta')\cap V_\theta''(p_\theta'')\neq\phi$, *then* $\Pi'\sim\Pi''$.

Proof. Since $V_\theta'(p_\theta')\cap V_\theta''(p_\theta'')\neq\phi$ $(\forall\theta,\ 0\leqslant\theta<\omega)$ for each index ν $(0\leqslant\nu<\omega)$ of Π', there are two indices $\lambda=\lambda(\nu)$ and $\delta=\delta(p_\nu'',\lambda)$ of the axiom (C^*). Therefore we get

$$V_\delta''(p_\delta'')\cap V_\lambda'(p_\lambda')\supseteqq V_\theta''(p_\theta'')\cap V_\theta'(p_\theta')\neq\phi \text{ when } \theta\equiv\text{Max}\ \{\delta,\lambda\}.$$

From (C^*) we have $V''_{\delta(\nu)}(p''_{\delta(\nu)})\subseteq V'_\nu(p'_\nu)$ and from $\lambda(\nu)\leqslant\nu$ we have $V'_{\lambda(\nu)}(p'_{\lambda(\nu)})\subseteq V_\nu'(p_\nu')$.

Hence

$$V_\theta''(p_\theta'')\cup V_\theta'(p_\theta')\subseteq V_\delta''(p_\delta'')\cup V_\lambda'(p_\lambda')\subseteq V_\nu'(p_\nu')$$

and $\Pi'\sim\Pi''$. (*Q.E.D.*)

Corollary. *For any* $\{U_\alpha(p_\alpha)\}$ *and* $\{V_\alpha(p_\alpha)\}$ *in* $\mathfrak{F}$, *we have* $\{U_\alpha(p_\alpha)\}\sim\{V_\alpha(p_\alpha)\}$.

In fact, for every θ, $0\leqslant\theta<\omega$, $U_\theta(p_\theta)\cap V_\theta(p_\theta)\neq\phi$ holds.

***Lemma* 3.** $\Pi\sim\Pi$. $\Pi\sim\Pi'\Rightarrow\Pi'\sim\Pi$. $\Pi^{(1)}\sim\Pi^{(2)}$ & $\Pi^{(2)}\sim\Pi^{(3)}\Rightarrow\Pi^{(1)}\sim\Pi^{(3)}$.

Proof. The first two properties of the equivalence relation are clear.

Suppose that $\Pi^{(1)}\sim\Pi^{(2)}$ & $\Pi^{(2)}\sim\Pi^{(3)}$. There are $\Pi^1=\{V_\alpha^1(p_\alpha^1)\}$, $\Pi^2=\{V_\alpha^2(p_\alpha^2)\}$ such that, by assumption, for every α, $0\leqslant\alpha<\omega$, there are two indices β and γ $(0\leqslant\beta,\gamma<\omega)$ such that

i) $$\begin{cases} V_\beta^{(1)}(p_\beta^{(1)})\cup V_\beta^{(2)}(p_\beta^{(2)})\subseteq V_\alpha^1(p_\alpha^1),\\ V_\gamma^{(2)}(p_\gamma^{(2)})\cup V_\gamma^{(3)}(p_\gamma^{(3)})\subseteq V_\alpha^2(p_\alpha^2).\end{cases}$$

Hence

$$\phi\neq\bigcap_{\nu=0}^{Max\{\beta,\gamma\}}V_\nu^{(2)}(p_\nu^{(2)})=V_\beta^{(2)}(p_\beta^{(2)})\cap V_\gamma^{(2)}(p_\gamma^{(2)})\subseteq V_\alpha^1(p_\alpha^1)\cap V_\alpha^2(p_\alpha^2)$$ and $\Pi^1\sim\Pi^2$ by

Lemma 2. Hence, for any ν, $0\leqslant\nu<\omega$, there are $p_\nu\in R$, $\{V_\nu(p_\nu)\}\in\mathfrak{F}$ and $\alpha=\alpha(\nu)$ $(0\leqslant\alpha<\omega)$ such that

ii) $$V_\alpha^1(p_\alpha^1)\cup V_\alpha^2(p_\alpha^2)\subseteq V_\nu(p_\nu).$$

From i) and ii) we have $$V_\beta^{(1)}(p_\beta^{(1)})\cup V_\beta^{(3)}(p_\beta^{(3)})\subseteq V_\nu(p_\nu)$$

and $\Pi^{(1)}\sim\Pi^{(3)}$. (*Q.E.D.*)

***Lemma* 4.** *If* $\Pi=\{V_\alpha(p_\alpha)\}\sim\Pi'=\{V_\alpha'(p_\alpha')\}$, *then there is an index* $\beta=\beta(\alpha)$, $0\leqslant\beta<\omega$, *such that* $V_\beta'(p_\beta')\subseteq V_\alpha(p_\alpha)$.

Proof. Since $\Pi\sim\Pi'$, there are $\{V_\alpha''(q_\alpha)\}\in\mathfrak{F}$, $\delta=\delta(\alpha)$ and $\beta=\beta(\alpha)$ such that

i) $$V_\beta(p_\beta)\cup V_\beta'(p_\beta')\subseteq V''_{\delta(\alpha)}(q_{\delta(\alpha)}).$$

Now,

ii) $$\begin{cases} \bigcap_{\nu=0}^{Max\{\beta,\lambda(\alpha)\}}V_\nu(p_\nu)\subseteq V_\beta(p_\beta)\subseteq V''_{\delta(\alpha)}(q_{\delta(\alpha)}),\\ \bigcap_{\nu=0}^{Max\{\beta,\lambda(\alpha)\}}V_\nu(p_\nu)\subseteq V_{\lambda(\alpha)}(p_{\lambda(\alpha)}).\end{cases}$$

Hence, we have

iii) $$V''_{\delta(\alpha)}(q_{\delta(\alpha)})\cap V_{\lambda(\alpha)}(p_{\lambda(\alpha)})\neq\phi.$$

Hence, by the axiom (C^*), we get

iv) $$V''_{\delta(\alpha)}(q_{\delta(\alpha)})\subseteq V_\alpha(p_\alpha).$$

Hence, from i), ii) and iv), we have

$$V_{\beta}'(p_{\beta'}) \subseteq V''_{\delta(\alpha)}(q_{\delta(\alpha)}) \subseteq V_\alpha(p_\alpha). \quad (Q.E.D.)$$

Remark. Given $u=\{U_\alpha(p_\alpha)\}$, $v=\{V_\alpha(q_\alpha)\}\in\mathfrak{F}$, we denote by $u\succ v$ the relation between u and v such that for every $U_\alpha(p_\alpha)$ there is a $V_\beta(q_\beta)$ contained in $V_\alpha(p_\alpha)$. If we define $u\approx v$ by $u\succ v$ & $u\prec v$ ([11]), then $u\sim v \Leftrightarrow u\approx v$. This shows that our completion concludes all completions by [5], [12], [6], [13], and [11].

By Lemma 3, we can construct the quotient set $\hat{R}\equiv\mathfrak{F}/\sim$.

Now, we will give a rank to $\hat{R}$ by the following method.

Let $\forall P\in\hat{R}$. Then $P\ni\exists\ \{V_\nu(p_\nu)\}$ such that $V_\nu(p_\nu)\in\mathfrak{B}_\nu$.

By the relations

$$\hat{V}(P)\equiv\{Q\in\hat{R}\mid Q\ni\{U_\nu(x_\nu)\}\,;\ \exists\theta'\ (0\leqslant\theta'<\omega)\text{ such that }U_{\theta'}(x_{\theta'})\subseteq V_\theta(p_\theta)\}$$

and

$$\text{the rank of }\hat{V}(P)\equiv\text{the rank of }V_\theta(p_\theta),$$

we define a neighborhood $\hat{V}(P)$ with rank of P in $\hat{R}$. And we denote by $\hat{\mathfrak{B}}_\alpha(P)$ the set of all neighborhoods with rank α of the point P. Every subset which contains a neighborhood with rank of P is called a neighborhood of P in $\hat{R}$.

***Lemma* 5.** *$\hat{R}$ is a ranked space, of depth $\omega(R)\geqslant\omega$, which satisfies the axioms (A), (B), (a) and (b).*

Proof. Axiom (a) : For any neighborhood $\hat{V}(P)$ of P in $\hat{R}$ there are a neighborhood $\hat{V}'(P)$ of P and an ordinal number γ, $0\leqslant\gamma<\omega$, such that $\hat{V}(P)\supseteq\hat{V}'(P)\in\hat{\mathfrak{B}}_\gamma$.

If $\alpha\leqslant\gamma$ then let $\beta=\gamma$. If $\alpha>\gamma$ then by the axiom (r) we have

$$V_\gamma'(p_\gamma)\supseteq\exists V_\alpha'(p_\alpha)\supseteq\exists V_\beta'(p_\beta)\ (\alpha\leqslant\beta<\omega).$$

From $\hat{U}(P)\equiv\{Q\in\hat{R}\mid\exists\vartheta\,;\ U_\vartheta(x_\vartheta)\subseteq V_\beta'(p_\beta)\}$, we get the followings :

(a). $\forall\hat{V}(P),\ \forall\alpha\,;\ \alpha\leqslant\exists\beta<\omega\ \&\ \hat{V}(P)\supseteq\exists U(P)\in\hat{\mathfrak{B}}_\beta$.

(A). By the definition of neighborhoods in $\hat{R}$, we get $\hat{V}(P)\ni P$.

(B). If $\hat{V}^{(i)}(P)$ $(i=1,2)$ are any neighborhoods of P, then $\hat{V}^{(i)}(P)\supseteq\exists\hat{U}^{(i)}(P)\in\hat{\mathfrak{B}}(\equiv\bigcup_{\alpha=0}^{\omega}\hat{\mathfrak{B}}_\alpha)$ $(i=1,2)$.

Now, let $\hat{U}^{(i)}(P)\equiv\{Q\in\hat{R}\mid\exists\theta_i'\,;\ U^{(i)}{}_{\theta_i}'(x^{(i)}{}_{\theta_i}')\subseteq V^{(i)}{}_{\theta_i}(p^{(i)}{}_{\theta_i})$ (i=1,2), then

$$\{V_\alpha{}^{(1)}(p_\alpha{}^{(1)})\}\sim\{V_\alpha{}^{(2)}(p_\alpha{}^{(2)})\}.$$

Hence there is an index ϑ, $0\leqslant\vartheta<\omega$, such that $V_{\theta_1}{}^{(1)}(p_{\theta_1}{}^{(1)})\supseteq V_\vartheta{}^{(2)}(p_\vartheta{}^{(2)})$.

Thus we have

$$V_{\theta_1}{}^{(1)}(p_{\theta_1}{}^{(1)})\cap V_\vartheta{}^{(2)}(p_\vartheta{}^{(2)})\supseteq V_\theta{}^{(2)}(p_\theta{}^{(2)})\text{ when }\theta=\mathrm{Max}\ \{\theta_2,\vartheta\}.$$

Therefore if we put

$$\hat{W}(P)\equiv\{Q\in\hat{R}\mid Q\ni\exists\{U_\nu(p_\nu)\},\ \exists\kappa\,;\ U_\kappa(p_\kappa)\subseteq V_\theta{}^{(2)}(p_\theta{}^{(2)})\},$$

then we have $\hat{U}^{(1)}(P)\cap\hat{U}^{(2)}(P)\supseteq\hat{W}(P)\in\hat{\mathfrak{B}}$. $(Q.E.D.)$

Notation. Let $\hat{\mathfrak{F}}$ be the set of all fundamental sequences of neighborhoods in $\hat{R}$.

***Lemma* 6.** For every $\{\hat{V}_\alpha(P)\,;\ 0\leqslant\alpha<\omega\}\in\hat{\mathfrak{F}}$ in $\{\hat{R},\hat{\mathfrak{B}}_\alpha\}$, we have always $\bigcap_{\alpha=0}^{\omega}\hat{V}_\alpha(P_\alpha)\neq\phi$.

Hence $\hat{R}$ is complete.

Proof. Let $\hat{V}_\alpha(P_\alpha)\equiv\{Q_\alpha\in\hat{R}\mid U_\alpha^{(\theta'_\alpha)}(p_\alpha^{(\theta'_\alpha)})\subseteq V_\alpha^{(\theta_\alpha)}(p_\alpha^{(\theta_\alpha)})\in\mathfrak{B}_{\gamma(\alpha)}\}$ and let $\forall y\in V_{\alpha+1}^{(\theta_\alpha+1)}(p_{\alpha+1}^{(\theta_\alpha+1)})$. From (C*), $\exists Q\ni\{u_\nu(y)\}$ & $Q\in\hat{V}_{\alpha+1}(P_{\alpha+1})$ holds. On the other hand, since $\hat{V}_\alpha(P_\alpha)\supseteq\hat{V}_{\alpha+1}(P_{\alpha+1})\ni Q$, there are $\{v_\nu(y_\nu)\}\in Q$ and θ', $0\leqslant\theta'<\omega$, and then $V_\alpha^{(\theta_\alpha)}(p_\alpha^{(\theta_\alpha)})\supseteq v_{\theta'}(y_{\theta'})$ holds.

Since $\{u_\nu(y)\}\sim\{v_\nu(y_\nu)\}$, there exists an index ϑ' such that $v_{\theta'}(y_{\theta'})\supseteq u_{\vartheta'}(y)\ni y$.

Hence we have

$$V_\alpha^{(0_\alpha)}(p_\alpha^{(0_\alpha)}) \supseteq V_{\alpha+1}^{(0_{\alpha+1})}(p_{\alpha+1}^{(0_{\alpha+1})}) \quad (\forall\alpha,\ 0 \leqslant \alpha < \omega).$$

Now, by the axiom (r') and the transfinite induction, there exists a fundamental sequence such that $\{V_\alpha^{(0_\alpha)}(p_\alpha^{(0_\alpha)})\ ;\ 0 \leqslant \alpha < \omega\}$ is its subsequence.

Let P be the class which concludes such fundamental sequence, then we have $\hat{V}_\alpha(P_\alpha) \ni P$ for every α, $0 \leqslant \alpha < \omega$. Therefore we have always $\bigcap_{\alpha=0}^{\omega} \hat{V}_\alpha(P_\alpha) \neq \phi$. $(Q.E.D.)$

Lemma 7. *The mapping* $f:\ R \ni p \longrightarrow f(p) \equiv P \ni \{V_\alpha(p)\}$ *of* $\{R, \mathfrak{B}_\alpha\}$ *onto a ranked space* $\{f(\hat{R}), \hat{\mathfrak{B}}_\alpha \cap f(R)\}$ *induced from* $\{\hat{R}, \hat{\mathfrak{B}}_\alpha\}$ *is one-to-one and bi-r-continuous.*

Proof. (1). f is one-to-one. In fact, let x, y be two distinct point of R and let $\{u_\alpha(x)\}$, $\{v_\alpha(y)\}$ any two fundamental sequence in R. If $\{u_\alpha(x)\} \sim \{v_\alpha(y)\}$ then by (T_0^*) there are $\{U_\alpha(x)\} \in \mathfrak{F}$ and μ, $0 \leqslant \mu < \omega$, such that $U_\mu(x) \not\ni y$. From $\{U_\alpha(x)\} \sim \{u_\alpha(x)\}$ we get $\{U_\alpha(x)\} \sim \{v_\alpha(y)\}$.
Hence there is an index ζ, $0 \leqslant \zeta < \omega$, such that $U_\mu(x) \supseteq v_\zeta(y) \ni y$. This is a contradiction.
Thus if $x \neq y$ then, for every $\{u_\alpha(x)\}$ and $\{v_\alpha(y)\}$, we have $\{u_\alpha(x)\} \not\sim \{v_\alpha(y)\}$. Hence, for two points X and Y in $\hat{R}$ such that $X \ni \exists\{u_\alpha(x)\}$ and $Y \ni \exists\{v_\alpha(y)\}$, we have $X = f(x) \neq f(y) = Y$. Thus f is one-to-one.

(2). For every $V(p) \in \mathfrak{B}_\alpha(p)$ we have $f(V(p)) = \hat{V}(P,\ f(R)) \in \hat{\mathfrak{B}}_\alpha(P,\ f(R))$ $(P = f(p))$.
Thus f is r-continuous.

In fact, from the axiom (a) there exists $\{V_\alpha(p)\} \in \mathfrak{F}$ such that $V(p)$ is its member.
Let P be the equivalence class which concludes $\{V_\alpha(p)\}$, then $f(p) = P$. Now, let

$$\hat{V}(P) = \{Q \in \hat{R} \mid Q \ni \{U_\nu(q_\nu)\},\ \exists\theta'\ ;\ U_{\theta'}(q_{\theta'}) \subsetneq v(p)\},$$

$\forall x \in V(p)$ and $\forall\{u_\nu(x)\} \in X$. Then, from (C^*), there is an index ϑ, $0 \leqslant \vartheta < \omega$, such that $u_\vartheta(x) \subseteq V(p)$. Therefore we get $X \in \hat{V}(P)$. On the other hand, as $f(x) = X \in f(R)$ we get $X \in f(R) \cap \hat{V}(P)$.
Thus,

$$V(p) \xrightarrow[into]{f} \hat{V}(P) \cap f(R) \equiv \hat{V}(P,\ f(R)),\ P = f(P).$$

Now, let $\forall Q \in \hat{V}(P) \cap f(R)$, then $Q \in f(R)$. Hence, $Q \ni \exists\{v_\nu(q)\}$ $(\exists q \in R)$. Moreover, from $Q \in \hat{V}(P)$, there exists an index θ', $0 \leqslant \theta' < \omega$, such that $q \in v_{\theta'}(q) \subsetneq V(p)$. Thus, it follows $V(p) \xrightarrow[onto]{f} \hat{V}(P,\ f(R))$.
Hence if $\{\lim_\alpha p_\alpha\} \ni p$, *i. e.*, $\exists\{V_\alpha(p)\} \in \mathfrak{F}$ such that $V_\alpha(p) \ni p_\alpha$ $(\forall\alpha,\ 0 \leqslant \alpha < \omega)$, then we have

$$f(V_\alpha(p)) = \hat{V}_\alpha(P,\ f(R)) = \hat{V}_\alpha(f(p),\ f(R)) \equiv \hat{V}_\alpha(f(p)) \cap f(R) \ni f(p_\alpha) \equiv Q_\alpha \text{ in } f(R),$$

and $\{\lim_\alpha f(p_\alpha)\} \ni f(p)$ in $f(R)$.
Thus f is an r-continuous mapping of R onto $f(R)$.

(3). f^{-1} is r-continuous.
Let $\{\hat{V}_\alpha(P,\ f(R))\}$ be a fundamental sequence in $f(R)$ such that $\hat{V}_\alpha(P,\ f(R)) \ni Q_\alpha$, *i. e.*, $\{\lim_\alpha Q_\alpha\} \ni P$ in $f(R)$. As f^{-1} is one-to-one, we have

$$\begin{aligned} f^{-1}(\hat{V}_\alpha(P,\ f(R)) &\equiv f^{-1}(\hat{V}_\alpha(P) \cap f(R)) = f^{-1}(\hat{V}_\alpha(P)) \cap f^{-1}(f(R)) = f^{-1}(\hat{V}_\alpha(P)) \cap R = f^{-1}(\hat{V}_\alpha(P)) \\ &= \exists V_\alpha(p_\alpha)\ (\hat{V}_\alpha(P) \equiv \{Q \in \hat{R} \mid Q \ni \{u_\nu(x_\nu)\},\ \exists\theta'\ ;\ u_{\theta'}(x_{\theta'}) \subsetneq V_\alpha(p_\alpha)\}) \\ &\ni \exists q_\alpha (\equiv f^{-1}(Q_\alpha)). \end{aligned}$$

Thus, for every $\hat{V}_\alpha(P,\ f(R)) \in \hat{\mathfrak{B}}_\alpha(P,\ f(R))$, there exists $p_\alpha \in R$ such that $f^{-1}(\hat{V}_\alpha(P,\ f(R)) \in \mathfrak{B}_\alpha(p_\alpha)$.

On the other hand, as $P \in f(R)$, we have $P \ni \exists\{v_\alpha(\exists p)\}$ and $f(p) = P$ (thus, $f^{-1}(P) = p$).

Now, from (b) and (r'), there exists $U_0(p) \in \mathfrak{B}(p)$ such that $R \supseteq U_0(p) \supseteq U_0(p_0)$.

Thus, from (r') there exists $U_1(p) \in \mathfrak{B}(p)$ such that $U_1(p) \supseteq V_1(p_1)$. Therefore, by the axiom (b), (r') and the transfinite induction, there exists $\{U_\alpha(p)\} \in \mathfrak{F}$ such that $U_\alpha(p) \supseteq V_\alpha(p_\alpha)$ for every α, $0 \leqslant \alpha < \omega$. Hence we have $U_\alpha(f^{-1}(P)) \ni f^{-1}(Q_\alpha)$ $(\forall\alpha,\ 0 \leqslant \alpha < \omega)$, *i. e.*, $\{\lim_\alpha f^{-1}(Q_\alpha)\} \ni f^{-1}(P)$ in R.
Thus f^{-1} is r-continuous. $(Q.E.D.)$

Lemma 8. $cl_r(f(R))=\hat{R}$, *i. e.*, $f(R)$ *is r-dense in* $\hat{R}$.

Proof. $\mathrm{cl}_r(f(R))\subseteq\hat{R}$ is clear. We will show that $\mathrm{cl}_r(f(R))\supseteq\hat{R}$. Let $\forall P\in\hat{R}$, $\forall V(P)\in\hat{\mathfrak{B}}(P)$ and let $\hat{V}(P)=\{Q\in\hat{R}\mid Q\ni\{U_\nu(q_\nu)\},\ \exists\theta';\ U_{\theta'}(q_{\theta'})\subseteq V_\theta(p_\theta)\}$.

As the indicator of R is ω, for an index θ of $\{V_\nu(p_\nu);\ 0\leqslant\nu<\omega\}\in\mathfrak{F}$ and for any ϑ, $0\leqslant\vartheta<\omega$, we have $V_\theta(p_\theta)\supseteq V_\vartheta(p_\vartheta)\neq\phi$. Thus there is a point $q\in R$ such that $q\in V_\vartheta(p_\vartheta)\subseteq V_\theta(p_\theta)$.

By (C*), for some $Q\in f(R)$ and for every $\{U_\nu(q)\}\in Q$ there exist two indices $\lambda(\theta)=\lambda(\theta,V_\theta(p_\theta),\ U_\theta(q))\geqslant 0$ and $\delta(\theta)=\delta(q,\lambda)$. Hence we have $U_{\delta(\theta)}(q)\subseteq V_\theta(p_\theta)$. Hence we have $Q=f(q)\in\hat{V}(P)$. Thus, for every $\{V_\alpha(P)\}\in\hat{\mathfrak{F}}$ there are points $Q_\alpha\in\hat{R}$ and $q_\alpha\in R$ such that $Q_\alpha=f(q_\alpha)\in\hat{V}_\alpha(P)$ $(\forall\alpha,0\leqslant\alpha<\omega)$. Therefore we have $\{\lim_\alpha Q_\alpha\}\ni P$, *i. e.*, $P\in\mathrm{cl}_r(f(R))$. Thus we have $\hat{R}\subseteq\mathrm{cl}_r(f(R))$. Hence $\hat{R}=\mathrm{cl}_r(f(R))$. (*Q.E.D.*)

Every neighborhood with rank of P in $\hat{R}$ had been defined by an element $\hat{\Pi}^P=\{V_\nu^{(1)}(p_\nu^{(1)})\}$ of P. Now, let $\tilde{R}$ be a ranked space defined by another element $\tilde{\Pi}^P=\{V_\nu^{(2)}(p_\nu^{(2)})\}$ of P. Then two spaces $\hat{R}$ and $\tilde{R}$ are r-equivalent. Namely, we have the following lemma :

Lemma 9. (Uniqueness of complete extension).

$$\hat{R}=\hat{cl}_r(f(R))=\tilde{cl}_r(f(R))=\tilde{R}.$$

Proof.

Let
$$\begin{cases}\forall P\in\hat{\mathrm{cl}}_r(f(R)).\ \exists\{\hat{V}_\alpha(P)\}\in\hat{\mathfrak{F}}.\ \exists\{\tilde{U}_\alpha(P)\}\in\tilde{\mathfrak{F}}.\\ \hat{V}_\alpha(P)\equiv\{Q^{(1)}\in\hat{R}\mid Q_\alpha^{(1)}\ni\{U_\nu^{(1)}(q_\nu^{(1)})\},\exists\theta_\alpha';\ U_{\theta_\alpha'}^{(1)}(q_{\theta_\alpha'}^{(1)})\subseteq V_{\theta_\alpha}^{(1)}(p_{\theta_\alpha}^{(1)})\}.\\ \tilde{U}_\alpha(P)\equiv\{Q_\alpha^{(2)}\in\hat{R}\mid Q_\alpha^{(2)}\ni\{U_\nu^{(2)}(q_\nu^{(2)})\},\ \exists\vartheta_\alpha';\ U_{\vartheta_\alpha'}^{(2)}(q_{\vartheta_\alpha'}^{(2)})\subseteq V_{\vartheta_\alpha}^{(2)}(p_{\vartheta_\alpha}^{(2)})\}.\end{cases}$$

As $P\in\hat{\mathrm{cl}}_r(f(R))$, there is a sequence $\{Q_\alpha^{(1)};\ 0\leqslant\alpha<\omega\}$ in $f(R)$ such that $Q_\alpha^{(1)}\in\hat{V}_\alpha(P)$ $(\forall\alpha,\ 0\leqslant\alpha<\omega)$. From $\hat{\Pi}^P\sim\tilde{\Pi}^P$ and Lemma 4, for a given index ϑ_α, $0\leqslant\vartheta_\alpha<\omega$, there exists an index ϑ_α'', $0\leqslant\vartheta_\alpha''<\omega$, such that $V_{\vartheta_\alpha''}^{(1)}(p_{\vartheta_\alpha''}^{(1)})\subseteq V_{\vartheta_\alpha}^{(2)}(p_{\vartheta_\alpha}^{(2)})$.

Now, let τ_α be the minimum of such ϑ_α''. Then we have $\hat{V}_{\tau_\alpha}(P)\subseteq\tilde{U}_\alpha(P)$ for every α, $0\leqslant\alpha<\omega$.

Hence we get

$$\hat{Q}_{\tau_\alpha}^{(1)}\in\hat{V}_{\tau_\alpha}(P)\subseteq\tilde{U}_\alpha(P)\quad(\forall\alpha,\ 0\leqslant\alpha<\omega).$$

Set $Q_\alpha^{(2)}\equiv Q_{\tau_\alpha}^{(1)}(\forall\alpha,\ 0\leqslant\alpha<\omega)$. Then we have $f(R)\ni Q_\alpha^{(2)}\in\tilde{U}_\alpha(P)$ and therefore

$$\{\lim_\alpha Q_\alpha^{(2)}\}\ni P \text{ in } f(R),\ i.\ e.,\ P\in\tilde{\mathrm{cl}}_r(f(R)).$$

Thus we have

$$\hat{\mathrm{cl}}_r(f(R))\subseteq\tilde{\mathrm{cl}}_r(f(R)).$$

By the similar way we get $\tilde{\mathrm{cl}}_r(f(R))\subseteq\hat{\mathrm{cl}}_r(f(R))$.
Hence we have

$$\hat{R}=\hat{\mathrm{cl}}_r(f(R))=\tilde{\mathrm{cl}}_r(f(R))=\tilde{R}.$$

Thus we get the uniqueness of completion. (*Q.E.D.*)
And we complete the proof of the Completion Theorem.

5. An example of the ranked space with non-indicator ω_0.

Such example has not been given yet. We will give a simple of the ranked space with indicator Ω (Ω is the ordinal of the second number class). Thus we can construct the ranked spaces with any indicator ω_ν.

Proposition. $R\equiv\{\xi\mid 0\leqslant\xi<\Omega\}$[3] *is a ranked space with indicator* Ω.

3) The set R is compact in the sense of M. Fréchet and also non-compact. (cf. K. Kunugi : Kaisekigaku Yōron, p. 223, (1933), Japan.)

Proof. We define a neighborhood $V_{\alpha'}(\alpha)$ with rank α' $(0 \leqslant \alpha' < \Omega)$ of a point α in R by the following relation :

$$V_{\alpha'}(\alpha) \overset{def.}{=} \{\xi \mid \alpha' \leqslant \xi < \Omega\} \text{ if } \alpha' \leqslant \alpha < \Omega,$$

$$\overset{def.}{=} \{\alpha\} \cup \{\xi \mid \alpha' \leqslant \xi < \Omega\} \text{ if } \alpha < \alpha' < \Omega.$$

Then R is a ranked space with indicator Ω and it satisfies the axioms (A), (B), (a) and (b). This is shown as follows :

Case. $\alpha' \leqslant \alpha < \Omega$; (A) and (b) are clear. (B) ; For every α' and α'' such that $\alpha', \alpha'' \leqslant \alpha < \Omega$ we have

$$V_{\alpha'}(\alpha) \cap V_{\alpha''}(\alpha) = V_{max(\alpha', \alpha'')}(\alpha).$$

(a) ; $\forall \alpha'', \alpha' \leqslant \alpha'' \leqslant \alpha < \Omega \Rightarrow V_{\alpha'}(\alpha) \supseteq V_{\alpha''}(\alpha)$.

Case. $\alpha < \alpha' < \Omega$; (A) and (b) are clear. (B) ; For every α' and α'' such that $\alpha < \alpha', \alpha'' < \Omega$ we have

$$V_{\alpha'}(\alpha) \cap V_{\alpha''}(\alpha) \supseteq V_{max(\alpha', \alpha'')}(\alpha).$$

(a) ; $\forall \alpha'', \alpha < \alpha' \leqslant \alpha'' < \Omega \Rightarrow V_{\alpha'}(\alpha) \supseteq V_{\alpha''}(\alpha)$.

Case. $\alpha' \leqslant \alpha \leqslant \alpha'' < \Omega$. ; (A) and (b) are clear. (B) ; $V_{\alpha'}(\alpha) \cap V_{\alpha''}(\alpha) = V_{\alpha''}(\alpha)$.

(a) ; $\forall \beta, \alpha' \leqslant \beta \leqslant \alpha'' < \Omega \Rightarrow V_{\alpha'}(\alpha) \supseteq V_{\beta}(\alpha)$. $(Q.E.D.)$

6. On $\sum$-spaces.

K. Nagami ([9]) introduced the notion of $\sum$-spaces as a generalization of M-spaces by K. Morita ([8]).

A perfectly normal space and a countably compact space are examples of P-spaces ([8]).
Every M-space or M^*-space or σ-space is a $\sum$-space ([9]). Every $\sum$-space is a P-space ([9]).

Definition. Let $\mathcal{F}$ be a covering of a Hausdorff space E and x a point of E.
Then we set

$$C(x, \mathcal{F}) = \cap \{F \mid x \in F \in \mathcal{F}\}.$$

A $\sum$-net of a space E is a sequence $\{\mathcal{F}_n\}_{n \in N}$ of locally finite closed coverings satisfying the following condition :

If $K_1 \supseteq K_2 \supseteq \cdots$ is a sequence of non-empty closed sets of E such that $K_n \subseteq C(x, \mathcal{F}_n)$ for some point x in E and for each $n \in N$, then $\bigcap_n K_n \neq \phi$. A space E is called a $\sum$-space if it has a $\sum$-net.

Now we have the following statement :

Proposition. Every $\sum$-space E has a structure as a ranked space with indicator ω_0.

In fact, for $\forall p \in E$ we set

$$v(n\,;\,p) = K_n \cup \{p\} \ (n \in N),\ \mathfrak{B}_n(p) = \text{the set of every } v(n\,;\,p) \text{ and } \mathfrak{B}_0(p) = \{E\}.$$

Then $\{E, \mathfrak{B}_n\}$ is a ranked space with indicator ω_0. $(Q.E.D.)$

Acknowledgment. The author is very grateful to Professor K. Kunugi, Member of the Japan Academy, and Professor T. Ando, Chief of Mathematics of the Research Institute of Applied Electricity, Hokkaido University, for their helpful advices of this study.

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