On Generalized Continuous Groups III

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Synopsis.

In this paper we will study an open problem of the closed-graph theorem offered by Alexander Grothendieck. We will show that the Linear Ranked Space satisfies his conjecture. A purpose of such study is to develope more general and unified functional analysis on wide spaces without metric or norm. We shall be able to develope such analysis on a Linear Ranked Space as a generalization of Banach space theory. For instance, we can extend the most part of results in ([1]) or ([2]) or ([20]) by such methods of ranked spaces. Moreovr every space considered for the conjecture has a structure as a linear ranked space.

§ 1. Introduction.**

Historically, the Linear Ranked Space has been introduced independently by H. Okano ([7]), M. Washihara ([16]), M. Yamaguchi ([19]) and T. Shintani ([12]). The author's studies ([12], [13]) are most general. These studies are applying to the theories of generalized integrals, of partial differential equations as an extension of L. Hörmander's theory, of nuclear spaces as an extension of I. M. Gel'fand's theory, and of the others by many mathematicians ([6], [18], [9],..., etc.).

Now, let E and F be two topological vector spaces. Let us consider a linear mapping f from a departure space E into an arrival space F. Then if the graph of f is closed then f is continuous under convenient assumption on E and F.

Conjecture (A. Grothendieck ([3])). *Let E be a Fréchet space. Then is there a class $\mathfrak{F}$ of arrival topological vector spaces with the following good properties?*

(*i*) *Every space $F \in \mathfrak{F}$ is wider than every Fréchet space.*

(*ii*) *The class $\mathfrak{F}$ is closed by the following operations :*

1) *The image of $F \in \mathfrak{F}$ by a continuous linear mapping belongs to $\mathfrak{F}$.*

2) *The closed subspace of $F \in \mathfrak{F}$ belongs to $\mathfrak{F}$.*

3) *The product space $F = \prod_n F_n$ of spaces $\{F_n\}_{n=1,2,\dots} \subset \mathfrak{F}$ belongs to $\mathfrak{F}$.*

4) *The inductive limit of spaces $\{F_n\}_{n=1,2,\dots} \subset \mathfrak{F}$ belongs to $\mathfrak{F}$.*

(*iii*) *For E and $F \in \mathfrak{F}$ the closed-graph theorem holds.*

These properties are fundamental and essential in the theory of Hilbert spaces or Banach spaces.

Recently, this conjecture is studying on wider spaces E and F by A. Martineau ([8]), L. Schwartz ([11]), W. Słowiskowski ([14]), M. De Wilde-H. G. Garnir ([1]), M. Nakamura ([10]) and many mathematicians.

§ 2. Notations and Some properties of Linear Ranked Spaces.***

For each point x of a ranked space $\mathfrak{B}_\alpha(x)$ denotes the system of all neighborhoods of x with rank α, $\boldsymbol{F}(x)$

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** Throughout this paper we shall use the same terminology that is introduced in [5] and [12].

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the set of all fundamental sequences of x.

***Definition* 1.** ([5], [12]). Let E and F be two ranked spaces with same indicator $\omega \geqslant \omega_0$.

(1) A mapping $f: E \underset{in}{\longrightarrow} F$ is said to be ***r-continuous*** at a point x in E if

$$\{\lim_\alpha x_\alpha\} \ni x \text{ in } E \Rightarrow \{\lim_\alpha f(x_\alpha)\} \ni f(x) \text{ in } F.$$ [1]

(2) A mapping $f: E \underset{in}{\longrightarrow} F$ is said to be ***R-continuous*** at a point x in E if for any fundamental sequence $\{V_\alpha(x)\}$ of x in E there is a fundamental sequence $\{U_\alpha(f(x))\}$ of $f(x)$ in F such that

$$f(V_\alpha(x)) \subsetneq U_\alpha(f(x)) \text{ for all } \alpha,\ 0 \leqslant \alpha < \omega.$$

Every R-continuous mapping is r-continuous, but the converse is not always true.

Definitfon* 2.** ([12], [16]). A linear space E over the real or complex field Φ is called a ***linear ranked space over Φ if this set E is a ranked space with indicator ω_0 and if the following conditions are fulfilled :

(1) The mapping $(x, y) \longrightarrow x+y$ of $E \times E$ into E is R-continuous.[1]

(2) For any $\{u_n(x)\} \in \boldsymbol{F}(x)$ and any sequence $\{\lambda_n\} \subset \Phi$ with limit $\lambda_n = \lambda$, there is a $\{V_n(\lambda x)\} \in \boldsymbol{F}(\lambda x)$ such that $\lambda_n \cdot u_n(x) \subseteq V_n(\lambda x)$ for all n, $0 \leqslant n < \omega_0$.

Hereafter we assume that every linear ranked space $\{E, \mathfrak{B}_n\}$ is homogeneous, namely, every neighborhood of x in E is written in the form of" $x+$a neighborhood of o in E" and $\mathfrak{B}_n(x) \equiv \mathfrak{B}_n + \{x\}$ ($\forall n$, $0 \leqslant n < \omega_0$) holds for each point x in E.

***Definition* 3.** ([12]). (i) A subset H of $\overset{RS}{E}$[2] becomes a ranked space by the induced rank from E, *i. e.*, by the relations $\mathfrak{B}_n(x; H) \underset{def.}{\equiv} \mathfrak{B}_n(x) \cap H$ $(x \in H)$ and

$$\{U_n(x)\} \in \boldsymbol{F}(x; H) \Leftrightarrow U_n(x) \equiv u_n(x) \cap H \quad (\{u_n(x)\} \in \boldsymbol{F}(x)).$$

Such H is called a ***subspace*** of ^{RS}E.

(ii) A subset M of $\overset{LRS}{E}$[3] is called a ***subspace*** (resp. a ***closed subspace***) of $\overset{LRS}{E}$ if M is a vector subspace (resp. an r-closed set[4] and also a vector subspace) of E.

***Proposition* 1.** *Let E be a linear ranked space and H an its vector subspace. Then we have the followings :*

(1) *H is a subspace of E.*

(2) *$cl_r(H)$ is a subspace of H. Hence a closed subspace of E is a linear ranked space.*

Proof. (1). $H \subsetneq E$ is a ranked space by the induced rank from E. For any $\{U_n(x)\}, \{V_n(y)\} \in \boldsymbol{F}(H)$, we have

$$\begin{aligned} U_n(x)+V_n(y) &= u_n(x) \cap H + v_n(y) \cap H \quad (\{u_n(x)\},\ \{v_n(y)\} \in \boldsymbol{F}) \\ &= (u_n(x)+v_n(y)) \cap H \\ &\subseteq w_n(x+y) \cap H \\ &\equiv W_n(x+y) \quad (\exists \{w_n(x+y)\} \in \boldsymbol{F},\ u_n(x)+v_n(y) \subseteq w_n(x+y)\ ;\ \{W_n(x+y)\} \in \boldsymbol{F}(H)). \end{aligned}$$

Moreover for any $\{\lambda_n\} \subset \Phi$ $(\lambda_n \to \lambda)$ and any $\{U_n(x)\} \in \boldsymbol{F}(H)$ there exists $\{V_n(\lambda x)\} \in \boldsymbol{F}(H)$ such that $\lambda_n \cdot U_n(x) \subseteq \lambda_n \cdot (u_n(x) \cap H) \subseteq \lambda_n \cdot u_n(x) \cap H \subseteq \exists v_n(\lambda x) \cap H \equiv V_n(\lambda x)$. Hence $H \subseteq E$ is a linear ranked space.

(2). $\mathrm{cl}_r(H)$ is a ranked space by the induced rank from E. Moreover we have the following facts :

$$\forall x,\ \forall y \in \mathrm{cl}_r(H) \Rightarrow x+y \in \mathrm{cl}_r(H).$$

$$\forall \lambda \in \Phi,\ \forall x \in \mathrm{cl}_r(H) \Rightarrow \lambda x \in \mathrm{cl}_r(H).$$

Thus $\mathrm{cl}_r(H)$ is a vector subspace of E. Hence $\mathrm{cl}_r(H)$ is a linear ranked space. (*Q.E.D.*)

***Proposition* 2.** *Let E and F be two linear ranked spaces over Φ and let a mapping*

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1) See [12].

2) RS=Ranked Space.

3) LRS=Linear Ranked Space.

4) $M(\subseteq \overset{RS}{E})$ is called an r-closed set in $\overset{RS}{E}$ if $\mathrm{cl}_r(M)=M$.

$f: \overset{LRS}{E} \longrightarrow \overset{LRS}{F}$ be *r-continuous. Then the image $f(E)$ of E is a linear ranked space.*

Because $f(E)$ is a ranked space by the induced rank from F and also $f(E)$ is a vector space. Hence $f(E)$ is a linear ranked space by Proposition 1, (1).

***Proposition* 3.** *Let $\{H_\lambda, \overset{(\lambda)}{\mathfrak{B}_n}\}$ $(\lambda \in \Lambda)$ be a system of (vector) subspaces of $\overset{LRS}{E}$. Then we have the followings :*

(1) *$\hat{H} \equiv \bigcap_{\lambda \in \Lambda} H_\lambda$ is a subspace of $\overset{LRS}{E}$.*

(2) *If $\bar{H} \equiv \bigcup_{\lambda \in \Lambda} H_\lambda$ is a vector subspace of E then $\bar{H}$ is a linear ranked space.*

In fact, $\hat{H}$ becomes a linear ranked space by the induced rank of E or by the relation $\hat{\mathfrak{B}_n} \equiv \bigcap_{\lambda \in \Lambda} \mathfrak{B}_n^{(\lambda)}$ $(n \in \boldsymbol{N})$.[5] By $\bar{\mathfrak{B}}_n \equiv \bigcup_{\lambda \in \Lambda} \mathfrak{B}_n^{(\lambda)}$ $(n \in \boldsymbol{N})$ $\bar{H}$ is a linear ranked space.

***Definition* 4.** ([12]) Let $\{E_\lambda, \mathfrak{B}_\alpha^{(\lambda)}(x_\lambda)\}$ $(x_\lambda \in E_\lambda, \lambda \in \Lambda, 0 \leqslant \alpha < \omega)$ be a system of ranked spaces with same indicator $\omega \geqslant \omega_0$. Then the direct product set $E = \prod_{\lambda \in \Lambda} E_\lambda$ is a ranked space by the following relations :

$$\mathfrak{B}_\alpha(x) \equiv \{\prod_{\lambda \in \Lambda} V^{(\lambda)}(x_\lambda) \mid V^{(\lambda)}(x_\lambda) \in \mathfrak{B}_{\alpha_\lambda}^{(\lambda)} (\alpha \leqslant \alpha_\lambda < \omega) \ \& \ \inf(\alpha_\lambda ; \lambda \in \Lambda) = \alpha\} \ (x = (x_\lambda)_{\lambda \in \Lambda} \in E, \ 0 \leqslant \alpha < \omega)$$

and

$$\{V_\alpha(x)\} \in \boldsymbol{F}(x ; E) \overset{def.}{\Leftrightarrow} V_\alpha(x) \equiv (V_\alpha^{(\lambda)}(x_\lambda))_{\lambda \in \Lambda} \ \& \ \{V_\alpha^{(\lambda)}(x_\lambda)\} \in \boldsymbol{F}(x_\lambda ; E_\lambda) \ (\lambda \in \Lambda).$$

Such space E is called the ***direct product ranked space*** of ranked spaces $\{E_\lambda\}$ $(\lambda \in \Lambda)$.

***Proposition* 4.** *The direct product space $E = \prod_{n=1}^{\infty} E_n$ of linear ranked spaces $\{E_n\}$ $(n \in \boldsymbol{N})$ over Φ is a linear ranked space over Φ.*

Proof. Since every E_n is a vector space, $E = \prod_{n=1}^{\infty} E_n$ is a vector space. And E is also a ranked space by the product rank.

1) For $\forall \{u_m^{(n)}(0)\}_m$, $\forall \{v_m^{(n)}(0)\}_m \in \boldsymbol{F}(0 ; E_n)$ there exists $\{w_m^{(n)}(0)\}_m \in \boldsymbol{F}(0 ; E_n)$ such that

$$u_m^{(n)}(0) + v_m^{(n)}(0) \subseteq w_m^{(n)}(0) \text{ for } \forall m, \forall n \in \boldsymbol{N}.$$

Hence we have

$$\prod_{n=1}^{\infty} (u_m^{(n)}(0) + v_m^{(n)}(0)) \subseteq \prod_{n=1}^{\infty} w_m^{(n)}(0) \quad (\forall m \in \boldsymbol{N}).$$

Thus the addition in E is R-continuous.

2) For $\forall \{\lambda_m^{(n)}\}_m \subset \Phi (\lambda_m^{(n)} \to 0)$ and $\forall \{u_m^{(n)}(0)\}_m \in \boldsymbol{F}(0 ; E_n)$ there exists $\{v_m^{(n)}(0)\}_m \in \boldsymbol{F}(0 ; E_n)$ such that

$$\lambda_m^{(n)} \cdot u_m^{(n)}(0) \subseteq v_m^{(n)}(0) \ (\forall m, \forall n \in \boldsymbol{N}).$$

Hence we have

$$\prod_{n=1}^{\infty} (\lambda_m^{(n)} \cdot u_m^{(n)}(0)) \subseteq \prod_{n=1}^{\infty} v_m^{(n)}(0) \ (\forall m \in \boldsymbol{N}).$$

Therefore $E = \prod_{n=1}^{\infty} E_n$ is a linear ranked space over Φ. (*Q.E.D.*)

Definition* 5.** In the direct product $E = \prod_k E_k$ of vector spaces E_k $(k \in \boldsymbol{N})$ over Φ if a vector space $\tilde{E}$ $(\subseteq E)$ is generated by $\bigcup_k E_k$ then $\tilde{E}$ is called the ***direct sum of vector spaces $E_k (k \in \boldsymbol{N})$ and it is denoted by $\sum_k \oplus E_k$. $\sum_k \oplus E_k$ is called the ***direct sum*** of $\overset{LRS}{E_k}$ $(k \in \boldsymbol{N})$ if every E_k $(k \in \boldsymbol{N})$ is a linear ranked space over Φ.

***Proposition* 5.** *If E_k $(k \in \boldsymbol{N})$ is a system of linear ranked spaces over Φ then $\sum_k \oplus E$ is a linear ranked space over Φ.*

Proof. $\tilde{E}(\subseteq E)$ is a vector space and a subspace of $\overset{RS}{E}$.

Let

$$E \ni \forall \tilde{x} \equiv (x_k)_{k \in \boldsymbol{N}} \text{ and } \tilde{V}_n(\tilde{x}) \equiv (V_n^{(k)}(x_k))_{k \in \boldsymbol{N}} \cap \tilde{E} \ (\{(V_n^{(k)}(x_k)\}_{n \in \boldsymbol{N}} \in \boldsymbol{F}(x_k ; E_k), \ \{\tilde{V}_n(\tilde{x})\}_{n \in \boldsymbol{N}} \in \boldsymbol{F}(\tilde{x} ; \tilde{E})).$$

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5) $\boldsymbol{N}$ denotes the set of all natural numbers.

For $\forall\{\tilde{u}_n(\tilde{x})\}$, $\forall\{\tilde{v}_n(\tilde{y})\}\in \boldsymbol{F}(\tilde{E})$ we have

$$\tilde{u}_n(\tilde{x})+\tilde{v}_n(\tilde{y})=(u_n{}^{(k)}(x_k))_{k\in N}\cap\tilde{E}+(v_n{}^{(k)}(y_k))_{k\in N}\cap\tilde{E}$$

$$=(u_n{}^{(k)}(x_k)+v_n{}^{(k)}(y_k))_{k\in N}\cap\tilde{E}$$

$$\subseteq(\exists w_n{}^{(k)}(x_k+y_k))_{k\in N}\cap\tilde{E}\ \ (\{w_n{}^{(k)}(x_k+y_k)\}\in\boldsymbol{F}(x_k+y_k\,;\,E_k)).$$

Morover for any $\{\lambda_n\}\subset\Phi$ $(\lambda_n\to\lambda)$ and any $\{\tilde{u}_n(\tilde{x})\}\in\boldsymbol{F}(\tilde{x}\,;\,\tilde{E})$ we have

$$\lambda_n\cdot\tilde{u}_n(\tilde{x})\subseteq(\lambda_n\cdot u_n{}^{(k)}(x_k))_{k\in N}\cap\tilde{E}\subseteq(\exists v_n{}^{(k)}(\lambda\cdot x_k))_{k\in N}\cap\tilde{E}\equiv\tilde{v}_n(\lambda\tilde{x})\in\boldsymbol{F}(\tilde{E})\ \ (\{v_n{}^{(k)}(\lambda x_k)\}_{n\in N}\in\boldsymbol{F}(E_k)).$$

Hence $\tilde{E}$ is a linear ranked space over Φ. (*Q.E.D.*)

Definition* 6.** Let $\dot{E}\equiv E/M$ be the quotient vector space over Φ by a vector subspace M of a given vector space E over Φ. $\dot{E}$ is called the ***quotient linear ranked space over Φ by M if E is a linear ranked space over Φ.

***Proposition* 6.** *Let E be a linear ranked space over Φ and M a vector subspace of E. Then the quotient vector space $\dot{E}=E/M$ is a linear ranked space over Φ.*

Proof. By the following relations :

$$\dot{\mathfrak{V}}_\alpha(\dot{x})^{6)}\ni\dot{V}(\dot{x})\ (\dot{x}\in\dot{E})\overset{def.}{\Leftrightarrow}\dot{V}(\dot{x})\equiv\{a+M\mid a\in V(x)\}\ (x\in\dot{x}\text{ and fix }x\in E)$$

and

$$\dot{\mathfrak{V}}_\alpha(\dot{x})\equiv\text{the set of all neighborhoods of }\dot{x}\text{ with rank }\alpha\ (0\leqslant\alpha<\omega_0)$$

$\dot{E}$ is a linear ranked space over Φ. (*Q.E.D.*)

Definition* 7.** Let $\{E_n\}_{n=1,2,\ldots}$ be a sequence of ranked spaces with same indicator $\omega\geqslant\omega_0$. Suppose that it satisfies the following condition: $\overset{RS}{E_n}$ is a closed subspaces of $\overset{RS}{E_{n+1}}$ for each n, $n\in\boldsymbol{N}$. If $E=\bigcup_{n=1}^{\infty}E_n$ is a ranked space then such ranked space E is called the ***inductive limit of ranked spaces E_n $(n=1,2,3,\ldots)$ and it is denoted by $E=\varinjlim E_n$.

***Proposition* 7.** *The inductive limit $E=\varinjlim E_n$ of linear ranked spaces over Φ $E_1\subset E_2\subset\cdots\subset E_n\subset\cdots$ is a linear ranked space over Φ.*

Proof. We set $E_n\equiv\{E_n,\ \mathfrak{V}_m{}^{(n)}\}$ $(n\in\boldsymbol{N})$ and $U^{(n)}(0)\in\mathfrak{V}_l{}^{(n)}(0)$ $(\exists l,\ 0\leqslant l<\omega_0)$. Since E_n is a closed subspace of E_{n+1} there exists $U^{(n+1)}(0)\in\mathfrak{V}_l{}^{(n+1)}(0)$ such that $U^{(n)}(0)=E_n\cap U^{(n+1)}(0)$. Inductively we have

$$\exists U^{(j+1)}(0)\in\mathfrak{V}_l{}^{(j+1)}(0)\text{ such that }U^{(j)}(0)=E_j\cap U^{(j+1)}(0)\text{ for all }j\geqslant n.$$

By the relation

$$U(0)\equiv\bigcap_{j\geqslant n}U^{(j)}(0)\ \&\ \mathfrak{V}_l(0)\ni U(0)\overset{def.}{\Leftrightarrow}\mathfrak{V}_l{}^{(j)}(0)\ni U^{(j)}(0)\quad(\forall j\geqslant n)$$

and

$$\{U_m(0)\}\in\boldsymbol{F}(E)\ (U_m(0)\equiv\bigcap_{j\geqslant n}U_m{}^{(j)}(0))\overset{def.}{\Leftrightarrow}\{U_m{}^{(j)}(0)\}\in\boldsymbol{F}(E_j)\ (\forall j\geqslant n),$$

we can show that E becomes a ranked space with indicator ω_0 :

(A) $U(0)\ni 0$ is clear.

(B) For any $U(0)$ and $V(0)$ we have

$$U(0)\cap V(0)=(\bigcap_{j=n}^{\infty}U^{(j)}(0))\cap(\bigcap_{j=m}^{\infty}V^{(j)}(0))\quad(n\leqslant m)$$

$$\supseteq\bigcap_{j=n}^{\infty}(U^{(j)}(0)\cap V^{(j)}(0))$$

$$\supseteq\bigcap_{j=n}^{\infty}W^{(j)}(0)\equiv\exists W(0)\quad(U^{(j)}(0)\cap V^{(j)}(0)\supseteq W^{(j)}(0),\ \exists W^{(j)}(0)\in\mathfrak{V}(E_j)).$$

(a) Since each E_n satisfies the axiom (a) there exists m, $l\leqslant m<\omega_0$, such that

$$U^{(j)}(0)\supseteq\exists V^{(j)}(0)\ \&\ V^{(j)}(0)\in\mathfrak{V}_m{}^{(j)}(0).$$

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6) $\dot{\mathfrak{V}}(\dot{x})\equiv\bigcup_{\alpha=0}^{\omega_0}\dot{\mathfrak{V}}_\alpha(\dot{x})$.

Hence if we set $V(0)=\bigcap_{j\geqslant n} V^{(j)}(0)$ then

$$U(0)\supseteqq V(0)\ \&\ V(0)\in\mathfrak{B}_m(0)\quad (l\leqslant m<\omega_0).$$

Thus E satisfies the axiom (a).

Hence $E=\bigcup_{n=1}^{\infty} E$ is a ranked space with indicator ω_0. Thus $\underrightarrow{lim}\ E_n$ is a ranked space with indicator ω_0.

Since every $E_n\ (n\in N)$ is a vector space, $\bigcup_{n=1}^{\infty} E_n$ is a vector space.

For any $\{U_n(0)\},\ \{V_n(0)\}\in \boldsymbol{F}(E)$ we have, for every $n\in N$,

$$U_n(0)+V_n(0)=\bigcap_{j\geqslant k} U_n{}^{(j)}(0)+\bigcap_{j\geqslant k'} V_n{}^{(j)}(0)\quad (k\geqslant k')$$
$$\subseteqq \bigcap_{j\geqslant k}(U_n{}^{(j)}(0)+V_n{}^{(j)}(0))$$
$$\subseteqq \bigcap_{j\geqslant k} W_n{}^{(j)}(0)\equiv \exists W_n(0)\ \ (\exists\{W_n{}^{(j)}(0)\}\in \boldsymbol{F}(E_j),\ U_n{}^{(j)}(0)+V_n{}^{(j)}(0)\subset W_n{}^{(j)}(0))$$

and $\{W_n(0)\}\in \boldsymbol{F}(E)$.

Moreover for any $\{\lambda_n\}\subset\Phi(\lambda_n\to 0)$ and any $\{U_n(0)\}\in \boldsymbol{F}(E)$ we have, for every $n\in N$,

$$\lambda_n\cdot U_n(0)=\lambda_n\cdot(\bigcap_{j\geqslant k} U_n{}^{(j)}(0))=\bigcap_{j\geqslant k}\lambda_n\cdot U_n{}^{(j)}(0)$$
$$\subseteqq\bigcap_{j\geqslant k} W_n{}^{(j)}(0)\equiv W_n(0)\ \ (\exists\{W_n{}^{(j)}(0)\}\in \boldsymbol{F}(E_j),\ \lambda_n\cdot U_n{}^{(j)}(0)\subseteqq W_n{}^{(j)}(0))$$

and $\{W_n(0)\}\in \boldsymbol{F}(E)$.

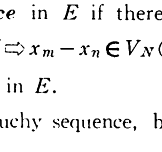
Hence $\underrightarrow{lim}\ E_n$ is a linear ranked space over Φ. (*Q.E.D.*)

Definition* 8.** Let Λ be an ordered set and $\{E_\lambda,\ \mathfrak{B}_\alpha{}^{(\lambda)}\}\ (\lambda\in\Lambda)$ a system of ranked spaces with same indicator. If an R-continuous mapping $\varphi_{\lambda,\mu}:\ E_\mu\longrightarrow E_\lambda$ is defined for every $\lambda,\ \mu\in\Lambda\ (\lambda<\mu)$ and if $\lambda<\mu<\nu\Rightarrow\varphi_{\lambda,\mu}\cdot\varphi_{\mu,\nu}=\varphi_{\lambda,\nu}$ holds, then $(\{E_\lambda,\ \mathfrak{B}_\alpha{}^{(\lambda)}\})$ is called the ***projective system of ranked spaces $\{E_\lambda,\ \mathfrak{B}_\alpha{}^{(\lambda)}\}\ (\lambda\in\Lambda)$. Let $E^*=\overset{RS}{\prod_{\lambda\in\Lambda}} E_\lambda$ be the direct product ranked space of $E_\lambda\ (\lambda\in\Lambda)$ and set

$$E=\{x\in E^*\mid x=(x_\lambda)_{\lambda\in\Lambda}\ \text{such that}\ x_\lambda=\varphi_{\lambda,\mu}(x_\mu)\ \text{if}\ \lambda<\mu\}.$$

Then $E(\subseteqq E^*)$ becomes a subspace of $\overset{RS}{E^*}$ by the induced rank. Such ranked space $\{E,\ \mathfrak{B}_\alpha\}$ is called the ***projective limit*** of $(\{E_\lambda,\ \mathfrak{B}_\alpha{}^{(\lambda)}\})$ and it is denoted by $(E,\ \varphi_\lambda)=\underleftarrow{lim}(E_\lambda,\ \varphi_{\lambda,\mu})$ or $E=\underleftarrow{lim}\ E_\lambda$. Since $pr_\lambda:\ E^*\longrightarrow E_\lambda$ is an R-continuous mapping, a mapping

$$\varphi_\lambda=pr_\lambda\mid E:\ E\longrightarrow E_\lambda$$



is also R-continuous. Such $\varphi_\lambda:\ E\longrightarrow E_\lambda$ is called the ***canonical mapping***. From the definition we have $\lambda<\mu\Rightarrow\varphi_{\lambda,\mu}\cdot\varphi_\mu=\varphi_\lambda$.

Definition* 9.** (i) A sequence $\{x_n\}\subset\overset{LRS}{E}$ is called an ***r-Cauchy sequence in E if there exists a fundamental sequence $\{V_n(0)\}\in \boldsymbol{F}(0)$ and if, for each $N\in N$, we have $m,n\geqslant N\Rightarrow x_m-x_n\in V_N(0)$.

(ii) $\overset{LRS}{E}$ is said to be ***r-complete*** if every r-Cauchy sequence in E is r-convergent in E.

Remark. In a linear ranked space, every r-convergent sequence is an r-Cauchy sequence, but the converse is not always true.

§ 3. Generalized Closed Graph Theorem.****

Let E and F be two homogeneous linear ranked spaces over Φ. In this section we suppose that every $V\in\mathfrak{B}(0\,;\,F)\equiv\bigcup_{n=0}^{\omega_1}\mathfrak{B}_n(0\,;\,F)$ is r-closed and convex.[7]

**** 1973年4月，日本数学会年会（於立教大学）にて一部講演.

7) We use only the fact $\frac{1}{2}V+\frac{1}{2}V\subseteqq V$ for $\forall V\in\mathfrak{B}(0\,;\,F)$.

Condition (☆). $\lambda V\in\mathfrak{B}_{[\frac{r}{\lambda}]}(0)$ holds for any $\lambda\in\Phi$ and any $V\in\mathfrak{B}_r(0)$ $(0\leqslant r<\omega_0)$.

***Theorem* 1.** (Closed Graph Theorem). *A linear mapping f of a linear ranked space E into an r-complete linear ranked space F satisfying the condition (☆)[8] is r-continuous and T-continuous provided*

(*i*) *the graph of f is r-closed in $\overset{LRS}{E\times F}$,*

(*ii*) *for each neighborhood $V(0)$ of in F the r-closure $cl_r(f^{-1}(V(0)))$ is a neighborhood of 0 in E,[9] and*

(*iii*) *(a^*) for each neighborhood $v(0)$ of 0 in E and each $\{v_n(0)\}\in\boldsymbol{F}(0\,;E)$ there exists an index $m\in\boldsymbol{N}$ such that $v_m(0)\subseteq v(0)$.[10]*

Proof. (I). We begin by showing that for each $V(0)\in\mathfrak{B}(0)$ we have $\mathrm{cl}_r(f^{-1}(\frac{1}{2}V(0)))\subseteq f^{-1}(V(0))$. For any $x\in\mathrm{cl}_r(f^{-1}(\frac{1}{2}V(0)))$ there exists an r-Cauchy sequence $\{y_n\}\subset F$ such that

$$y_0=0 \text{ and } x\in\mathrm{cl}_r(f^{-1}(y_n+U_n)) \text{ for all } n,\ 0\leqslant n<\omega_0\ (U_0\equiv\frac{1}{2}V(0),\ U_n\equiv\frac{1}{2^n}U_0).$$

We will show this by mathematical induction. For $n=0$ this is clear. Suppose that y_n is defined for each $n\geqslant 0$. Then we have

$$x\in\mathrm{cl}_r(f^{-1}(y_n+U_n))\subseteq f^{-1}(y_n+U_n)+\mathrm{cl}_r(f^{-1}(\frac{1}{2}U_n)).$$

Hence there exists a point x' in E such that

$$f(x')\in y_n+U_n \ \&\ x\in x'+\mathrm{cl}_r(f^{-1}(\frac{1}{2}U)).$$

Now,

$$x'+\mathrm{cl}_r(f^{-1}(\frac{1}{2}U_n))=\mathrm{cl}_r(x'+f^{-1}(\frac{1}{2}U_n))\subseteq\mathrm{cl}_r(f^{-1}(f(x'))+f^{-1}(\frac{1}{2}U_n))\subseteq\mathrm{cl}_r(f^{-1}(f(x')+\frac{1}{2}U_n)).$$

Set $y_{n+1}=f(x')$. Then we have

$$y_{n+1}\in y_n+U_n \text{ and } x\in\mathrm{cl}_r(f^{-1}(y_{n+1}+U_{n+1})) \text{ for all } n,\quad 0\leqslant n<\omega_0.$$

Now set $W_n=y_{n+1}+U_n$ for all n, $0\leqslant n<\omega_0$. Then we have

$$\begin{aligned} W_{n+1}&\equiv y_{n+2}+U_{n+1}\subseteq(y_{n+1}+U_{n+1})+U_{n+1}\ (\text{By } y_{n+2}\in y_{n+1}+U_{n+1})\\ &=y_{n+1}+(\frac{1}{2}U_n+\frac{1}{2}U_n)\\ &\subseteq y_{n+1}+U_n\equiv W_n\ (\text{Because } U_n \text{ is convex}).^{7)} \end{aligned}$$

Thus for all n, $0\leqslant n<\omega_0$, we have

$$W_n\supseteq W_{n+1}\ \&\ W_n\ni y_{n+1},\ y_{n+2},\ y_{n+3},\ldots\ldots.$$

From $W_n\supseteq W_{n+1}$ $(\forall n,\ 0\leqslant n<\omega_0)$ we get

$$y_i-y_{n+1}\in U_n-U_{n+1}\ (\forall i>n) \text{ and } y_{n+1}-y_j\in U_{n+1}-U_n\ (\forall j>n).$$

Thus there exists a $\{U_n'\}\in\boldsymbol{F}(0)$ such that

$$y_i-y_j=(y_i-y_{n+1})+(y_{n+1}-y_j)\in(U_n-U_{n+1})+(U_{n+1}-U_n)\subseteq U_n' \text{ for } \forall i,\ \forall j>n.$$

Therefore $\{y_n\}$ is an r-Cauchy sequence in $\boldsymbol{F}$. Thus we get $\{\lim_n y_n\}\ni\exists y$ in $\boldsymbol{F}$. (Because $\boldsymbol{F}$ is r-complete.) Therefore, for all $n\geqslant 0$, we get the following fact :

$$y\in\bigcap_{\forall n}\mathrm{cl}_r(W_n)\subseteq\mathrm{cl}_r(W_0)=\mathrm{cl}_r(y_1+U_0)\subseteq\mathrm{cl}_r(U_0+U_0)\ (\text{As } y_1\in y_0+U_0\subseteq U_0)$$

$$=\mathrm{cl}_r(\frac{1}{2}V(0)+\frac{1}{2}V(0))\subseteq\mathrm{cl}_r(V(0))=V(0)\quad(\text{Since } V(0) \text{ is } r\text{-closed}),$$

i. e., $y\in V(0)$.

Now, as $y_n\xrightarrow{r}y$ in F (as $n\to\infty$) there exist $\{V_n'(y)\}$, $\{W_n(y)\}\in\boldsymbol{F}(y)$ such that for all n, $0\leqslant n<\omega_0$,

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8) Such $\overset{LRS}{\boldsymbol{F}}$ has a structure as an extension of every locally convex, pseudo-metrizable and semi-normable *TVS*. In general F is not pseudo-metrizable.

9) If we define a topology as a *TVS* in E by a system of neighborhoods with ranks, if E is complete in the sense of ranked spaces ([5]) and if F is a pseudo-metrizable *TVS*, then E is 2nd Category ([5]). Such E satisfies above condition (ii) (Kelley-Namioka [4]). Therefore Theorem 1 is an extension of the theorem of Kelley-Namioka [4].

10) First countability axiom $\underset{\Leftarrow}{\Rightarrow}$ (a^*).

$$W_n=y_{n+1}+U_n\subseteq \exists V_n'(y)+U_n\subseteq W_n(y) \quad (V_n'(y)\ni y_{n+1} \text{ for each } n).$$

Hence, for all n, $0\leqslant n<\omega_0$, we have

$$y_{n+1}+U_{n+1}=y_{n+1}+\frac{1}{2}U_n\subseteq y_{n+1}+(\frac{1}{2}U_n+\frac{1}{2}U_n)\subseteq y_{n+1}+U_n=W_n\subseteq W_n(y).$$

Thus we get

$$x\in \mathrm{cl}_r(f^{-1}(y_{n+1}+U_{n+1}))\subseteq \mathrm{cl}_r(f^{-1}(W_n(y))) \text{ for all } n,\ 0\leqslant n<\omega_0.$$

Therefore we have the followings :

$$\exists x_k^{(n)}\in f^{-1}(W_n(y)),\ i.\ e.,\quad f(\exists x_k^{(n)})\in W_n(y) \ (\text{for } k=0,1,2,\dots \text{ and each } n,\ 0\leqslant n<\omega_0,)$$

such that $x_k^{(n)}\xrightarrow{r} x$ (as $k\to\infty$).

Thus there are $\{u_k^{(n)}(x)\}_{k=0,1,2,\dots}\in \boldsymbol{F}(x)$ $(u_k^{(n)}(x)\ni x_k^{(n)})$ for each n, $0\leqslant n<\omega_0$.

Therefore we get

$$(u_k^{(n)}(x)\times W_n(y))\cap(\text{the graph of } f)\neq\phi\ (k=0,1,2,\dots;\ n=0,1,2,\dots).$$

Now, by condition (a^*) and the homogenity for E, we can pick out a fundamental sequence of (x, y) in $\overset{LRS}{E\times F}$

$$\{u_{k(n)}^{(n)}(x)\times W_n(y)\}_{n=0,1,2,\dots} \quad (k(n)<k(n+1))$$

from the family $\{u_k^{(n)}(x)\times W_n(y)\mid k=0,1,2,\dots;\ n=0,1,2,\dots\}$.

Therefore we have

$$(u_{k(n)}^{(n)}(x)\times W_n(y))\cap(\text{the graph of } f)\neq\phi \text{ for all } n,\ 0\leqslant n<\omega_0.$$

By assumption, the graph of f is r–closed set in $\overset{LRS}{E\times F}$.

Thus we get $\quad f(x)=y\in V(0)$

and $\quad x\in f^{-1}(y)\subseteq f^{-1}(V(0))$.

Therefore $\quad \mathrm{cl}_r(f^{-1}(\frac{1}{2}V(0))\subseteq f^{-1}(V(0))$ is proved.

(II). Next, we will show that f is r–continuous on E. Let $\forall x\in E$ and let $\{\lim_n x_n\}\ni x$ in E. From $\{\lim_n x_n\}\ni x$ in E, there exists $\{u_n(x)\}\in \boldsymbol{F}(x)$ such that $u_n(x)\ni x_n$ for all n, $0\leqslant n<\omega_0$. For any $\{V_n(0)\}\in \boldsymbol{F}(0;F)$, by (I), we have $\mathrm{cl}_r(f^{-1}(\frac{1}{2}V_N(0))\subseteq f^{-1}(V_N(0))$. Since $V'_N(0)\equiv\frac{1}{2}V_N(0)$ is a neighborhood of 0 in F, using homogenity for E, we get that $\mathrm{cl}_r(f^{-1}(V'_N(0)))+x$ is a neighborhood of x in E. Now, by homogenity for F, since $V_N(f(x))\equiv V_N(0)+f(x)$ is a neighborhood of $f(x)$ in F we have $\{V_N(f(x))\}\in \boldsymbol{F}(F)$ and

$$\mathrm{cl}_r(f^{-1}(V'_N(0)))+x\subseteq f^{-1}(V_N(0))+x\subseteq f^{-1}(V_N(0)+f(x))=f^{-1}(V_N(f(x)).$$

By (a^*) there exists an $m(N)\in \boldsymbol{N}$ and we have

$$x_{m(N)}\in u_{m(N)}(x)\subseteq \mathrm{cl}_r(f^{-1}(V'_N(0)))+x\subseteq f^{-1}(V_N(f(x))).$$

Hence $\quad f(x_j)\in V_N(f(x))$ for all $j\geqslant m(N)$.

Now, as $f(x_{m(N)})\in V_N(f(x))\supseteq V_{N+1}(f(x))\ni f(x_{m(N+1)})$ we set

$$V_N(f(x))\equiv\widetilde{V}_{m(N)}(f(x))\equiv\widetilde{V}_{m(N)+1}(f(x))\equiv\widetilde{V}_{m(N)+2}(f(x))\equiv\cdots\cdots$$

$$\cdots\cdots\equiv\widetilde{V}_{m(N)+l(N)-1}(f(x))\supseteq\widetilde{V}_{m(N)+l(N)}(f(x))\equiv V_{N+1}(f(x))$$

and $m(N+1)=m(N)+l(N)$.

Then we have

$$f(x_{m(N)+p})\in\widetilde{V}_{m(N)+p}(f(x)) \text{ for all } p,\ 0\leqslant p\leqslant l(N).$$

Hence there exists $\{\widetilde{V}_n(f(x))\}\in \boldsymbol{F}(f(x);F)$ such that $\widetilde{V}_n(f(x))\ni f(x_n)$ for all n, $0\leqslant n<\omega_0$.

Thus we have

$$\{\lim_n x_n\}\ni x \text{ in } E\Rightarrow\{\lim_n f(x_n)\}\ni f(x) \text{ in } F$$

and f is r–continuous on E.

(III). Finally, we will show that f is T–continuous on E. Let $V(0)$ be an arbitrary neighborhood of 0 in F. By (I) and (a), for a $V'(0)\in\mathfrak{B}(0)$ in F we have

$$\mathrm{cl}_r(f^{-1}(\frac{1}{2}V'(0)))\subseteq f^{-1}(V'(0))\subseteq f^{-1}(V(0)).$$

By assumption, $\mathrm{cl}_r(f^{-1}(\frac{1}{2}V'(0)))$ is a neighborhood of 0 in E. Hence, by axiom (a), there exists $v(0)\in\mathfrak{V}(0)$

in E such that $\quad v(0)\subseteq \mathrm{cl}_r(f^{-1}(\frac{1}{2}V(0)))\subseteq f^{-1}(V(0))$.

Thus we have $\quad f(v(0))\subseteq V(0)$.

Hence, $\quad f(v(0)+x)=f(v(0))+f(x)\subseteq V(0)+f(x)$ for each $x\in E$.

Therefore f is T-continuous on E.

This completes the proof of the generalized closed-graph theorem. ($Q.E.D.$)

Let $\{F_n, \mathfrak{V}_l^{(n)}\}_{n=1,2,3,\dots}$ be an increasing sequence of linear ranked spaces such that every F_n satisfies the condition[7] (☆) and every $U\in\mathfrak{V}^{(n)}(0)$ is r-closed and convex in *each* F_n. Then we have next theorem as a generalization of Theorem 1.

***Theorem* 2.** *A linear mapping f of a linear ranked space E into an r-complete $F^*=\underrightarrow{\lim}\, F_n$ is r-continuous and T-continuous provided three conditions (i)—(iii) of Theorem 1 for E and F^*.*

Proof. For each $U(0)\in\mathfrak{V}_l(0)$ we have $U(0)=\bigcap_{j\geqq n}U^{(j)}(0)$ and $U^{(j)}(0)\in\mathfrak{V}_l^{(j)}(0)$ $(j\geqq n)$. Moreover for each $\{U_k(0)\}_{k=0,1,2,\dots}\in\boldsymbol{F}(0)$ we have $U_k(0)=\bigcap_{j\geqq n}U_k^{(j)}(0)$ $(k=0,1,2,\dots)$ and $\{U_k^{(j)}(0)\}\in\boldsymbol{F}(0\,;\,F_j)$. Hence F^* satisfies the condition (☆). Thus we get this theorem by the method as is taken in the proof for Theorem 1. ($Q.E.D.$)

§ 4. Examples of Linear Ranked Spaces.

***Proposition* 8.** *The following spaces are linear ranked spaces :*

(a) *Pseudo-metrizable TVS*

(b) *(Semi-) Normed space*

(c) *Countably normed space*

(d) *Dual (or union) space of countably normed space*

(e) D, D', $\hat{D}$, $\hat{D}'$

(f) *Nuclear space and its dual*

(g) *Space of (E.R.) integrable functions*

(h) *Space with a system of semi-norms*

(i) *LCTVS and its conjugate*

(j) *Inductive limit[11] of LCTV spaces*

(k) *Fréchet space, Banach space, LF-space, Bornological space, Barrelled space, Montel space, L. Hörmander's space $\mathcal{F}(\Omega)$,..., etc*[11]

(l) *Inductive limit[11] for each space in (k).*

In fact, we get this proposition from the result in [12].[12] These spaces are included in the space F.[13] Moreover (a*) and (☆) are held for these spaces.[13]

Definition* 10.** ([1]). A ***netted space E is a $LCTVS$ where exists a net of subsets of E

$$e_{n_1\cdots n_k}\ ;\ k,\ n_1,\dots,\ n_k=1,2,\dots,$$

such that

(i) $e_{n_1\cdots n_k}$ are absolutely convex,

(ii) $E=\bigcup_{n_1}e_{n_1},\ e_{n_1}=\bigcup_{n_2}e_{n_1n_2},\dots,\ e_{n_1\cdots n_k}=\bigcup_{n_{k+1}}e_{n_1\cdots n_{k+1}},\dots,$

(iii) to every sequence n_k corresponds a sequence $\lambda_k>0$ such that any series

$$\sum_{k=1}^{\infty}\lambda_k f_k,\ f_k\in e_{n_1\cdots n_k},$$

converges in E,

(iv) a series of the preceding form is such that

$$\sum_{k=1}^{\infty}\lambda_k f_k\in e_{n_1},\ \sum_{k=2}^{\infty}\lambda_k f_k\in e_{n_1n_2},\dots,\sum_{k=p}^{\infty}\lambda_k f_k\in e_{n_1\cdots n_p},\dots.$$

11) These are locally convex (cf. [20], p. 28).

12) See On Generalized Continuous Groups II, pp. 63–64. And replace $<$ by $\leqq$. Then, for $\forall\, v(0)\in\mathfrak{V}$, we have $\mathrm{cl}_r(v(0))=\overline{v(0)}=v(0)$.

13) For (a) see Kelley–Namioka [4], p. 97.

***Proposition* 9.** *Every netted space has a struature as a linear ranked space. Thus every netted space is included in the space F of Theorem* 1.

Proof. Let E be a netted space defined by M. De Wilde.

We set $v(n, k\,;\,0) \equiv \overline{e_{n_0 \cdots n_k}(0)}$,

$$e_{n_0 n_1 \cdots n_k}(0) \equiv \beta_{n_0} \cap e_{n_1 \cdots n_k}(0) \text{ (See H. G. Garnir [1], p. 368)}$$

$$(\beta_{n_0} \equiv \{f \mid p(f) \leq \frac{\rho}{n_0}\} (\rho > 0),\ c_{n_1 \cdots n_k}(0) \equiv e_{n_1 \cdots n_k} \cup \{0\}),$$

$$\max(n_0, \ldots, n_k) = n \quad \text{for any } n_0, \ldots, n_k,\ k \in N,$$

$$\mathfrak{V}_n(0) \equiv \{v(n, k\,;\,0) \mid k \in N\},\ \mathfrak{V}_0(0) = \{E\},\ \mathfrak{V}_n(f) \equiv \mathfrak{V}_n(0) + \{f\}\ (\forall f \in E)$$

and $$\lambda \cdot v(n, k\,;\,0) \in \mathfrak{V}_{[\frac{n}{\lambda}]}(0)\ (n = 0, 1, 2, \ldots) \text{ for any } \lambda \in \Phi.$$

Then we have

$$\overline{e_{n_0 \cdots n_k}(0)} \ni 0 \text{ and } \overline{e_{n_0 \cdots n_k}(0) \cap e_{m_0 \cdots m_l}(0)} \supseteq \overline{e_{n_0 \cdots n_k}(0) \cap e_{m_0 \cdots m_l}(0)} \in \mathfrak{V}_{max(n_0, \ldots, n_k, m_0, \ldots, m_l)}(0).$$

Axiom (a) is clear. Moreover we have

$$\overline{e_{n_0 \cdots n_k}(0)} + \overline{e_{m_0 \cdots m_l}(0)} \subseteq \overline{e_{n_0 \cdots n_k}(0) + e_{m_0 \cdots m_l}(0)} \in \mathfrak{V}(0) \text{ and } \lambda_m \cdot \overline{e_{n_0 \cdots n_k}(0)} \in \mathfrak{V}(0) \text{ for}$$

$\forall \{\lambda_m\}$ $(\lambda_m \longrightarrow 0)$ in Φ.

Moreover we have

$$\mathrm{cl}_r\left(\overline{e_{n_0 \cdots n_k}(0)}\right) = \overline{\overline{e_{n_0 \cdots n_k}(0)}} = \overline{e_{n_0 \cdots n_k}(0)}.$$

Thus E is a linear ranked space. (*Q.E.D.*)

Definition* 11.** ([10]). (i) A filter Φ in a space E is said to be an ***S-filter if Φ has a countable basis $\{S_n\}$ such that $\bigcap_{n=1}^{\infty} S_n = \phi$.

(ii) A filter Φ in a linear space E is said to be an ***LS-filter*** if Φ is generated by the complements of all the finite union of linear subspaces E_n $(n=1,2,\ldots)$ of E such that $E = \bigcup_{n=1}^{\infty} E_n$.

(iii) A subset A of a linear space E is said to be ***linearly open*** if for any straight line L in E, $E \cap A$ is open in L by its usual topology.

(iv) A filter Φ in a linear space E is called a ***P-filter*** if for every x in E there exists a linearly open set A such that either A is disjoint[14] from Φ or Φ_A, considered as a filter in E, is finer than an ***LS-filter.***

***Proposition* 10.** (1) *If a space E has a sequence of S-filters* Φ_i $(i=1,2,\ldots)$ *then E has a structure as a ranked space with indicator* ω_0.

(2) *If a linear space E has a sequence of LS-filters* Φ_i $(i=1,2,\ldots)$ *then E has a structure as a linear ranked space.*

(3) *If a linear space E has a sequence of P-filters* Φ_i $(i=1,2,\ldots)$ *then E has a structure as a linear ranked space.*

Proof. (1). Suppose that each S-filter Φ_i is generated by subsets $S_k^{(i)}$ $(k=1,2,\ldots)$ such that $\bigcap_{k=1}^{\infty} S_k^{(i)} = \phi$. We set, for each point $p \in E$,

$$v(p\,;\,n) \equiv \bigcap_{i=1}^{\infty} \bigcap_{k=1}^{n} S_k(i) \cup \{p\},\ \mathfrak{V}_n(p) \equiv \{v(p\,;\,n)\}\ (n \in N),\ \mathfrak{V}_0(p) \equiv \{E\}.$$

Then $\{E, \mathfrak{V}_n\}$ fulfills the axioms (A), (B), (a) and (b). Hence E is a ranked space with indicator ω_0.

(2). Suppose that each LS-filter Φ_i is generated by the complements of all the finite union of linear subspaces $E_k^{(i)}$ $(k=1,2,\ldots)$ of E such that $E = \bigcup_{k=1}^{\infty} E_k^{(i)}$. We set, for each $x \in E$,

$$v(0\,;\,n) \equiv \bigcap_{i=1}^{\infty} \bigcap_{k=1}^{n} E_k^{(i)c} + 0,\ v(x\,;\,n) \equiv v(0\,;\,n) + x,\ \mathfrak{V}_n(x) \equiv \{v(x\,;\,n)\}\ (n \in N),\ \mathfrak{V}_0(x) \equiv \{E\}.$$

Then $\{E, \mathfrak{V}_n\}$ fulfills the axiom (A), (B), (a) and (b). Now, for any $\{u_n(0)\}$, $\{v_n(0)\} \in \boldsymbol{F}(0)$

14) Let A be a subset of a set E and Φ a filter in E. We say that A is disjoint from Φ if there is B in Φ such that $A \cap B = \phi$.

($u_n(0) \in \mathfrak{B}_{r(n)}(0)$, $v_n(0) \in \mathfrak{B}_{s(n)}(0)$), we have

$$u_n(0)+v_n(0)=\bigcap_{i=1}^{\infty}\bigcap_{k=1}^{r(n)} E_k{}^{(i)c}+\bigcap_{i=1}^{\infty}\bigcap_{k=1}^{s(n)} E_k{}^{(i)c} \subseteq \bigcap_{i=1}^{\infty}\bigcap_{k=1}^{t(n)} E_k{}^{(i)c} \equiv w_n(0)\ (t(n)=\mathrm{Min}\{r(n),\ s(n)\})$$

and $\{w_n(0)\} \in F(0)$. Moreover, for any $\{\lambda_n\} \subset \Phi$ ($\lambda_n \xrightarrow{T} 0$) and $\{u_n(0)\} \in F(0)$, we have

$$\lambda_n \cdot u_n(0)=\lambda_n \cdot \bigcap_{i=1}^{\infty}\bigcap_{k=1}^{r(n)} E_k{}^{(i)c}=\bigcap_{i=1}^{\infty}\bigcap_{k=1}^{r(n)} \lambda_n \cdot E_k{}^{(i)c}=v_n(0)(\exists\{v_n(0)\} \in F(0)).$$

Therefore E is a linear ranked space.

(3). For any $x \in E$ there exists a linearly open set $A \ni x$ such that $\Phi_i \cap A = \phi$ or $\Phi_{iA} \supseteq \exists \Phi_{iA}'$ ($i=1,2,3,\ldots$). We assume $\Phi_i \cap A \neq \phi$ without losing generality. Let each Φ_{iA}' be generated by the complements of all the finite union of linear subspaces $E_k{}^{(i)}$ such that $E=\bigcup_{k=1}^{\infty} E_k{}^{(i)}$ ($i=1,2,3,\ldots$).

We will give a rank for E by the method as is taken in (2).

Then E becomes a linear ranked space. By using (1), (2) and (3) we can show this Proposition. (*Q.E.D.*)

M. Nakamura proved the following facts ([10]):

(1) Netted spaces by M. De Wilde–H. G. Garnir ⇨ *GN*–spaces.

(2) $\alpha\beta\gamma$–representable spaces by W. Słowiskowski ⇨ *GN*–spaces.

(3) Souslin spaces by A. Martineau–L. Schwartz ⇨ K–Souslin spaces by A. Martineau ⇔ quasi–Souslin spaces by M. Nakamura.

(4) Spaces with nets of type P by M. De Wilde ⇨ *G*–spaces.

Since *G. M.* -space, *GN*–space, quasi–Souslin space and *G*–space are defined by using *L*–covering or filters of Definition 11, these spaces have the structures as the linear ranked spaces. Moreover these spaces hold the condition (a^*) and (☆).

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References

[1] H. G. Garnir, M. De Wilde, J. Schmets: Analyse Fonctionelle, Tome I, Birkhäuser, Bâle, 1968; H. G. Garnir: Some new results in classical functional analysis. Proc. Intern. Conf. Functional Analysis and Related topics, Tokyo (1970), 361–367.

[2] I. M. Gel'fand: Generalized Functions. Vol. 1–5, Academic Press, New York (1964).

[3] A. Grothendieck: Produits Tensoriels Topologiques et Espaces Nucléaires, Mem. Amer. Math. Soc. **16** (1955).

[4] J. L. Kelley–I. Namioka: Linear Topological Spaces, Van Nostrand Co., New York, (1963); G. Köthe: Topologische lineare Räume I, Springer–Verlag, Berlin (1960).

[5] K. Kunugi: Sur les Espaces Complet et Régulièrement Complets, I–III, Proc. Japan Acad., **30** (1954), 553–556, 912–916, **31** (1955), 49–53; Sur la méthode des espaces rangés I, II, Ibid., **42** (1966), 318–322, 549–554.

[6] K. Kunugi, H. Okano, S. Nakanishi: Their works on (*E. R.*) integrals, Ibid., (1956–1970).

[7] H. Okano: Sur les Intégrales (*E. R.*) et ses Applications, Osaka Math. Jour., **11** (1959), 187–212.

[8] A. Martineau: Sur le théorèm du graphe fermé, C. R. Acad. Sc. paris, **263**, A (1966), 870–871; Sur des théorèms de S. Banach et de L. Schwartz concernant le graphe fermé, Studia Math. **30** (1968), 43–51.

[9] Y. Nagakura: The Theory of Nuclear Spaces Treated by the Method of Ranked Space I–VI, Proc. Japan Acad., (1971–1973).

[10] M. Nakamura: Her works on closed graph theorem, Ibid., (1970–1972); On closed graph theorem, The requested lecture to the Symosium on Functional Analysis, Math. Soc. Japan, (1970); Lectures to Math. Soc. Japan, (1970–1973).

[11] L. Schwartz : Sur le theorêm du graphe fermé, C. R. Acad. Sc. Paris, **263**, A (1966), 602–605.

[12] T. Shintani : On Ranked Groups, Lecture to Math. Soc. Japan, Shizuoka University, (1970) ; A direct product decomposition of the Ranked Group, Lecture to Math. Soc. Japan, Tokyo Metropolitan University, (1971) ; On Generalized Continuous Groups I, II, Mem. Tomakomai Tech. College, **6** (1971), 45–60, **7** (1972), 59–66 ; Some considerations in the Ranked Spaces, Ibid., **6** (1971), 61–65.

[13] ———— : A study of the Linear Ranked Spaces—On A. Grothendieck's problem, The requested lecture to the Symposium on Ranked Spaces and its Applications, Aug. 26, 1972, Nishinomiya, Japan.

[14] W. Słowiskowski : On continuity of inverse operators, Bull. Amer. Math. Soc., **67**, **5**, (1961), 467–470.

[15] F. Treves : Locally Convex Spaces and Linear Partial Differential Equations, Springer-Verlag, Berlin, (1967).

[16] M. Washihara : On ranked spaces and linearity II, Proc. Japan Acad., **45** (1969), 238–242.

[17] M. De Wilde : Sur le theorem du graphe ferme, C. R. Acad. Sc. Paris, **265**, A (1967), 376–379 ; Theoreme du graphe ferme et espaces a reseau absorbant, Bull. Math. Soc. Sc. Math. Phys. R. P. Roumanie, **11** (1967), 225–238 ; Reseaux dans les espaces lineaires a semi-normes, Mem. Soc. royale sc. Iiege, **5**, **18**, Fasc. 2.

[18] H. Yamagata : Extension of $\mathcal{F}(\Omega)$ and its application (to appear), (1972).

[19] M. Yamaguchi : Calculus in Ranked Vector Spaces I–V, Ibid., **44** (1968), 207–218, 307–317, 424–433 ; On a Ranked Vector Space, Ibid., **46** (1970), 422–426.

[20] T. Yamanaka : Linear Topological Spaces and Generalized Functions (In Japanese), Tokyo, (1966).