On the Completions of the Linear Ranked Spaces

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Synopsis

In this paper we shall construct the completions of the ranked groups and of the linear ranked spaces. These problems have not been solved yet. Our completion theorems are extensions of the completion theorem by G. Birkhoff. The nuclear space is an example of such completion.

Throughout this paper we shall use the same terminology that is introduced in [4] and [5]. We assume (a^*).

§1. The Completion of the Ranked Groups.

Let G be a group and e the identity of G.

Definition 1. ([5]). *For a ranked space* $\{G, \mathfrak{V}_\alpha\}$ *with any indicator* $\omega \geqq \omega_0$ *if G is a group and if the group operation*

$$\{G \times G, \mathfrak{V}_\alpha \times \mathfrak{V}_\alpha\} \ni \forall (x, y) \longmapsto xy^{-1} \in \{G, \mathfrak{V}_\alpha\}$$

is R-continuous, then $\{G, \mathfrak{V}_\alpha\}$ *is called a* ***ranked group*** *with indicator* ω. *We denote this by* $(G, \mathfrak{V}_\alpha)$.

Definition 2. ([5]). (i) *A sequence* $\{x_\alpha\} \subset (G, \mathfrak{V}_\alpha)$ *is called an* ***r-Cauchy sequence*** *of points in* $(G, \mathfrak{V}_\alpha)$ *iff*

$$\{\lim_{\alpha,\beta \uparrow \omega} x_\alpha x_\beta^{-1}\} \ni e \qquad \text{in}\,(G, \mathfrak{V}_\alpha).$$

(ii) $(G, \mathfrak{V}_\alpha)$ *is said to be* ***r-complete*** *if every r-Cauchy sequence of points in* $(G, \mathfrak{V}_\alpha)$ *is r-convergent in* $(G, \mathfrak{V}_\alpha)$.

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In a ranked group every r-convergent sequence is an r-Cauchy sequence, but the converse is not always true.

A ranked group is called a **T_0-group** iff for each pair x and y of distinct points there is a neighborhood of one point to which the other does not belong.

Let $\mathfrak{M}$ be the set of all r-Cauchy sequences of points in $(G, \mathfrak{B}_\alpha)$.

As an extension of G. Birkhoff's completion theorem [2], we have

Completion Theorem 1. *Let $(G, \mathfrak{B}_\alpha)$ be a given T_0 -group with any indicator $\omega \geqslant \omega_0$. Then there exists the completion $(\hat{G}, \hat{\mathfrak{B}}_\alpha)$ of $(G, \mathfrak{B}_\alpha)$ iff*

$$\{x_\alpha\} \in \mathfrak{M} \ \& \ \{\lim_{\beta} y_\beta\} \ni e \Rightarrow \{\lim_{\alpha,\beta\uparrow\omega} x_\alpha^{-1} y_\beta x_\alpha\} \ni e.$$

Proof. Suppose that $(G, \mathfrak{B}_\alpha)$ is r-complete i. e. we have

$$\{\lim_{\alpha} x_\alpha\} \ni \exists x \text{ in } (G, \mathfrak{B}_\alpha)$$

for all $\{x_\alpha\} \in \mathfrak{M}$. Since the group operation is R-continuous we have $\{\lim_{\alpha,\beta\uparrow\omega} x_\alpha^{-1} y_\beta x_\alpha\} \ni e$.

Let us verify the conversion. We will call two members $\{x_\alpha\}$ and $\{y_\beta\}$ of $\mathfrak{M}$ equivalent and write $\{x_\alpha\} \sim \{y_\beta\}$ iff

$$\{\lim_{\alpha,\beta\uparrow\omega} x_\alpha y_\beta^{-1}\} \ni e \text{ in } (G, \mathfrak{B}_\alpha).$$

$\sim$ is an equivalence relation on $\mathfrak{M}$. Set $E = \{x_\alpha\} (x_\alpha \equiv e$ for all α, $0 \leqslant \alpha < \omega)$.

By the relations

$$X \cdot Y \overline{\overline{\ }} \{x_\alpha \cdot y_\beta\} \text{ for all } X = \{x_\alpha\}, Y = \{y_\beta\} \in \mathfrak{M}$$

and

$$X^{-1} \overline{\overline{\ }} \{x_\alpha^{-1}\} \text{ for all } X = \{x_\alpha\} \in \mathfrak{M},$$

we obtain easily that $\mathfrak{M}$ forms a group and that E is the identity of $\mathfrak{M}$. Moveover, by the relations

$$V^*(E) \overline{\overline{\ }} \{\{x_\alpha\} \in \mathfrak{M} \mid x_\beta \in V(e) \ (\forall \beta \geqslant \exists \alpha_0)\}$$

and

$$V^*(E) \in \mathfrak{B}_\gamma^*(E) (0 \leqslant \gamma < \omega) \underset{def.}{\Longleftrightarrow} V(e) \in \mathfrak{B}_\gamma(e) (0 \leqslant \gamma < \omega),$$

$\mathfrak{M}$ becomes a ranked group with indicator ω.

Let $\mathfrak{N}$ ($\subseteq\mathfrak{M}$) denote the subset consisting of all r–Cauchy sequences which r–converge to the identity $e\in(G, \mathfrak{V}_\alpha)$. Then $\mathfrak{N}$ ($\ni E$) is a subgroup of ($\mathfrak{M}, \mathfrak{V}_\alpha{}^*$). Moreover, by the assumption we get that $\mathfrak{N}$ is a normal subgroup of ($\mathfrak{M}, \mathfrak{V}_\alpha{}^*$). Therefore, $\hat{G}\equiv\mathfrak{M}/\mathfrak{N}$ forms a quotient ranked group with indicator ω (See [5]) and it is written by $(\hat{G}, \hat{\mathfrak{V}}_\alpha)$.

By $\hat{x}$ we denote the class which contains an r–Cauchy sequence $\{x_\alpha\}$ for which

$$x_0=x_1=x_2=\cdots=x_\alpha=\cdots=x\in(G, \mathfrak{V}_\alpha).$$

By the mapping $f: x\longmapsto\hat{x}$ of $(G, \mathfrak{V}_\alpha)$ onto a subgroup $f(G)$ of $(\hat{G}, \hat{\mathfrak{V}}_\alpha)$ we obtain that G is isomorphic to the subgroup $f(G)\equiv\{\hat{x}\in\hat{G} \mid \forall x\in G\}$ of $\hat{G}$ and also that f is an r–isomorphism i. e. f is one–to–one and bi–R–continuous by the induced rank from $\hat{G}$ (See [5] and [6]).

Moreover, $f(G)$ is r–dense in $(\hat{G}, \hat{\mathfrak{V}}_\alpha)$ i. e. we have

$$\forall X\in(\hat{G}, \hat{\mathfrak{V}}_\alpha) \;\&\; X\ni\{x_\alpha\}\Rightarrow X\in\{\lim_\alpha \hat{x}_\alpha\}(\hat{x}_\alpha=f(x_\alpha) \text{ for } \{x_\alpha\}\subset G).$$

Finally, we will show that $(\hat{G}, \hat{\mathfrak{V}}_\alpha)$ is r–complete. For any r–Cauchy sequence $\{X_\alpha\}\subset\hat{G}$, there exists a sequence $\{x_\beta\}\subset(G, \mathfrak{V}_\alpha)$ such that

$$\{\lim_{\alpha,\beta\uparrow\omega} X_\alpha\cdot\hat{x}_\beta^{-1}\}=\{\lim_{\alpha,\beta\uparrow\omega} f(f^{-1}(x_\alpha)\cdot x_\beta^{-1})\}\ni\hat{e}.$$

We get easily $\{x_\beta\}\in\mathfrak{M}$. Since $\{\hat{x}_\beta\}\subset(\hat{G}, \hat{\mathfrak{V}}_\alpha)$ r–converges to the class which contains $\{x_\beta\}$, $\{X_\alpha\}$ is r–convergent in $(\hat{G}, \hat{\mathfrak{V}}_\alpha)$. Therefore, we get that $(\hat{G}, \hat{\mathfrak{V}}_\alpha)$ is r–complete. This completes the proof of the theorem.

§ 2. The Completion of the Linear Ranked Spaces.

Let E be a linear space over the real or complex field Φ and 0 the origin point of E.

Definition 3. ([5]). *Let $\mathcal{F}(f)$ denote the set of all fundamental sequences of x in $\{E, \mathfrak{V}_n\}(0\leqslant n<\omega_0)$. A linear space E over the real or complex field Φ is called a* ***linear ranked space*** *over Φ if this set E is a ranked space with* *indicator* ω_0 *and if the following conditions are fulfilled:*

(1) The mapping $(x, y) \longmapsto x+y$ of $E \times E$ into E is R-continuous.

(2) For any $\{u_n(x)\} \in \mathcal{F}(x)$ and for any sequence $\{\lambda_n\} \subset \Phi$ with $\lim_{n\to\infty} \lambda_n = \lambda$, there exists a $\{v_n(\lambda x)\} \in \mathcal{F}(\lambda x)$ such that $\lambda_n \cdot u_n(x) \subseteq v_n(\lambda x)$ for all n, $0 \leqslant n < \omega_0$.

We denote this by $[E, \mathfrak{V}_n]$.

Definition 4. ([5]). *A sequence* $\{f_n\} \subset [E, \mathfrak{V}_n]$ *is called an* ***r-Cauchy sequence*** *in* $[E, \mathfrak{V}_n]$ *iff*

$$\{\lim_{n,m\uparrow\infty} (x_m - x_n)\} \ni 0 \text{ in } [E, \mathfrak{V}_n].$$

$[E, \mathfrak{V}_n]$ *is said to be* ***r-complete*** *if every r-Cauchy sequence in* $[E, \mathfrak{V}_n]$ *is r-convergent in* $[E, \mathfrak{V}_n]$.

In a linear ranked space every r-convergent sequence is an r-Cauchy sequence, but the converse is not always true.

Completion Theorem 2. *For a given linear ranked space* $[E, \mathfrak{V}_n]$ *satisfying the axiom* (T_0), *there exists the completion* $[E^\wedge, \mathfrak{V}_n^\wedge]$ *of* $[E, \mathfrak{V}_n]$.

Proof. Let $\mathfrak{M}$ denote the set of all r-Cauchy sequences of points in $[E, \mathfrak{V}_n]$. We will call two members $X=\{f_n\}$ and $Y=\{g_n\}$ of $\mathfrak{M}$ equivalent and write $X \approx Y$ iff

$$\{\lim_{m,n\uparrow\infty} (f_m - g_n)\} \ni 0 \text{ in } [E, \mathfrak{V}_n].$$

$\approx$ is an equivalence relation on $\mathfrak{M}$.

By the relations

$$X+Y \overline{\overline{\leftarrow}} \{f_n+g_n\} \text{ for all } X=\{f_n\},\ Y=\{g_n\} \in \mathfrak{M},$$

$$\lambda \cdot x \overline{\overline{\leftarrow}} \{\lambda \cdot f_n\} \text{ for all } \lambda \in \Phi \text{ and for all } X=\{f_n\} \in \mathfrak{M}$$

and

$$\mathfrak{O} \overline{\overline{\leftarrow}} \{f_n\} (f_n = 0 \text{ for all } n,\ 0 \leqslant n < \omega_0),$$

$\mathfrak{M}$ forms a linear ranked space over Φ and $\mathfrak{O}$ is the origin point of $\mathfrak{M}$.

Let $\mathfrak{N}$ be the subset of $\mathfrak{M}$ consisting of all r-Cauchy sequences which

r-converges to the origin point $0\in[E,\mathfrak{V}_n]$. Then $\mathfrak{N}$ forms a linear subspace of $\mathfrak{M}$ and also $E^\wedge\equiv\mathfrak{M}/\mathfrak{N}$ forms a linear ranked space over Φ. It is denoted by $[E, \mathfrak{V}_n^\wedge]$.

By $f^\wedge$ we will denote the class which contains an r-Cauchy sequence $\{f_n\}$ for which

$$f_0=f_1=f_2=\cdots=f_n=\cdots=f\in[E, \mathfrak{V}_n].$$

By the mapping $\varphi: f\longmapsto f^\wedge$ of $[E, \mathfrak{V}_n]$ onto $\varphi(E)\subseteq E^\wedge$ we get that E is isomorphic to the ranked linear subspace $\{f^\wedge\in E \mid \forall f\in E\}$ of $[E^\wedge, \mathfrak{V}_n^\wedge]$ and also φ is an r-isomorphism by the induced rank from $[E^\wedge, \mathfrak{V}_n^\wedge]$. Moreover, $\varphi(E)$ is r-dense in $E^\wedge$ i. e. we have $E^\wedge\ni\forall X\ni\{f_n\}\Rightarrow X\in\{\lim_{n\to\infty} f_n^\wedge\}(f_n^\wedge=\varphi(f_n)$ for $\{f_n\}\subset E)$.

Finally, we will show that $[E^\wedge, \mathfrak{V}_n^\wedge]$ is r-complete. For every r-Cauchy sequence $\{A_m\}\subset[E^\wedge, \mathfrak{V}_n^\wedge]$, there exists a sequence $\{\alpha_n\}\subset[E, \mathfrak{V}_n]$ such that

$$\{\lim_{m,n\to\infty}(A_m-\alpha_n^\wedge)\}=\{\lim_{m,n\to\infty}\varphi(\varphi^{-1}(A_m)-\alpha_n)\}\ni 0^\wedge.$$

We get easily $\{\alpha_n\}\in\mathfrak{M}$. As $\{\alpha_n^\wedge\}\subset[E^\wedge, \mathfrak{V}_n^\wedge]$ r-converges to the class which contains $\{\alpha_n\}$, $\{A_m\}$ is r-convergent in $[E^\wedge, \mathfrak{V}_n^\wedge]$. Thus we have that $[E^\wedge, \mathfrak{V}_n^\wedge]$ is r-complete. This completes the proof of the theorem.

Finally, notice that the nuclear space is an example of such completion.

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